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Strong Differential Sandwich Results for Analytic Functions Associated with Wanas Differential Operator

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Abstract
In this article, we introduce and study two new families of analytic functions by using strong differential subordinations and superordinations associated with Wanas differential operator/. We also give and establish some important properties of these families.

Keywords: Strong subordinations, Convex function, Strong superordinations, Best dominant, Wanas operator and Best subordinant.

نتائج الساندوج التفاضلية القوية للدوال التحليلية المرتبطة بمؤثر وناس التفاضلي

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الخلاصة:
في هذا البحث، نقدم وندرس عائلتين جديدتين من الدوال التحليلية باستخدام التابعية وفوق التابعية التفاضلية القوية المرتبطة بمؤثر وناس التفاضلي ونبرهن بعض الخواص المهمة لهذه العوائل.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the open unit disk and the closed unit disk of the complex plane, respectively. We assume that $\mathrm{H}(U \times \overline{U})$ is the set of analytic functions in $U \times \overline{U}$. For a positive integer n and $a \in \mathbb{C}$, suppose that
$\mathrm{H}[a, \mathrm{n}, \tau] = \{f \in \mathrm{H}(U \times \overline{U}): f(z, \tau) = a + a_{\mathrm{n}}(\tau)z^{\mathrm{n}} + a_{\mathrm{n}+1}(\tau)z^{\mathrm{n}+1} + \cdots, z \in U, \tau \in \overline{U}\}$,
where $a_j(\tau)$ are holomorphic functions in $\overline{U}$ for $j \geq \mathrm{n}$.
Let A_τ be the collection of functions of the shape:

$$f(z, \tau) = z + \sum_{\kappa=2}^{\infty} a_\kappa(\tau) z^\kappa, \quad (z \in U, \tau \in \overline{U}), \tag{1.1}$$

which are analytic in $U \times \overline{U}$ and $a_\kappa(\tau)$ are holomorphic functions in $\overline{U}$ for $\kappa \geq 2$.

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Definition (1.1) [1]: Let Q_τ be analytical and injective functions set on $\bar{U} \times \bar{U} \backslash E(f,\tau)$ such that

$E(f,\tau) = \left\{ r \in \partial U : \lim_{z \to r} f(z,\tau) = \infty \right\}$,

with $f_z'(r,\tau) \neq 0$ for $r \in \partial U \times \bar{U} \backslash E(f,\tau)$. The subclass of Q_τ for any of them $f(0,\tau) = a$ is represented by $Q_\tau(a)$.

Definition (1.2) [2]: Let $f(z,\tau), F(z,\tau)$ be analytic functions in $U \times \bar{U}$. The function $f(z,\tau)$ is said to be strongly subordinate to $F(z,\tau)$ if there exists a function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z,\tau) = F(w(z),\tau)$ for all $\tau \in \bar{U}$. In such case, we write $f(z,\tau) \prec\prec F(z,\tau), z \in U, \tau \in \bar{U}$.

Remark (1.1) [2]:

(1) Since $f(z,\tau)$ is analytic in $U \times \bar{U}$ for each $\tau \in \bar{U}$ and univalent in U for all $\tau \in \bar{U}$, then the Definition (1.2) is the same as to $f(0,\tau) = F(0,\tau)$ for each $\tau \in \bar{U}$ and $f(U \times \bar{U}) \subset F(U \times \bar{U})$.

(2) If $f(z,\tau) = f(z)$ and $F(z,\tau) = F(z)$, then the strong subordination becomes the usual notion of subordination.

If $f(z,\tau)$ is strongly subordinate to $F(z,\tau)$, then $F(z,\tau)$ is strongly superordinate to $f(z,\tau)$.

Lemma (1.1) [3]: Let $h(z,\tau)$ be a univalent where $h(0,\tau) = a$. for each $\tau \in \bar{U}$ and let $\mu \in \mathbb{C} \setminus \{0\}$ with $Re\,(\mu) \geq 0$. If $p \in H[a,1,\tau]$ and

$$p(z,\tau) + \frac{1}{\mu} z p_z'(z,\tau) \prec\prec h(z,\tau), \quad (z \in U, \tau \in \bar{U}), \tag{1.2}$$

then

$p(z,\tau) \prec\prec q(z,\tau) \prec\prec h(z,\tau), \quad (z \in U, \tau \in \bar{U})$,

where $q(z,\tau) = \mu z^{-\mu} \int_0^z h(y,\tau)\, y^{\mu-1} dy$ is convex and the most dominant of (1.2).

