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# Coefficient Estimates for Subclasses of Regular Functions

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**Abstract**
The aim of this paper is to introduce and investigate new subclasses of regular functions defined in $\mathfrak{U}$. The coefficients estimate$|a_2|, |a_3|$and $|a_3 - \mu a_2^2|$ for functions in these subclasses are determined. Many of new and known consequences are shown as particular cases of our outcomes.

**Keywords:** regular functions, Univalent function, Subordination, Majorization, Fekete-Szego.

# تقديرات المعامل لفئات جزئيه من الدوال التحليليه

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**الخلاصة**
الهدف من هذا البحث هو تقديم واستقصاء فئات جزئيه جديدة من الدوال التحليليه المعرفه في قرص الوحدة . تقدير المعاملات $|a_2|, |a_3|$ و$|a_3 - \mu a_2^2|$ للدوال في هذه الفئات جزئيه تم تحديدها. تظهر العديد من النتائج الجديدة والمعروفة على أنها حالات خاصة لنتائجنا.

## 1. Introudction

Let $\mathcal{A}$ be the class of all regular functions $f$ in the unit disk $\mathfrak{U} = \{z: |z| < 1\}$,of the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

and normalized by $f'(0)=1$ and $f(0)=0$ .

A function $f(z) \in \mathcal{A}$ is subordinate to regular function $F(z)$ if there is Schwarz function $\mathcal{k}(z)$ which is regular satisfying $\mathcal{k}(0)=0, |\mathcal{k}(z)|<1$ in $\mathfrak{U}$, and

$$f(z) = F(\mathcal{k}(z)),$$

in this case we write

$$f \prec F \text{ or } f(z) \prec F(z) \quad (z \in \mathfrak{U}). \tag{1.2}$$

Furthermore, if the $F$ is univalent in$\mathfrak{U}$, then $f \prec F$ is equivalent to $f(0) = F(0)$ and $f(\mathfrak{U}) \subset F(\mathfrak{U})$.For more details on the notion of subordination, (see [1]).

Let $f(z)$ and $F(z)$ be regular in the open unit disk$\mathfrak{U}$. Then we say that $f$ is majorized by $F$ in $\mathfrak{U}$ (see [2]) and write

$$f(z) \prec\prec F(z) \quad (z \in \mathfrak{U}),$$

if there exists a regular function $\phi(z)$ in $\mathfrak{U}$, such that $|\phi(z)| \leq 1$ and $f(z) = \phi(z) F(z)$ $(z \in \mathfrak{U})$.

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Ma and Minda [3], defined the classes as follow:

$$K(\phi):=\left\{h\in\mathcal{A}: 1+\frac{zh''(z)}{h'(z)}\prec\phi(z);\ z\in\mathfrak{U}\right\}$$

$$S*(\phi):=\left\{h\in\mathcal{A}: \frac{zh'(z)}{h(z)}\prec\phi(z);\ z\in\mathfrak{U}\right\},$$

We suppose that the function $\phi(z)$ is a regular and univalent with a positive real part in the disk $\mathfrak{U}$, satisfying $\phi(0)=1$, $\phi'(0)>0$ and $\phi(\mathfrak{U})$ is starlike region with the respect to 1 and symmetric with the respect to the real axis. The classes K(ϕ) and $S^*(\phi)$, are called convex of Ma-Minda type and starlike of Ma-Minda type respectively.

