On Large-Lifting and Large-Supplemented Modules

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Abstract
In this paper, we introduce the concepts of Large-lifting and Large-supplemented modules as a generalization of lifting and supplemented modules. We also give some results and properties of this new kind of modules.

Keywords. L-small, L-lifting module, L-supplemented module.

1. Introduction
Throughout this paper, we assume that $R$ is a commutative ring with identity. A submodule $N$ of an $R$-module $M$ is called Large (essential) submodule in $M$ if for every nonzero submodule $K$ of $M$, then $N \cap K \neq 0$ [1]. A proper submodule $N$ of an $R$-module $M$ is called small ($N \ll M$), if for any submodule $K$ of $M$ such that $N + K = M$ implies that $K = M$ [1]. Assume that $N$ and $K$ are submodules of $M$, where $M$ is an $R$ module, then $N$ is called supplement of $K$ in $M$, if $N$ is minimal with respect to the property $M = N + K$. This is equivalent to $M = N + K$ and $N \cap K \ll N$, if every submodule of $M$ has a supplement in $M$, then $M$ is called supplemented module [2].

An $R$-module $M$ is called lifting, if for every submodule $N$ of $M$ there exists a submodule $K$ of $M$ such that $M = K \square H$ and $N \cap H \ll H$ where $H$ be a submodule of $M$, equivalently $M$ is called lifting, if and only if for every submodule $N$ of $M$ there exists a submodule $K$ of $N$ such that $M = K \square H$ and $N \cap H \ll M$ [2]. In [3], we give the concept of Large-small (L-small) submodule, it is given as follows; Let $N$ be a proper submodule of $M$, then $N$ is called L-small submodule of $M (N \ll L M)$, if $N + K = M$ where $K \leq M$, then $K$ is essential submodule of $M$ ($K \leq e M$). In [4], we also give the concept of Large-coessential (L-coessential) submodule. It is given as follows; Let $M$ be an $R$-module and $K, N$ are submodules of $M$ such that $K \leq N \leq M$, then $K$ is said to be Large-coessential submodule, if $N \ll L K$. This paper consists two sections, in section one we

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give the concept of Large-lifting (L-lifting) modules and some of its properties, such that an R-module $M$ is said to be L-lifting, if for every submodule $N$ of $M$ there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $N \cap H \ll_L M$ where $H$ is a submodule of $M$. In section two we introduce the concept of Large-supplemented (L-supplemented) modules, such that an R-module $M$ is called L-supplemented, if every submodule of $M$ has L-supplement in $M$, where a submodule $N$ is called L-supplement of $K$ in $M$, if $M = N + K$ and $N \cap K \ll_L N$. In Lemma(1.1), Lemma(1.2) and Lemma(1.3) we give some properties in [3] and [4] that we need in this paper.

**Lemma 1.1[3]:** (1) Let $M$ be an R-module and $K, N$ be submodules of $M$ such that $K \leq N \leq M$, if $N \ll_L M$ then $K \ll_L M$.

(2) Let $f: M \to M'$ be an epimorphism where $M$ and $M'$ are R-modules such that $N \ll_L M'$ then $f^{-1}(N) \ll_L M$.

(3) Let $M$ be an R-module and $K, N$ be submodules of $M$ where $K$ is a closed in $M$ such that $K \leq N \leq M$, if $N \ll_L M$ then $K \ll_L M$ and $\frac{N}{K} \ll_L \frac{M}{K}$.

(4) Let $M$ be an R-module and $K, N$ be submodules of $M$ such that $K \leq N \leq M$, and $N$ is direct summand of $M$, if $K \ll_L M$, then $K \ll_L N$.

**Lemma 1.2[3]:** (1) Let $M = \bigoplus_{i \in I} M_i$ be a fully stable module, if $N_i \ll_L M_i$ then $\bigoplus_{i \in I} N_i \ll_L \bigoplus_{i \in I} M_i$.

(2) Let $M$ be an R-module such that $M$ is faithful, finitely generated and multiplication module and let $I$ be an ideal of $R$ then $I \ll_L R$ if and only if $IM \ll_L M$.