Lemma (1.2) [1]: Let $h(z,\tau)$ be a convex where $h(0,\tau) = a$ for each $\tau \in \bar{U}$ and let $\mu \in \mathbb{C} \setminus \{0\}$ with $Re\,(\mu) \geq 0$. If $p \in H[a,1,\tau] \cap Q_\tau$, $p(z,\tau) + \frac{1}{\mu} z p_z'(z,\tau)$ is univalent in $U \times \bar{U}$ and

$$h(z,\tau) \prec\prec p(z,\tau) + \frac{1}{\mu} z p_z'(z,\tau), \quad (z \in U, \tau \in \bar{U}). \tag{1.3}$$

Then

$q(z,\tau) \prec\prec p(z,\tau), \quad (z \in U, \tau \in \bar{U})$,

in which $q(z,\tau) = \mu z^{-\mu} \int_0^z h(y,\tau)\, y^{\mu-1} dy$ is the best subordinate of and it is convex (1.3).

Recently, Wanas et al. [4] introduced the following operator, which is called Wanas, operator) $W_{\alpha,\beta}^{l,\delta} : A \longrightarrow A$ which is defined in U by

$$W_{\alpha,\beta}^{l,\delta} f(z) = z + \sum_{\kappa=2}^{\infty} \left[\sum_{m=1}^{l} \binom{l}{m} (-1)^{m+1} \left(\frac{\alpha^m + \kappa \beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_\kappa z^\kappa .$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_0^+ = \mathbb{R} \cup \{0\}$ with $\alpha + \beta > 0$, $l \in \mathbb{N}$, $\delta \in \mathbb{N}$.

Special cases of this operator can be found in [5,6,7,8,9,10,11,12,13,14,15]. For more details, see [16,17].

Now, we consider the Wanas operator in $U \times \bar{U}$. For $f \in A_\tau$, we have

$$W_{\alpha,\beta}^{l,\delta} f(z,\tau) = z + \sum_{\kappa=2}^{\infty} \left[\sum_{m=1}^{l} \binom{l}{m} (-1)^{m+1} \left(\frac{\alpha^m + \kappa\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_\kappa(\tau) z^\kappa \quad (z \in U, \tau \in \bar{U}). \quad (1.4)$$

It can be easily checked through (1.4) that

$$z\left(W_{\alpha,\beta}^{l,\delta} f(z,\tau)\right)' = \varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau) - \psi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta} f(z,\tau), \quad (1.5)$$

where $\varphi(\alpha,\beta,m,l)$ and $\psi(\alpha,\beta,m,l)$ have the following binomial series representations:

$$\varphi(\alpha,\beta,m,l) = \sum_{m=1}^{l} \binom{l}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta}\right)^m + 1 \right) \quad (1.6)$$

and

$$\psi(\alpha,\beta,m,l) = \sum_{m=1}^{l} \binom{l}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta}\right)^m. \quad (1.7)$$

2. Main Results

Definition (2.1): Let $\vartheta(z,\tau)$ be an analytic function in $U \times \bar{U}$ together with $\vartheta(0,\tau) = 1$ for every $\tau \in \bar{U}$ and $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_0^+ = \mathbb{R} \cup \{0\}$ with $\alpha + \beta > 0$, $l, \delta \in \mathbb{N}$. A function $f \in A_\tau$ is said to be a member of the class $\mathcal{G}(\lambda,\alpha,\beta,m,l,\delta;\vartheta)$ if the deep differential subordination is satisfied

$$\frac{1}{z}\left[\left(1 - \lambda\, \varphi(\alpha,\beta,m,l)\right) W_{\alpha,\beta}^{l,\delta} f(z,\tau) + \lambda\, \varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau)\right] \prec\prec \vartheta(z,\tau).$$

A function $f \in A_\tau$ is said to be a member of the class $\mathcal{T}(\lambda,\alpha,\beta,m,l,\delta;\vartheta)$ if the deep differential superordination is satisfied

$$\vartheta(z,\tau) \prec\prec \frac{1}{z}\left[\left(1 - \lambda\, \varphi(\alpha,\beta,m,l)\right) W_{\alpha,\beta}^{l,\delta} f(z,\tau) + \lambda\, \varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau)\right],$$

where $\varphi(\alpha,\beta,m,l)$ is given by (1.6).