At this work, it is supposed that

$$k(z)=k_1z+k_2z^2+k_3z^3+\cdots$$

and

$$\phi(z)=1+d_1z+d_2z^2+d_3z^3+\cdots,\quad d_1>0,$$

where ϕ is a regular in $\mathfrak{U}$ and ϕ(0)=1. Motivated by the work in [4], we introduce the classes as follows**.**

**Definition (1.1)**.Let the class $\mathcal{Z}_\beta(\phi)(0\le\beta\le1)$ consist of functions $f\in\mathcal{A}$ satisfying the subordination condition

$$\frac{\beta zf'(z)}{(1-\beta)z+\beta f(z)}+(1-\beta)\left[\frac{zf''(z)}{f'(z)}+1\right]\prec\phi(z).$$

**Definition (1.2)** Let the class $\mathcal{L}_\lambda(\beta,\phi)$, $(0\le\beta<1, 0\le\lambda\le1)$ consist of functions $f\in\mathcal{A}$ satisfying the subordination condition

$$(1-\lambda)\frac{zf'(z)}{f(z)}\left[\frac{f(z)}{z}\right]^\beta+\lambda\left[\frac{zf''(z)}{f'(z)}+1\right]^{1-\beta}\prec\phi(z).$$

**Definition (1.3)** Let the class $\mathcal{B}_\alpha(\phi)$ $(0\le\alpha\le1)$ consist of functions $f\in\mathcal{A}$ satisfying the subordination condition

$$\alpha\frac{zf''(z)}{f'(z)}+\frac{f'(z)+zf''(z)}{f'(z)+\alpha zf''(z)}\prec\phi(z).$$

**Definition (1.4)** Let the class $\mathcal{A}_\alpha^\gamma(\beta,\phi)$ $(\alpha>0, \beta\ge0, 0\le\gamma\le1)$, consist of functions $f\in\mathcal{A}$ satisfying the subordination condition

$$\left[\frac{zf'(z)}{f(z)}\right]^\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]^\beta+\gamma(f'(z)-1)\prec\phi(z).$$

In this paper, the Fekete-Szego inequality for the functions in these subclasses are obtained. More details of Fekete-Szego coefficient for various classes (see [5, 6, 7, 8, 9])

To prove our results, we shall use the next lemma.

**Lemma (1.5)** [9]**.**Let $w$ be regular function normalized by $|w(z)|<1$, $w(0)=0$, and

$$w(z)=w_1z+w_2z^2+w_3z^3+\cdots.$$

Then

$|w_2-\mu w_1^2|\le\max\{1,|\mu|\}$, where μ is complex number.

**2. Main Results.**

**Theorem (2.1).** Let $f\in\mathcal{A}$ belongs to $\mathcal{Z}_\beta(\phi)$ **.** Then

$$|a_2|\le\frac{d_1}{2-\beta^2},\ |a_3|\le\frac{d_1}{|6-3\beta-\beta^2|}\max\left\{1,\left|\frac{\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2}d_1-\frac{d_2}{d_1}\right|\right\}$$

and

$$|a_3-\mu a_2^2|\le\frac{d_1}{|6-3\beta-\beta^2|}\max\left\{1,\left|\frac{\mu(6-3\beta-\beta^2)+\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2}d_1-\frac{d_2}{d_1}\right|\right\}.$$

**Proof**: Since $f\in\mathcal{Z}_\beta(\phi)$, there exist regular function $w$ with $|w(z)|<1$and w(0)=0 such that:

$$\frac{\beta z f'(z)}{(1-\beta)z+\beta f(z)} + (1-\beta)\left[\frac{zf''(z)}{f'(z)}+1\right] = \phi\big(w(z)\big)\,. \qquad (2.3)$$

Since

$$\frac{\beta z f'(z)}{(1-\beta)z+\beta f(z)} = \beta+(2\beta-\beta^2)\, a_2 z+[(3\beta\text{-}\beta^2)a_3\text{-}\beta(2\beta-\beta^2)z_2^2]+\cdots \qquad (2.4)$$

and

$$(1-\beta)\left(\frac{zf''(z)}{f'(z)}+1\right) = (1-\beta)+2(1-\beta)a_2 z+6(1-\beta)(a_3 \text{ -}4\, a_2^2)z^2 \,\ldots.. \qquad (2.5)$$