**Lemma 1.3[4]:** (1) Let $M$ be an R-module and $K, N$ be submodules of $M$ such that $K \leq N \leq M$, if $\frac{N}{K} \ll_L \frac{M}{K}$ then $N \ll_L M$.

(2) Let $M$ be an R-module and $K, N$, and $U$ be submodules of $M$ such that $K \leq N \leq U \leq M$, then $N \ll_L U$ in $M$ if and only if $\frac{N}{K} \ll_L \frac{U}{K}$ in $\frac{M}{K}$.

Now, we need to prove the following lemma.

**Lemma 1.4:** Let $M = M_1 \oplus M_2$ then $N_1 \ll_L M_1$ and $N_2 \ll_L M_2$ if and only if, $N_1 \cap N_2 \ll_L M_1 \oplus M_2$.

**Proof:** ($\Rightarrow$) Let $U_1 \oplus U_2$ be a submodule of $M_1 \oplus M_2$ such that $N_1 \oplus N_2 \subseteq U_1 \oplus U_2 = M_1 \oplus M_2$. So that $(N_1 + U_1) \oplus (N_2 + U_2) = M_1 \oplus M_2$ and hence $N_1 + U_1 = M_1$ and $N_2 + U_2 = M_2$. Since $N_1 \ll_L M_1$ and $N_2 \ll_L M_2$, then $U_1 \subseteq e_1 M_1$ and $U_2 \subseteq e_2 M_2$, this implies that $U_1 \oplus U_2 \subseteq \ll_L M_1 \oplus M_2$ by [1], and therefore $N_1 \oplus N_2 \ll_L M_1 \oplus M_2$.

($\Leftarrow$) Let $N_1 \oplus N_2 \subseteq \ll_L M_1 \oplus M_2$. Since $N_1 \leq N_1 \oplus N_2 \ll_L M_1 \oplus M_2 = M$ then by Lemma(1.1), we have $N_1 \ll_L M$ and since $N_1 \leq M_1 \leq M$ and $M_1$ is direct summand of $M$ then by Lemma(1.1) we get $N_1 \ll_L M_1$. Similarly we have $N_2 \ll_L M_2$.

2. **Large-Lifting modules.**

In this section we introduce the concept of Large-lifting modules and some properties of it are considered.

**Definition 2.1:** An R-module $M$ is called Large-lifting (L-lifting), if for every submodule $N$ of $M$ there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $N \cap H \ll_L M$ where $H$ is a submodule of $M$.

**Remarks and Examples 2.2:**

(1) Every lifting is L-lifting.

**Proof:** Let $M$ be a lifting module and $N \leq M$, then $M = K \oplus H$ where $K \leq N$ and $N \cap H \ll M$ so $N \cap H \ll_L M$ where $H \leq M$ by [3].

(2) The following example shows that the converse of (1) is not true.

Example: $Z$ as $Z$-module is L-lifting since for $N = nZ \leq Z$, there exists $\{0\}$ direct summand of $nZ$ such that $M = Z = \{0\} + Z$ and $nZ \cap Z = nZ \ll_L Z$ by [3], also if $N = Z \leq Z$, let $K = Z \leq N$ such that $Z = Z \oplus \{0\}$ and $Z \cap \{0\} = \{0\} \ll_L Z$, but $Z$ is not lifting since $nZ$ no
t small submodule in Z.

(3) $Z_{24}$ as Z-module is not L-lifting since, Let $N = Z_{24}$, the only direct summand of $Z_{24}$ are \{0\} and $3Z_{24}$, $8Z_{24}$ such that $Z_{24} = K \oplus H$. If $K = \{0\}$ thus $H = Z_{24}$ and $N \cap H = Z_{24} \cap Z_{24} = Z_{24}$ which is not L-small in $Z_{24}$ and if $K = 3Z_{24}$ thus $H = 8Z_{24}$ and $Z_{24} \cap 8Z_{24} = 8Z_{24}$ which is not L-small in $Z_{24}$ and if $K = 8Z_{24}$ thus $H = 3Z_{24}$ and $Z_{24} \cap 3Z_{24} = 3Z_{24}$ which is not L-small in $Z_{24}$.