Theorem (2.1): Let $\vartheta(z,\tau)$ be a convex function in $U \times \bar{U}$, the range $\vartheta(0,\tau) = 1$ for each $\tau \in \bar{U}$ and $\lambda > 0$. If $f \in \mathcal{G}(\lambda,\alpha,\beta,m,l,\delta;\vartheta)$. Then there exists a convex function $q(z,\tau)$ that is equal to $q(z,\tau) \prec\prec \vartheta(z,\tau)$ and $f \in \mathcal{G}(0,\alpha,\beta,m,l,\delta;q)$.

Proof: Suppose that

$$p(z,\tau) = \frac{W_{\alpha,\beta}^{l,\delta} f(z,\tau)}{z}$$

$$= 1 + \sum_{\kappa=2}^{\infty} \left[\sum_{m=1}^{l} \binom{l}{m} (-1)^{m+1} \left(\frac{\alpha^m + \kappa\beta^m}{\alpha^m + \beta^m} \right) \right]^{\delta} a_\kappa(\tau) z^{\kappa-1}, \quad (z \in U, \tau \in \bar{U}). \quad (2.1)$$

It is clear that $p \in H[1,1,\tau]$.

Since $f \in \mathcal{G}(\lambda,\alpha,\beta,m,l,\delta;\vartheta)$, then we have

$$\frac{1}{z}\left[\left(1 - \lambda\, \varphi(\alpha,\beta,m,l)\right) W_{\alpha,\beta}^{l,\delta} f(z,\tau) + \lambda\, \varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau)\right] \prec\prec \vartheta(z,\tau). \quad (2.2)$$

From (2.1) and (2.2), we get

$$\frac{1}{z}\left[\left(1 - \lambda\, \varphi(\alpha,\beta,m,l)\right) W_{\alpha,\beta}^{l,\delta} f(z,\tau) + \lambda\, \varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau)\right] = p(z,\tau) + \lambda z p_z'(z,\tau) \prec\prec \vartheta(z,\tau).$$

When Lemma (1.1) is used with $\mu = \frac{1}{\lambda}$, the result is
$p(z,\tau) \prec\prec q(z,\tau) \prec\prec \vartheta(z,\tau)$.
Using (2.1), we get
$$\frac{W_{\alpha,\beta}^{l,\delta} f(z,\tau)}{z} \prec\prec q(z,\tau) \prec\prec \vartheta(z,\tau),$$
where
$$q(z,\tau) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \vartheta(y,\tau)\, y^{\frac{1}{\lambda}-1} dy$$
is convex and has the strongest dominant. This gives the required results.

Theorem (2.2): Let $\vartheta(z,\tau)$ be a convex function in$U \times \overline{U}$ the context of $\vartheta(0,\tau) = 1$ for each $\tau \in \overline{U}$ and $\lambda > 0$. If $f \in \mathcal{T}(\lambda,\alpha,\beta,m,l,\delta;\vartheta)$, $\frac{W_{\alpha,\beta}^{l,\delta} f(z,\tau)}{z} \in H[1,1,\tau] \cap Q_\tau$ and
$$\frac{1}{z}\left[\left(1 - \lambda\, \varphi(\alpha,\beta,m,l)\right) W_{\alpha,\beta}^{l,\delta} f(z,\tau) + \lambda\, \varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau)\right]$$
which is univalent in $U \times \overline{U}$, then there exists a convex function $q(z,\tau)$ such that $f \in \mathcal{T}(0,\alpha,\beta,m,l,\delta;q)$.

Proof: Assume the function $p(z,\tau)$ has the Definition (2.1). This is obvious. $p \in H[1,1,\tau] \cap Q_\tau$. Following a quick calculation and taking into account$f \in \mathcal{T}(\lambda,\alpha,\beta,m,l,\delta;\vartheta)$, we can conclude that
$\vartheta(z,\tau) \prec\prec p(z,\tau) + \lambda z p_z'(z,\tau)$.
When Lemma (1.2) is used with $\mu = \frac{1}{\lambda}$, the result is
$q(z,\tau) \prec\prec p(z,\tau)$.
And through (2.1), we get
$$q(z,\tau) \prec\prec \frac{W_{\alpha,\beta}^{l,\delta} f(z,\tau)}{z},$$
where
$$q(z,\tau) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \vartheta(y,\tau)\, y^{\frac{1}{\lambda}-1} dy$$
is the best subordinant since it is convex.
We can obtain the following strong differential "Sandwich Theorem" by combining the results of Theorem (2.1) and Theorem (2.2).