from (2.4) and (2.5), we get the following

$$\frac{\beta z f'(z)}{(1-\beta)z+\beta f(z)} + (1-\beta)\left[\frac{zf''(z)}{f'(z)}+1\right] =$$

$$1+(2-\beta^2)\, a_2 z+[(6-3\beta-\beta^2\,)a_3-(24-24\beta+2\beta^2\text{-}\beta^3)a_2^2]z^2\ldots \qquad (2.6)$$

and

$$\phi(w(z) = 1+d_1 w_1 z+(d_1 w_2+d_2\, {w_1}^2)z^2\ldots \qquad (2.7)$$

Putting (2.6) and (2.7) in (2.3) and equating coefficient both sides, we get

$$a_2 = \frac{d_1 w_1}{2-\beta^2}$$

and

$$a_3 = \frac{d_1}{6-3\beta-\beta^2}\left[d_1 w_2+d_2\, {w_1}^2 - \frac{\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2}{d_1}^2{w_1}^2\right]$$

By using the well-known inequality, $|w_1| \le 1$, we obtain

$$|a_2| \le \frac{d_1}{2-\beta^2}.$$

Further

$$a_3-\mu a_2^2 = \frac{d_1}{6-3\beta-\beta^2}\left[d_1 w_2+d_2\, {w_1}^2 - \frac{\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2}{d_1}^2{w_1}^2\right]\text{-}\ \mu\frac{{d_1}^2{w_1}^2}{(2-\beta^2)^2}$$

Applying Lemma (1.5) to

$$\left|w_2-\left\{\frac{\mu(\,6-3\beta-\beta^2\,)+\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2}d_1-\frac{d_2}{d_1}\right\}{w_1}^2\right|.$$

.

We conclude that

$$|a_3-\mu a_2^2| \le \frac{d_1}{|6-3\beta-\beta^2|}\max\left\{1,\left|\frac{\mu(\,6-3\beta-\beta^2\,)+\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2}d_1-\frac{d_2}{d_1}\right|\right\}.$$

For $\mu$=0, the above relation will give estimate of $|a_3|$.

**Remark (2.2):** For $\beta$= 0, we have

$$\mathcal{Z}_0(\phi) \coloneqq \mathcal{K}(\phi),$$

Also for $\beta$= 1, we have

$$\mathcal{Z}_1(\phi) \coloneqq S^*(\phi),$$

In this case, $\mathcal{K}(\phi)$ and $S^*(\phi)$, were studied by Ma and Minda (see [3]).

We observe that on choosing $\beta = \frac{1}{2}$ in previous theorem, we obtain the next corollary.

**Corollary (2.3):** Let $f$ be in the class $\mathcal{Z}_{\frac{1}{2}}(\phi)$.Then

$$|a_2| \le \frac{4}{7}d_1,$$

$$|a_3| \le \frac{4d_1}{17}\max\left\{1,\left|\frac{99d_1}{8}+\frac{d_2}{d_1}\right|\right\}$$

and

$$|a_3-\mu a_2^2| \le \frac{4d_1}{17}\max\left\{1,\left|\frac{17}{4}\mu-\frac{99d_1}{8}-\frac{d_2}{d_1}\right|\right\}.$$

**Theorem (2.4)** If $f \in \mathcal{A}$ satisfies

$$\frac{\beta z f'(z)}{(1-\beta)z+\beta f(z)} + (1-\beta)\left[\frac{zf''(z)}{f'(z)}+1\right] \ll \phi(z),$$

then

$$|a_2| \le \frac{d_1}{2-\beta^2}, |a_3| \le \frac{d_1}{|6-3\beta-\beta^2|} \left|\frac{\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2} d_1 - \frac{d_2}{d_1}\right|$$

and

$$|a_3 - \mu a_2^2| \le \frac{d_1}{|6-3\beta-\beta^2|} \left|\frac{\mu(6-3\beta-\beta^2)+\beta^3-2\beta^2+24\beta-24}{(2-\beta^2)^2} d_1 - \frac{d_2}{d_1}\right|.$$