(4) Every semisimple module is lifting [2], hence L-lifting by (1). Thus $Z_6$ as Z-module is L-lifting.

(5) Let $M$ be a semisimple module, then $M$ is lifting if and only if $M$ is L-lifting.

(6) Every hollow module is lifting [2], hence L-lifting by (1). Thus $Z_4$ as Z-module is hollow, so it is L-lifting.

Recall that an R-module $M$ is called L-hollow module if every proper submodule of $M$ is L-small submodule in $M$ [3].

**Remark 2.3:** Every L-hollow module is L-lifting.

**Proof:** Let $M$ be an L-hollow module and $N$ be a proper submodule of $M$ and let $M = \{0\} \oplus M$ and $N \cap M = N \ll L M$, so that $M$ is L-lifting. The converse of previous remark is not true, the following example: $Z_6$ as Z-module is L-lifting by (4) but not L-hollow by [3].

**Remark 2.4:** Every Local module is hollow so L-hollow [3], hence it is L-lifting by Remark(2.3), where an R-module $M$ is called local if it is hollow and has a unique maximal submodule [5].

**Proposition 2.5:** Let $M$ be an indecomposable, then $M$ is L-hollow if and only if $M$ is L-lifting.

**Proof:** Clear from Remark (2.3).

Let $M$ be L-lifting and $N$ be a proper submodule of $M$ and let $K \leq N$ such that $M = K \oplus H$ where $H \leq M$ and $N \cap H \ll L M$, since $M$ is indecomposable, then either $K = 0$ or $K = M$. If $K = M$ then $N = M$ and this is a contradiction, so that $K = 0$, and hence $H = M$, so $N = N \cap M = N \cap H \ll L M$. Therefore $M$ is L-hollow.

The characterization of L-lifting module is given by the next theorem.

**Theorem 2.6:** Let $M$ be an R-module, then the following statements are equivalent:

1- $M$ is L-lifting module .

2- Every submodule $N$ of $M$ can be written as $N = V \oplus W$ where $V$ direct summand of $M$ and $W \ll L M$.

3- Every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $K \leq N$ and $\frac{N}{K} \ll L \frac{M}{K}$.

**Proof:** (1) $\Rightarrow$ (2) Let $N$ be a submodule of $M$ then there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $N \cap H \ll L M$ where $H$ is a submodule of $M$. Now $N = N \cap M = N \cap (K \oplus H) = K \oplus (N \cap H)$ by modular law. Let $V = K$ and $W = N \cap H$, so $N = V \oplus W$ where $V$ direct summand of $M$ and $W \ll L M$.

(2) $\Rightarrow$ (3) Let $N$ be a submodule of $M$ and $N = V \oplus W$ where $V$ direct summand of $M$ and $W \ll L M$. It is enough to show that $\frac{N}{V} \ll L \frac{M}{V}$. Let $\frac{U}{V} \leq L \frac{M}{V}$ such that $\frac{U}{V} + \frac{V}{V} = \frac{M}{V}$ so $\frac{V}{V} \oplus \frac{W}{V} + \frac{U}{V} = \frac{M}{V}$, hence $M = V + W + U = W + U$. Since $W \ll L M$, then $U \ll L M$, and since $V$ direct summand of $M$ then $V$ is closed in $M$, from [6-10], we have $\frac{U}{V} \leq L \frac{M}{V}$, so that $\frac{N}{V} \ll L \frac{M}{V}$.

(3) $\Rightarrow$ (1) Let $N$ be a submodule of $M$ then there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $\frac{N}{K} \ll L \frac{M}{K}$. By Lemma(1.3), we have $N \ll L M$ by and since $N \cap H \leq N \leq M$ so we get $N \cap H \ll L M$ by Lemma(1.1).
Proposition 2.7: Let $M$ be an indecomposable module, then $M$ is not $L$-lifting for every nontrivial submodule $N$ of $M$.