Theorem (2.3): Let $\vartheta_1(z,\tau)$ and $\vartheta_2(z,\tau)$ be convex functions in $U \times \overline{U}$ where $\vartheta_1(0,\tau) = \vartheta_2(0,\tau) = 1$ for all $\tau \in \overline{U}$ and $\lambda > 0$. If $f \in \mathcal{G}(\lambda,\alpha,\beta,m,l,\delta;\vartheta_1) \cap \mathcal{T}(\lambda,\alpha,\beta,m,l,\delta;\vartheta_2)$, $W_{\alpha,\beta}^{l,\delta} f(z,\tau) \in H[1,1,\tau] \cap Q_\tau$ and
$$\frac{1}{z}\left[\left(1 - \lambda\, \varphi(\alpha,\beta,m,l)\right) W_{\alpha,\beta}^{l,\delta} f(z,\tau) + \lambda\, \varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau)\right]$$
is univalent in $U \times \overline{U}$, then
$f \in \mathcal{G}(0,\alpha,\beta,m,l,\delta;q_1) \cap \mathcal{T}(0,\alpha,\beta,m,l,\delta;q_2)$
where

$$q_1(z,\tau) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \vartheta_1(y,\tau)\, y^{\frac{1}{\lambda}-1} dy$$

and

$$q_2(z,\tau) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \vartheta_2(y,\tau)\, y^{\frac{1}{\lambda}-1} dy$$

Both q_1 and q_2 are convex functions.

Theorem (2.4): Let $\vartheta(z,\tau)$ be a convex functions in $U \times \bar{U}$ where $\vartheta(0,\tau) = 1$ for all $\tau \in \bar{U}$ and

$$G(z,\tau) = \frac{\epsilon+2}{z^{\epsilon+1}} \int_0^z y^{\epsilon} f(y,\tau) dy, \quad (z \in U, \tau \in \bar{U}, Re(\epsilon) > -2\ ). \tag{2.3}$$

If $f \in \mathcal{G}(1,\alpha,\beta,m,l,\delta;\vartheta)$, then there exists a convex function $q(z,\tau)$ with the property that $q(z,\tau) \prec\prec \vartheta(z,\tau)$ and $G \in \mathcal{G}(1,\alpha,\beta,m,l,\delta;q)$.

Proof: Suppose that

$$p(z,\tau) = \left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z', \quad (z \in U, \tau \in \bar{U}). \tag{2.4}$$

Then, $p \in H[1,1,\tau]$.

From (2.3), we get (2.5)

$$z^{\epsilon+1} G(z,\tau) = (\epsilon+2) \int_0^z y^{\epsilon} f(y,\tau) dy. \tag{2.5}$$

When we differentiating both sides of (2.5) by z, we get

$$(\epsilon+2) f(z,\tau) = (\epsilon+1) G(z,\tau) + z G_z'(z,\tau)$$

and

$$(\epsilon+2) W_{\alpha,\beta}^{l,\delta} f(z,\tau) = (\epsilon+1) W_{\alpha,\beta}^{l,\delta} G(z,\tau) + z \left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z'.$$

By differentiating the last reference with respect to z, we have

$$\left(W_{\alpha,\beta}^{l,\delta} f(z,\tau)\right)_z' = \left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z' + \frac{z}{\epsilon+2} \left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_{z^2}''. \tag{2.6}$$

Since $f \in \mathcal{G}(1,\alpha,\beta,m,l,\delta;\vartheta)$, then we have

$$\frac{1}{z}\left[\varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau) - \psi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta} f(z,\tau)\right] \prec\prec \vartheta(z,\tau), \tag{2.7}$$

where $\varphi(\alpha,\beta,m,l)$ and $\psi(\alpha,\beta,m,l)$ have the forms (1.6) and (1.7), respectively.