**Proof**: The required proof is obtained by setting w($z$)= $z$ in the previous proof.

**Theorem (2.5)** If $f$ is given by (1.1) belong to $\mathcal{L}_\lambda(\beta,\phi)$ **,** then

$$|a_2| \le \frac{d_1}{|\beta-3\lambda\beta+\lambda+1|}, \tag{2.8}$$

$$|a_3| \le \frac{d_1}{|\beta-7\lambda\beta+4\lambda+2|} \max\left\{1, \left|\frac{\frac{3}{2}\lambda\beta^2+\frac{3}{2}\lambda\beta-3\lambda+\frac{1}{2}\beta^2+\frac{1}{2}\beta-1}{(\beta-3\lambda\beta+\lambda+1)^2} d_1 + \frac{d_2}{d_1}\right|\right\}.$$

and

$$|a_3 - \mu a_2^2| \le \frac{d_1}{|\beta-7\lambda\beta+4\lambda+2|} \max\left\{1, \left|\frac{\frac{3}{2}\lambda\beta^2+\frac{3}{2}\lambda\beta-3\lambda+\frac{1}{2}\beta^2+\frac{1}{2}\beta-1}{(\beta-3\lambda\beta+\lambda+1)^2} d_1 - \frac{\mu(\beta-7\lambda\beta+4\lambda+2)d_1}{(\beta-3\lambda\beta+\lambda+1)^2} - \frac{d_2}{d_1}\right|\right\}. \tag{2.9}$$

**Proof**: Let $f \in \mathcal{L}_\lambda(\beta,\phi)$.Then there exists a regular function w with w(0)=0 and $|w(z)|$<1 such that:

$$(1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\beta + \lambda\left(\frac{zf''(z)}{f'(z)}+1\right)^{1-\beta} = \phi(w(z)). \tag{2.10}$$

Since

$$(1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\beta + \lambda\left(\frac{zf''(z)}{f'(z)}+1\right)^{1-\beta} =$$

$$1+(\beta-3\lambda\beta+\lambda+1)\ a_2 z + [(\beta-7\lambda\beta+4\lambda+2)a_3 + \{\tfrac{3}{2}\lambda\beta^2+\tfrac{3}{2}\lambda\beta-3\lambda+\tfrac{1}{2}\beta^2+\tfrac{1}{2}\beta-1\}a_2^2\ ]z^2 + \ldots\ldots \tag{2.11}$$

Putting (2.7) and (2.11) in (2.10) and equating coefficients both sides, we get

$$a_2 = \frac{d_1 w_1}{\beta-3\lambda\beta+\lambda+1},$$

By using the well-known inequality, $|w_1| \le 1$, on $a_2$, we obtain (2.8)**.**

Also

$$a_3 - \mu a_2^2 = \frac{d_1}{\beta-7\lambda\beta+4\lambda+2}\left[w_2 - \left(\frac{\frac{3}{2}\lambda\beta^2+\frac{3}{2}\lambda\beta-3\lambda+\frac{1}{2}\beta^2+\frac{1}{2}\beta-1}{(\beta-3\lambda\beta+\lambda+1)^2} d_1 - \frac{\mu(\beta-7\lambda\beta+4\lambda+2)d_1}{(\beta-3\lambda\beta+\lambda+1)^2} - \frac{d_2}{d_1}\right) w_1^2\right]$$

Applying Lemma (1.5) in previous relation, we obtain (2.9).

For μ=0, in (2.9), we get the upper bound to $|a_3|$.

**Remark (2.6):** Setting β= 0, and λ=0, we have

$$\mathcal{L}_0(0,\phi) \coloneqq S^*(\phi),$$

and for β= 0, and λ=1, we obtain

$$\mathcal{L}_1(0,\phi) \coloneqq \mathcal{K}(\phi),$$

These classes were introduced by Ma and Minda see [3].