Proof: Suppose that $M$ is $L$-lifting and by theorem (2.6), let $N = K + H$ where $K$ direct summand of $M$ and $H \ll_L M$, since $M$ be an indecomposable then $K = 0$, hence $N = H \ll_L M$ and this is contradiction, so $M$ is not $L$-lifting for every nontrivial submodule $N$ of $M$.

Proposition 2.8: Any direct summand of $L$-lifting module is $L$-lifting.

Proof: Let $M$ be $L$-lifting and assume that $M = M_1 \oplus M_2$. In order to show $M_1$ is $L$-lifting, let $N \leq M_1$ so that $N \leq M$ and by theorem (2.6), let $N = V \oplus W$ where $V$ direct summand of $M$ and $W \ll_L M$ hence $W \ll_L M_1$ by Lemma(1.1). Now, $M = V \oplus H$ where $H \leq M$, since $V$ direct summand of $M$, then we get the result if we prove $V$ direct summand of $M_1$, so $M_1 = M_1 \cap M = M_1 \cap (V \oplus H) = V \oplus (M_1 \cap H)$ by modular law, hence $V$ direct summand of $M_1$, so $M_1$ is $L$-lifting.

Theorem 2.9: Let $M$ be an $R$-module, then the following statements are equivalent:

1- $M$ is $L$-lifting module.

2- For each submodule $N$ of $M$, there exists $\phi \in \text{End}(M)$ such that $\phi^2 = \phi$, $\phi(M) \leq N$ and $(1 - \phi)(N) \ll_L M$.

Proof: (1) $\Rightarrow$ (2) Let $N$ be a submodule of $M$ then there exists a submodule $K$ of $N$ such that $M = K \oplus H$ and $N \cap H \ll_L M$ where $H$ be a submodule of $M$. Let $\phi : M \rightarrow K$ be a projection map clearly $\phi^2 = \phi$ and $M = K \oplus H = \phi(M) \oplus (1 - \phi)(M)$, $\phi(M) \leq N$. Now $(1 - \phi)(N) = N \cap (1 - \phi)(M) = N \cap H \ll_L M$, so $(1 - \phi)(N) \ll_L M$.

(2) $\Rightarrow$ (1) Let $N$ be a submodule of $M$ then there exists $\phi \in \text{End}(M)$ such that $\phi^2 = \phi$, $\phi(M) \leq N$ and $(1 - \phi)(N) \ll_L M$. Clearly that $M = \phi(M) \oplus (1 - \phi)(M)$, let $K = \phi(M)$ and $H = (1 - \phi)(M)$, hence $N \cap H = N \cap (1 - \phi)(M)$. To show $N \cap (1 - \phi)(N) = (1 - \phi)(N)$, let $u = (1 - \phi)(v) \in N \cap (1 - \phi)(M)$, since $(1 - \phi)^2 = (1 - \phi)$ so $u = (1 - \phi)^2(v) = (1 - \phi)(v) \in (1 - \phi)(N)$. Now let $u = (1 - \phi)(v) \in (1 - \phi)(N)$; $v \in N$, then $u \in (1 - \phi)(M)$, $u = (1 - \phi)(v) \in N$, hence $u \in N \cap (1 - \phi)(M)$ so $N \cap H = N \cap (1 - \phi)(M) = (1 - \phi)(N) \ll_L M$, hence $N \cap H \ll_L M$, so $M$ is $L$-lifting module.

Remark 2.10: The following example shows that if $M$ is $L$-lifting module and $N$ is a submodule of $M$, then $M/N$ need not to be $L$-lifting module.

Example: Let $Z$ be $L$-lifting module and $24Z \leq Z$ but $Z_{24Z}$ is not $L$-lifting by (2.2).

Now, we introduce the following proposition in which $M/N$ be $L$-lifting module.

Proposition 2.11: Let $M$ be $L$-lifting module and $W$ be a submodule of $M$ such that for every direct summand $K$ of $M$, $K + W$ direct summand of $M/W$, then $M/W$ is $L$-lifting.