Now, from (1.5), the (2.7) is the same as

$$\left(W_{\alpha,\beta}^{l,\delta} f(z,\tau)\right)_z' \prec\prec \vartheta(z,\tau). \tag{2.8}$$

From (2.6) and (2.8), we get

$$\left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z' + \frac{z}{\epsilon+2} \left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_{z^2}'' \prec\prec \vartheta(z,\tau). \tag{2.9}$$

Substituting (2.4) by (2.9), we get the following

$$p(z,\tau) + \frac{1}{\epsilon+2} z p_z'(z,\tau) \prec\prec \vartheta(z,\tau).$$

When Lemma (1.1) is used with $\mu = \epsilon + 2$, the result is

$$p(z,\tau) \prec\prec q(z,\tau) \prec\prec \vartheta(z,\tau).$$

We get the following using (2.4)

$$\left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z' \prec\prec q(z,\tau) \prec\prec \vartheta(z,\tau),$$

where

$$q(z,\tau) = (\epsilon+2)z^{-(\epsilon+2)} \int_0^z \vartheta(y,\tau)\, y^{\epsilon+1} dy$$

It has a convex shape and is the most dominant.

Theorem (2.5): Let $\vartheta(z,\tau)$ be a convex the function in $U \times \overline{U}$ with $\vartheta(0,\tau) = 1$ for every $\tau \in \overline{U}$ and $G(z,\tau)$ is given by (2.3). If $f \in \mathcal{T}(1,\alpha,\beta,m,l,\delta;\vartheta)$, $\left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z' \in H[1,1,\tau] \cap Q_\tau$ and

$$\frac{1}{z}\left[\varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau) - \psi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta} f(z,\tau)\right]$$

If is univalent in $U \times \overline{U}$, then there exists a convex function $q(z,\tau)$ that gives $G \in \mathcal{T}(1,\alpha,\beta,m,l,\delta;q)$.

Proof: Suppose that the function $p(z,\tau)$ is defined by (2.4). It is evident that $p \in H[1,1,\tau] \cap Q_\tau$. After a short calculation and considering $f \in \mathcal{T}(1,\alpha,\beta,m,l,\delta;\vartheta)$, we can conclude that

$$\vartheta(z,\tau) \prec\prec p(z,\tau) + \frac{1}{\epsilon+2} z p_z'(z,\tau).$$

When Lemma (1.2) is used with $\mu = \epsilon + 2$, the result is

$$q(z,\tau) \prec\prec p(z,\tau).$$

We get the following using (2.4)

$$q(z,\tau) \prec\prec \left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z',$$

where

$$q(z,\tau) = (\epsilon+2)z^{-(\epsilon+2)} \int_0^z \vartheta(y,\tau)\, y^{\epsilon+1} dy$$

is the best subordinant since it is convex.

The following strong differential " Sandwich Theorem" is obtained by combining the results of Theorems(2.4) and Theorem (2.5).

Theorem (2.6): Let $\vartheta_1(z,\tau)$ and $\vartheta_2(z,\tau)$ be convex functions in $U \times \overline{U}$ with $\vartheta_1(z,\tau) = \vartheta_2(z,\tau) = 1$ for every $\tau \in \overline{U}$ and $G(z,\tau)$ is given by (2.3). If $f \in \mathcal{G}(1,\alpha,\beta,m,l,\delta;\vartheta_1) \cap \mathcal{T}(1,\alpha,\beta,m,l,\delta;\vartheta_2)$, $\left(W_{\alpha,\beta}^{l,\delta} G(z,\tau)\right)_z' \in H[1,1,\tau] \cap Q_\tau$ and

$$\frac{1}{z}\left[\varphi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta+1} f(z,\tau) - \psi(\alpha,\beta,m,l) W_{\alpha,\beta}^{l,\delta} f(z,\tau)\right]$$

is univalent in $U \times \overline{U}$, then

$$f \in \mathcal{G}(1,\alpha,\beta,m,l,\delta;q_1) \cap \mathcal{T}(1,\alpha,\beta,m,l,\delta;q_2)$$

where

$$q_1(z,\tau) = (\epsilon+2)z^{-(\epsilon+2)} \int_0^z \vartheta_1(y,\tau)\, y^{\epsilon+1} dy$$

and

$$q_2(z,\tau) = (\epsilon+2)z^{-(\epsilon+2)} \int_0^z \vartheta_2(y,\tau)\, y^{\epsilon+1} dy$$

The functions q_1 and q_2 are convex.

Conclusion

In this work, two suggested families of analytic functions by using strong differential subordinations and superordinations associated Wanas differential operator are introduced and studied. Also, many important results and properties of these families of analytic functions are established and discussed.

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