For λ=1, we get the class $\mathcal{L}_1(\beta,\phi)$: $=\mathcal{L}(\beta,\phi)$, and for λ=0, we obtain the class $\mathcal{L}_0(\beta,\phi)$:=$\mathcal{L}^\beta(\phi)$, in this case, we obtain the next corollaries.

**Corollary (2.7):** Let $f$ be in the class $\mathcal{L}(\beta,\phi)$, .Then

$$|a_2| \le \frac{d_1}{|2-2\beta|},\ \ a_3| \le \frac{d_1}{|6-6\beta|} \max\left\{1, \left|\frac{2\beta^2+2\beta-4}{(2-2\beta)^2} d_1 + \frac{d_2}{d_1}\right|\right\}.$$

And

$$|a_3 - \mu a_2^2| \le \frac{d_1}{|6-6\beta|} \max\left\{1, \left|\frac{2\beta^2+2\beta-4}{(2-2\beta)^2} d_1 - \frac{\mu(6-6\beta)d_1}{(2-2\beta)^2} - \frac{d_2}{d_1}\right|\right\}.$$

**Corollary (2.8):** Let $f$ of the form (1.1), belong to the class $\mathcal{L}^\beta(\phi)$.Then

$$|a_2| \le \frac{d_1}{|\beta+1|},\ |a_3| \le \frac{d_1}{|\beta+2|} \max\left\{1, \left|\frac{\frac{1}{2}\beta^2+\frac{1}{2}\beta-1}{(\beta+1)^2} d_1 + \frac{d_2}{d_1}\right|\right\}.$$

and

$$|a_3-\mu a_2^2|\le\frac{d_1}{|\beta+2|}\max\left\{1,\left|\frac{\frac{1}{2}\beta^2+\frac{1}{2}\beta-1}{(\beta+1)^2}d_1-\frac{\mu(\beta+2)d_1}{(\beta+1)^2}-\frac{d_2}{d_1}\right|\right\}.$$

Setting $\lambda=\frac{1}{2}$ in previous theorem we get the next corollary.

**Corollary (2.9):** Let $f$ be in the class $\mathcal{L}_{\frac{1}{2}}(\beta,\phi)$ .Then

$$|a_2|\le\frac{d_1}{\frac{3}{2}-\frac{1}{2}\beta},\ |a_3|\le\frac{d_1}{4-\frac{5}{2}\beta}\max\left\{1,\left|\frac{\frac{5}{4}\beta^2+\frac{5}{4}\beta-\frac{5}{2}}{(\frac{3}{2}-\frac{1}{2}\beta)^2}d_1-\frac{d_2}{d_1}\right|\right\}$$

and

$$|a_3-\mu a_2^2|\le\frac{d_1}{4-\frac{5}{2}\beta}\max\left\{1,\left|\left(\frac{\frac{5}{4}\beta^2+\frac{5}{4}\beta-\frac{5}{2}+\mu\left(4-\frac{5}{2}\beta\right)}{(\frac{3}{2}-\frac{1}{2}\beta)^2}\right)d_1-\frac{d_2}{d_1}\right|\right\}.$$

**Theorem (2.10):** If $f\in\mathcal{A}$ satisfies

$$(1-\lambda)\frac{zf'(z)}{f(z)}\left[\frac{f(z)}{z}\right]^{\beta}+\lambda\left[\frac{zf''(z)}{f'(z)}+1\right]^{1-\beta}\ll\phi(z),$$

then

$$|a_2|\le\frac{d_1}{|\beta-3\lambda\beta+\lambda+1|},\ |a_3|\le\frac{d_1}{|\beta-7\lambda\beta+4\lambda+2|}\left|\frac{\frac{3}{2}\lambda\beta^2+\frac{3}{2}\lambda\beta-3\lambda+\frac{1}{2}\beta^2+\frac{1}{2}\beta-1}{(\beta-3\lambda\beta+\lambda+1)^2}d_1+\frac{d_2}{d_1}\right|$$