Proof: Let $N/W \leq M/W$, since $M$ is $L$-lifting, then by theorem (2.6), there exists $K \leq N$ such that $M = K \oplus H$; $H \leq M$ and $N/K \ll_L M$, because of $K + W$ is direct summand of $M$, we have $N/K + W \ll_L M/K + W$, so $K + W \ll_L N$ in $M$ and by Lemma(1.3), we get $K + W \ll_L N/W$ in $M/W$, hence $N/W \ll (K + W)/W$, therefore $M/W$ is $L$-lifting.

An $R$-module is called distributive, if for all submodules $K$, $N$ and $U$ of $M$, then $K \cap (N + U) = (K \cap N) + (K \cap U)$ [9].

Corollary 2.12: Let $M$ be $L$-lifting and distributive module and let $W$ be a submodule of $M$ then $M/W$ is $L$-lifting.

Proof: Let $K$ be a direct summand of $M$, such that $M = K \oplus U$ for some submodule $U$ of $M$, hence $M/W = (K \oplus U)/W = K + W/W + U/W$ and since $M$ is distribution module, then $(K + W) \cap$
\[(U + W) = ((K + W) \cap U) + ((K + W) \cap W) = (K \cap U) + (W \cap U) + (K \cap W) + W = W,\] hence \[\frac{M}{W} = \frac{K + W}{W} \oplus \frac{U + W}{W}\] and by proposition \((\ref{prop11})\), we get \(\frac{M}{W}\) is \(L\)-lifting.

**Lemma 2.13** [6]: Let \(M = M_1 \oplus M_2\) be an \(R\)-module, then \(\frac{M}{A} = \frac{A + M_1}{A} \oplus \frac{A + M_2}{A}\) for every fully invariant submodule \(A\) of \(M\).

**Corollary 2.14**: Let \(M\) be \(L\)-lifting module if \(W\) is fully invariant submodule of \(M\) then \(\frac{M}{W}\) is \(L\)-lifting.

**Proof**: It directly comes from Lemma \((\ref{lem213})\) and proposition \((\ref{prop11})\).

### 3. Large-Supplemented modules

In this section we introduce the concept of Large-supplemented modules. Some results are also given.

**Definition 3.1**: Let \(M\) be an \(R\)-module and \(N, K\) are submodules of \(M\), then \(N\) is called Large-supplement (\(L\)-supplement) of \(K\) in \(M\), if \(M = N + K\) and \(N \cap K \ll_L N\). If every submodule of \(M\) has \(L\)-supplement, then \(M\) is called \(L\)-supplemented module.

**Remarks and Examples 3.2:**

1. Every supplemented module is \(L\)-supplemented.

**Proof**: Let \(M\) be a supplemented module and \(N\) be a submodule of \(M\), then \(N\) is a supplement of \(K\) in \(M\), so \(M = N + K\) and \(N \cap K \ll_L N\) by \([3]\), so \(N\) is \(L\)-supplement of \(K\) in \(M\), hence \(M\) is \(L\)-supplemented.

2. Next example indicates that the converse of \((\ref{prop1})\) is not true.

Example: \(Z\) as \(Z\)-module is \(L\)-supplemented since let \(n, m \in N\), \(nZ\) is \(L\)-supplement of \(mZ\) since \(Z = nZ + mZ\) and \(nZ \cap mZ = (nm)Z \ll_L nZ\), but \(Z\) is not supplemented since \(nZ\) is not supplement in \(Z\) since \(Z = nZ + mZ\) and \(nZ \cap mZ = (nm)Z\) but \((nm)Z\) is not small in \(nZ\), since \(\{0\}\) is the only small submodule.

3. Let \(M\) be a semisimple module, then \(M\) is supplemented if and only if, \(M\) is \(L\)-supplemented.

4. Next example shows that if \(N\) and \(K\) are submodules of \(M\), and \(N\) is \(L\)-supplement of \(K\) in \(M\), then it is not necessary that \(K\) is \(L\)-supplement of \(N\) in \(M\).

Example: In \(Z_4\) as \(Z\)-module, \(Z_4\) is \(L\)-supplement of \(\{0, 2\}\) in \(Z_4\) since \(Z_4 = Z_4 + \{0, 2\}\) and \(Z_4 \cap \{0, 2\} = \{0, 2\} \ll_L Z_4\) but \(\{0, 2\}\) is not \(L\)-supplement of \(Z_4\) in \(Z_4\) since \(Z_4 = \{0, 2\} + Z_4\) and \(\{0, 2\} \cap Z_4 = \{0, 2\}\) but \(\{0, 2\}\) is not \(L\)-small in \(\{0, 2\}\).

5. In \(Z_6\) as \(Z\)-module where \(Z_6 = \{0, 3\} \oplus \{0, 2, 4\}\) then \(\{0, 3\}\) is \(L\)-supplement of \(\{0, 2, 4\}\) since \(Z_6 = \{0, 3\} + \{0, 2, 4\}\) and \(\{0, 3\} \cap \{0, 2, 4\} = \{0\} \ll_L \{0, 3\}\) also \(\{0, 2, 4\}\) is \(L\)-supplement of \(\{0, 3\}\).

6. Every semisimple module is \(L\)-supplemented.

7. In \([2]\), authors proved that every direct summand of \(M\) is supplement submodule of \(M\), hence it is \(L\)-supplement by \((\ref{prop1})\).

8. Let \(M\) be an \(R\)-module and \(N\) be \(L\)-hollow of \(M\), then \(N\) is \(L\)-supplement of each proper submodule \(K\) of \(M\) such that \(M = N + K\).

**Proof**: Let \(K\) be a proper submodule of \(M\) such that \(M = N + K\). It is clear that \(N \cap K \neq N\), since if \(N \cap K = N\), then \(N \leq K\) hence \(K = M\) and this is a contradiction. Since \(N\) is \(L\)-hollow then \(N \cap K \ll_L N\), so \(N\) is \(L\)-supplement of \(K\) in \(M\).

9. Let \(M\) be an \(R\)-module, then every \(L\)-small submodule of \(M\) has \(L\)-supplement in \(M\).

**Proof**: Let \(N\) be \(L\)-small submodule of \(M\), so that \(M = N + M\) and \(N \cap M = N \ll_L M\), therefore \(M\) is \(L\)-supplement of \(N\) in \(M\).

10. The converse of \((\ref{prop9})\) is not true, for example \(Z_6\) as \(Z\)-module.

**Proposition 3.3**: Let \(M\) be an \(R\)-module and \(N, K\) be submodules of \(M\) such that \(N \leq K \leq M\) and \(N\) is closed in \(K\), if \(K\) is \(L\)-supplement of \(H\) in \(M\) then \(\frac{K}{N}\) is \(L\)-supplement of \(\frac{H+N}{N}\) in \(\frac{M}{N}\).

**Proof**: Since \(K\) is \(L\)-supplement of \(H\) in \(M\), then we have \(M = K + H\) and \(K \cap H \ll_L K\). Now
\[
\frac{M}{N} = \frac{K+H}{N} = \frac{K}{N} + \frac{H+N}{N}, \text{ we have to show that } \frac{K}{N} \cap \frac{H+N}{N} \ll_L \frac{K}{N}, \text{ so that } \frac{K}{N} \cap \frac{H+N}{N} = \frac{K(\cap H+N)}{N} = \frac{K}{N} \text{ by modular law. Let } \frac{U}{N} \leq \frac{K}{N} \text{ where } U \leq K \text{ and } N \leq U \text{ such that } \frac{(K(\cap H+N))}{N} + \frac{U}{N} = \frac{K}{N}, \text{ so } \frac{U}{N} \leq e \frac{K}{N}, \text{ therefore } \frac{K}{N} \text{ is } L\text{-supplement of } \frac{H+N}{N} \text{ in } \frac{M}{N}.
\]

**Proposition 3.4:** Let \( f: M \rightarrow M' \) be an epimorphism, if \( M' \) is \( L\)-supplemented module then \( M \) is \( L\)-supplemented.