and

$$|a_3-\mu a_2^2|\le\frac{d_1}{|\beta-7\lambda\beta+4\lambda+2|},\left|\left(\frac{\frac{3}{2}\lambda\beta^2+\frac{3}{2}\lambda\beta-3\lambda+\frac{1}{2}\beta^2+\frac{1}{2}\beta-1-\mu(\beta-7\lambda\beta+4\lambda+2)}{(\beta-3\lambda\beta+\lambda+1)^2}\right)d_1-\frac{d_2}{d_1}\right|.$$

**Proof**: The required proof is obtained by setting $w(z)=z$ in the previous proof.

**Theorem (2.11)** If $f$ is given by (1.1) belong to $\mathcal{B}_{\alpha}(\phi)$ **,** then

$$|a_2|\le\frac{d_1}{2(2\alpha+1)},\tag{2.12}$$

$$|a_3|\le\frac{d_1}{8}\max\left\{1,\left|\frac{\alpha^2-\alpha-1}{(2\alpha+1)^2}d_1-\frac{d_2}{d_1}\right|\right\}.$$

and

$$|a_3-\mu a_2^2|\le\frac{d_1}{8}\max\left\{1,\left|\frac{2\mu+\alpha^2-\alpha-1}{(2\alpha+1)^2}d_1-\frac{d_2}{d_1}\right|\right\}.\tag{2.13}$$

**Proof**: Let $f\in\mathcal{B}_{\alpha}(\phi)$ .Then there exists regular function with $|w(z)|<1$ and $w(0)=0$ such that:

$$\alpha\frac{zf''(z)}{f'(z)}+\frac{f'(z)+zf''(z)}{f'(z)+\alpha zf''(z)}=\phi(w(z))\tag{2.14}$$

Since

$$\alpha\frac{zf''(z)}{f'(z)}+\frac{f'(z)+zf''(z)}{f'(z)+\alpha zf''(z)}=1+2(2\alpha+1)a_2z+4[2a_3-(1+\alpha-\alpha^2)a_2^2]z^2\ldots,\tag{2.15}$$

putting (2.7) and (2.15) in (2.14) and equating coefficient both sides, we get

$$a_2=\frac{d_1w_1}{2(2\alpha+1)}.$$

By using the well-known inequality, $|w_1|\le 1$, on $a_2$, we obtain (2.12).

Also

$$a_3-\mu a_2^2=\frac{d_1}{8}\left\{w_2+\left(\frac{1+\alpha-\alpha^2}{(2\alpha+1)^2}d_1+\frac{d_2}{d_1}\right)w_1^2\right\}-\frac{\mu\, {d_1}^2w_1^2}{4(2\alpha+1)^2},$$

applying Lemma (1.5) to previous relation, we obtain (2.13).

For μ=0, the above will reduce to the estimate of $|a_3|$.

**Remark (2.12):** For α= 0, 1, in Theorem (2.11), we have

$$\mathcal{B}_0(\phi):=\mathcal{K}(\phi),\mathcal{B}_1(\phi):=\mathcal{K}(\phi),$$

This class was introduced by Ma and Minda see [3].

Putting $\alpha=\frac{1}{2}$ in previous theorem, we obtain the following corollary.