**Proof:** Let \( H \leq M \), then \( f(H) \leq M' \), since \( f \) is \( L\)-supplemented then there exists \( K \) is \( L\)-supplement of \( f(H) \) in \( M' \), so \( M' = K + f(H) \) and \( K \cap f(H) \ll_L K \). Now \( f^{-1}(K + f(H)) = f^{-1}(K) + H = M \) and since \( K \cap f(H) \ll_L K \) then \( f^{-1}(K \cap f(H)) \ll_L f^{-1}(K) \) by Lemma(1.1), hence \( f^{-1}(K) \cap H \ll_L f^{-1}(K) \), so \( f^{-1}(K) \) is \( L\)-supplement of \( H \) in \( M \), hence \( M \) is \( L\)-supplemented.

**Proposition 3.5:** Let \( M \) be an \( R\)-module and \( N, K \) are submodules of \( M \) such that \( K \) is \( L\)-supplement of \( N \) in \( M \), if \( M = H + K \) for some submodule \( H \) of \( N \), then \( K \) is \( L\)-supplement of \( H \) in \( M \).

**Proof:** Suppose \( M = H + K \) for some submodule \( H \) of \( K \) and \( K \) is \( L\)-supplement of \( N \) in \( M \), so we have \( M = N + K \) and \( N \cap K \ll_L K \), and since \( H \cap K \leq N \cap K \ll_L K \), then \( H \cap K \ll_L K \) by Lemma(1.1), hence \( K \) is \( L\)-supplement of \( H \) in \( M \).

**Proposition 3.6:** Let \( M \) be an \( R\)-module and \( N, K \) and \( U \) are submodules of \( M \) such that \( N \leq K \), if \( N \) is \( L\)-supplement of \( U \) in \( M \) then \( N \) is \( L\)-supplement of \( U \cap K \) in \( K \).

**Proof:** Since \( N \) is \( L\)-supplement of \( U \) in \( M \) then we have, \( M = N + U \) and \( N \cup U \ll_L N \). Now \( K = M \cap K = (N + U) \cap K = N + (U \cap K) \) by modular law, and since \( (U \cap K) \leq N \cap U \ll_L N \), so we get \( N \cap (U \cap K) \ll_L N \) by Lemma(1.1), hence \( N \) is \( L\)-supplement of \( U \cap K \) in \( K \).

**Proposition 3.7:** Let \( M = M_1 \oplus M_2 \), if \( N_1 \) is \( L\)-supplement of \( N_2 \) in \( M_1 \) and \( K_1 \) is \( L\)-supplement of \( K_2 \) in \( M_2 \), then \( N_1 \oplus K_1 \) is \( L\)-supplement of \( N_2 \oplus K_2 \) in \( M \).

**Proof:** Since \( N_1 \) is \( L\)-supplement of \( N_2 \) in \( M_1 \) and \( K_1 \) is \( L\)-supplement of \( K_2 \) in \( M_2 \), then we have \( M_1 = N_1 + N_2 \) and \( N_1 \cap N_2 \ll_L N_1 \), we also have \( M_2 = K_1 + K_2 \) and \( K_1 \cap K_2 \ll_L K_1 \), so \( M = M_1 \oplus M_2 = (N_1 + N_2) \oplus (K_1 + K_2) = (N_1 \oplus K_1) + (N_2 \oplus K_2) \). Since \( N_1 \cap N_2 \ll_L N_1 \) and \( K_1 \cap K_2 \ll_L K_1 \), then by Lemma(1.4), we have \( (N_1 \cap N_2) \oplus (K_1 \cap K_2) \ll_L N_1 \oplus K_1 \). Clearly \( (N_1 \oplus K_1) \cap (N_2 \oplus K_2) = (N_1 \cap N_2) \oplus (K_1 \cap K_2) \ll_L N_1 \oplus K_1 \), hence \( N_1 \oplus K_1 \) is \( L\)-supplement of \( N_2 \oplus K_2 \) in \( M \).

**Proposition 3.8:** Let \( M \) be faithful, finitely generated and multiplication module over commutative ring \( R \) and \( N \) be a submodule of \( M \), if \( N \) is \( L\)-supplement of \( IM \) in \( M \), then \( J \) is \( L\)-supplement of \( I \) in \( R \), where \( I, J \) are ideals of \( R \).

**Proof:** Since \( N \) is \( L\)-supplement of \( IM \) in \( M \), then we have \( M = N + IM \) and \( N \cap IM \ll_L N \), since \( M \) is multiplication then \( N \cap IM = JM \). Now \( M = RM = IM + JM = (I + J)M \) and since \( M \) is faithful, finitely generated and multiplication, then \( M \) is cancellation by [8], so \( R = I + J \) also we have \( IM \cap N = IM \cap JM = (I \cap J)M \ll_L N = JM \), hence \( (I \cap J)M \ll_L JM \). To show \( I \cap J \ll_L J \), let \( H \) be an ideal of \( R \) such that \( (I \cap J) + H = J \), so \( (I \cap J)M + HM = JM \) and since \( (I \cap J)M \ll_L JM \), then \( HM \ll_L JM \), so \( H \ll_L J \) so we get the result, and hence \( J \) is \( L\)-supplement of \( I \) in \( R \).

The characterization of \( L\)-supplement submodules is given in the next theorem.

**Theorem 3.9:** Let \( M \) be an \( R\)-module and \( N, K \) are submodules of \( M \), then the following statements are equivalent:

1. \( K \) is \( L\)-supplement of \( N \) in \( M \).
2. \( M = N + K \) and for every non-essential submodule \( H \) of \( K \), then \( M \neq N + H \).
Proof: (1) $\Rightarrow$ (2) Assume $K$ is $L$-supplement of $N$ in $M$, so we have $M = N + K$ and $N \cap K \ll L K$ and suppose $M = N + H$ where $H$ is non-essential submodule of $K$, so $K = K \cap M = K \cap (N + H) = H + (N \cap K)$ by modular law, and since $N \cap K \ll L K$ so we have $H \leq K$ and this is a contradiction, so that $M \neq N + H$.

(2) $\Rightarrow$ (1) From (2) $M = N + K$, we must show $N \cap K \ll L K$. Let $U \leq K$ such that $(N \cap K) + U = K$, if $U$ is non-essential submodule of $K$, then by assumption $M \neq N + U$, so $M = N + K = N + (N \cap K) + U = N + U$ and this is a contradiction, so that $U \leq K$ , hence $N \cap K \ll L$, and we get $K$ is $L$-supplement of $N$ in $M$.

Proposition 3.10: Let $M$ be an $R$-module and $M_1$, $H$ are submodules of $M$, such that $M_1$ is $L$-supplemented module, if $M_1 + H$ has $L$-supplement in $M$ then $H$ has $L$-supplement in $M$.

Proof: By assumption $M_1 + H$ has $L$-supplement in $M$, so there exists $U \leq M$ such that $M_1 + H + U = M$ and $(M_1 + H) \cap U \ll L U$, since $M_1$ is $L$-supplemented then $(H + U) \cap M_1 \leq M_1$ has $L$-supplement in $M_1$, so there exists $V \leq M_1$ such that $((H + U) \cap M_1) + V = M_1$ and $(H + U) \cap V \ll L V$. Now $M = M_1 + H + U = ((H + U) \cap M_1) + V + H + U = H + (V + U)$. One can easily show $H \cap (V + U) \leq ((H + V) \cap U) + ((H + U) \cap V) \ll L U + V$ by Lemma (1.4), so $H \cap (V + U) \ll L U + V$ and $V + U$ is $L$-supplement of $H$ in $M$, hence $H$ has $L$-supplement in $M$.

Proposition 3.11: Let $M = M_1 \oplus M_2$ such that $M_1$ and $M_2$ are $L$-supplemented modules then $M$ is $L$-supplemented module.

Proof: Let $H \leq M$ and since $M_1 + M_2 + H = M$, so it is trivial has $L$-supplement in $M$. By proposition (3.10) and since $M_1$ is $L$-supplemented, then $M_2 + H$ has $L$-supplement in $M$, again by proposition (3.10) and since $M_2$ is $L$-supplemented, then $H$ has $L$-supplement in $M$, and hence $M$ is $L$-supplemented module.

References