**Corollary (2.13):** Let $f$ be in the class $\mathcal{B}_{\frac{1}{2}}(\phi)$. Then

$$|a_2|\le\frac{d_1}{4},\ |a_3|\le\frac{d_1}{8}max\left\{1,\left|\frac{5}{16}d_1+\frac{d_2}{d_1}\right|\right\}.$$

and

$$|a_3 - \mu a_2^2| \le \frac{d_1}{8} max\left\{1, \left|\frac{2\mu-\frac{5}{4}}{4} d_1 + \frac{d_2}{d_1}\right|\right\}.$$

**Theorem (2.14):** If $f \in \mathcal{A}$ satisfies

$$\alpha\frac{zf''(z)}{f'(z)} + \frac{f'(z) + zf''(z)}{f'(z) + \alpha zf''(z)} \ll \phi(z),$$

then

$$|a_2| \le \frac{d_1}{2(2\alpha+1)},\ |a_3| \le \frac{d_1}{8}\left|\frac{\alpha^2-\alpha-1}{(2\alpha+1)^2} d_1 + \frac{d_2}{d_1}\right|.$$

and

$$|a_3 - \mu a_2^2| \le \frac{d_1}{8}\left|\frac{2\mu+\alpha^2-\alpha-1}{(2\alpha+1)^2} d_1 - \frac{d_2}{d_1}\right|.$$

**Proof**: The required proof is obtained by setting w($z$)= $z$ in the previous proof.

**Theorem (2.15)** If $f$ is given by (1.1) belong to$\mathcal{A}_\alpha^\gamma(\beta,\phi)$**,** then

$$|a_2|\le\frac{d_1}{2|\gamma|+(\alpha+2\beta)}, \tag{2.16}$$

$$|a_3|\le\frac{2d_1}{3|\gamma|+4(\alpha+3\beta)} \max\left\{1, \left|\frac{2d_1((\alpha+2\beta)^2-3(\alpha+4\beta))}{2[\gamma+(\alpha+2\beta)]^2} - \frac{2d_2}{d_1}\right|\right\}.$$

and

$$|a_3 - \mu a_2^2|\le\frac{2d_1}{3|\gamma|+4(\alpha+3\beta)} \max\left\{1, \left|\frac{2d_1((\alpha+2\beta)^2-3(\alpha+4\beta))+\mu d_1(3\gamma+4(\alpha+3\beta))}{2[\gamma+(\alpha+2\beta)]^2} - \frac{2d_2}{d_1}\right|\right\}. \tag{2.17}$$

**Proof**: Let f $\in \mathcal{A}_\alpha^\gamma(\beta,\phi)$.Then there is a regular function w with $|w(z)|$<1 and w(0)=0 such that:

$$\left[\frac{zf'(z)}{f(z)}\right]^\alpha \left[1 + \frac{zf''(z)}{f'(z)}\right]^\beta + \gamma(f'(z) - 1) \prec \phi(w(z)). \tag{2.18}$$

Since

$$\left[\frac{zf'(z)}{f(z)}\right]^\alpha \left[1 + \frac{zf''(z)}{f'(z)}\right]^\beta + \gamma(f'(z) - 1)=$$

$$1+((\alpha+2\beta)+2\gamma)\, a_2 z + \frac{1}{2}\,[((\alpha+2\beta)^2-3(\alpha+4\beta))\, a_2^2 + (4(\alpha+3\beta)+3\gamma)\, a_3]z^2 + \ldots \tag{2.19}$$

Putting (2.7) and (2.19) in (2.18) and equating coefficient both sides, we get

$$a_2 = \frac{d_1 w_1}{(\alpha+2\beta)+2\gamma},$$

By using the well-known inequality, $|w_1| \le 1$, on $a_2$, we obtain (2.16)**.**
Also

$$a_3 - \mu a_2^2 = \frac{2d_1}{4(\alpha+3\beta)+3\gamma}[w_2 - \left\{\frac{(\alpha+2\beta)^2 d_1 - 3(\alpha+4\beta)d_1}{2[(\alpha+2\beta)+2\gamma]^2} - \frac{d_2}{d_1}\right\} w_1^2 - \frac{\mu {d_1}^2 w_1^2}{((\alpha+2\beta)+2\gamma)^2}$$

Applying Lemma (1.5) to previous relation, we obtain (2.17).
For μ=0, the above relation will reduce to the estimate of $|a_3|$.

**Remark (2.16):** When γ = β =0, α=1, and $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$ in Theorem (2.15), then we get the estimates in [10, Corollary (3.3)]. For γ = 0, Theorem (2.15) gives a special case of the estimates [11, Theorem (2.7)], for k=1.

Taking α=1,β=1 and γ = 1 in Theorem (2.15), we get the following corollary.

**Corollary (2.17):** Let $f$ be in the class$\mathcal{A}(\phi)$. Then

$$|a_2|\le\frac{d_1}{5},\ |a_3|\le\frac{2d_1}{19} \max\left\{1, \left|\frac{18d_1-15}{32} - \frac{2d_2}{d_1}\right|\right\}.$$

and

$$|a_3 - \mu a_2^2|\le\frac{2d_1}{19} \max\left\{1, \left|\frac{18d_1-15+19\mu d_1}{32} - \frac{2d_2}{d_1}\right|\right\}.$$

For β=0 in previous theorem, we get the following corollary.

**Corollary (2.18):** Let $f$ be in the class $\mathcal{A}_\alpha^\gamma(\phi)$ .Then

$$|a_2|\le\frac{d_1}{2|\gamma|+\alpha},\ |a_3|\le\frac{2d_1}{3|\gamma|+4\alpha} \max\left\{1, \left|\frac{2d_1(\alpha^2-3\alpha)}{2[\gamma+\alpha]^2} - \frac{2d_2}{d_1}\right|\right\}$$

and

$$|a_3 - \mu a_2^2|\le \frac{2d_1}{3|\gamma|+4\alpha} max\left\{1, \left|\frac{2d_1(\alpha^2-3\alpha)+\mu d_1(3\gamma+4\alpha)}{2[\gamma+\alpha]^2} - \frac{2d_2}{d_1}\right|\right\}.$$

Put γ = 1 and β=0 in previous theorem, we obtain the next corollary.

**Corollary (2.19)** If $f$ is given by (1.1) belong to $\mathcal{A}_\alpha(\phi)$**,** then

$$|a_2|\leq\frac{d_1}{2+\alpha},\ |a_3|\leq\frac{2d_1}{3+4\alpha}\max\left\{1,\left|\frac{2d_1(\alpha^2-3\alpha)}{2(1+\alpha)^2}-\frac{2d_2}{d_1}\right|\right\}.$$

and

$$|a_3-\mu a_2^2|\leq\frac{2d_1}{3+4\alpha}\max\left\{1,\left|\frac{2d_1(\alpha^2-3\alpha)+\mu d_1(3+4\alpha)}{2(1+\alpha)^2}-\frac{2d_2}{d_1}\right|\right\}.$$

**Theorem (2.20):** If $f\in\mathcal{A}$ satisfies

$$\left[\frac{zf'(z)}{f(z)}\right]^\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]^\beta+\gamma(f'(z)-1)\ll\phi(z),$$

then

$$|a_2|\leq\frac{d_1}{2|\gamma|+(\alpha+2\beta)},\ |a_3|\leq\frac{2d_1}{3|\gamma|+4(\alpha+3\beta)}\left\{\left|\frac{2d_1((\alpha+2\beta)^2-3(\alpha+4\beta))}{2[\gamma+(\alpha+2\beta)]^2}-\frac{2d_2}{d_1}\right|\right\}.$$

and

$$|a_3-\mu a_2^2|\leq\frac{2d_1}{3|\gamma|+4(\alpha+3\beta)}\left\{\left|\frac{2d_1((\alpha+2\beta)^2-3(\alpha+4\beta))+\mu d_1(3\gamma+4(\alpha+3\beta))}{2[\gamma+(\alpha+2\beta)]^2}-\frac{2d_2}{d_1}\right|\right\}.$$

**Proof**: The result follows by taking w($z$)= $z$ in the proof of Theorem(2.15).

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