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A Class of Harmonic Multivalent Functions for Higher Derivatives Associated with General Linear Operator

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Abstract
The main goal of this paper is to introduce the higher derivatives multivalent harmonic function class, which is defined by the general linear operator. As a result, geometric properties such as coefficient estimation, convex combination, extreme point, distortion theorem and convolution property are obtained. Finally, we show that this class is invariant under the Bernandi-Libera-Livingston integral for harmonic functions.

Keywords: Harmonic, Multivalent functions, Higher derivatives, Linear operator, Integral operator.

صنف من الدوال التوافقية متعددة التكافؤ للمشتقات العليا المرتبطة مع المؤثر الخطي العام

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الخلاصة
الهدف الرئيسي من هذا البحث هو تقديم صنف من الدوال التوافقية متعددة التكافؤ للمشتقات العليا ، والتي تم تحديدها بواسطة المؤثر الخطي العام. نتيجة لذلك ، تم الحصول على الخصائص الهندسية مثل تقدير المعامل ، الجمع المحدب ، النقطة المتطرفة ، نظرية التشويه ، خاصية الالتواء. أخيرًا ، تم برهان أن هذا الصنف ثابت ضمن تكامل Bernandi–Libera–Livingston للدوال التوافقية .

1. Introduction

The function $f = u + iv$ is said to be continuous in the complex domain $F \subset \mathbb{C}$ harmonic, if both u and v are real harmonic functions in F, we can write $f = h + \bar{g}$ in any simply connected domain F , where h and g are analytic functions in F. We call h the analytic part and g the co-analytic part of f . See Clunie and Shil-small[1].

Let $H(p)$ be the family of all harmonic function $f = h + \bar{g}$ that are sense preserving in the open unit disk $U = \{z: |z| < 1\}$, where

$$h(z) = z^p + \sum_{w=p+n}^{\infty} a_w z^w \quad , g(z) = \sum_{w=p+n-1}^{\infty} b_w z^w , |b_{p+n-1}| < 1. \tag{1.1}$$

For $g(z) = 0$, $H(p)$ reduces to $\mathcal{A}(p)$ the class of all multivalent analytic functions if the co-analytic part of f is zero. For $f(z) \in \mathcal{A}(p)$, we introduce the linear operator $H_l^{c,\alpha,n}(\alpha_i, d_j)(f): \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ as follows:

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$$H_l^{c,\alpha,n}(\alpha_i,d_j)f(z)=z^p+\sum_{w=p+n}^{\infty}\frac{(\alpha_1,l)_{w-1}\dots(\alpha_t,l)_{w-1}}{(l,l)_{w-1}\dots(d_1,l)_{w-1}\dots(d_r,l)_{w-1}}\left(\frac{1+\alpha}{w+\alpha}\right)^c a_w z^w, \tag{1.2}$$

where $z\in U,\alpha\in\mathbb{C}/\{0,-1,-2,\dots\},c\in\mathbb{C},\alpha_i,d_j\in\mathbb{C},d_j\in\mathbb{C}/\{0,-1,-2,\dots\},p,n\geq 1,|l|<1$ and $t=r+1$.

For complex parameters $\alpha_1,\dots,\alpha_t$ and $d_1,\dots,d_r\big(d_j\in\mathbb{C}/\{0,-1,-2,\dots\},j=1,\dots,r,|l|<1\big)$, the $l-$hypergeometric $\Psi=\sum_{w=p+n-1}^{\infty}\frac{(\alpha_1,l)_{w-1}\dots(\alpha_t,l)_{w-1}}{(l,l)_{w-1}\dots(d_1,l)_{w-1}\dots(d_r,l)_{w-1}}z^w$ (1.3)

$$(t=r+1;t,r\in\mathbb{N}_0=\{0,1,2,\dots\};p,n\in\mathbb{N}=\{1,2,\dots\},z\in U).$$

The $l-$shifted factorial for $\alpha\in\mathbb{C}$ and $w\in\mathbb{N}$ is defined by
$(\alpha,l)_0=1\ and\ (\alpha,l)_w=(1-l)(1-\alpha l)(1-\alpha l^2)\dots(1-\alpha l^{w-1})$.

It should be noted that the linear operator (1.2) generalizes many operators that are studied by some authors as follows:

1- If $n=1$, then we obtain the linear operator that is considered by Abdul Ameer and Juma [2].

2- If $p=1,n=1$, then we obtain the linear operator that is considered by Juma and Darus[3].

3- If $p=1,n=1,c=0$, then $H_l^{0,\alpha,1}(\alpha_i,d_j)f(z)=\mathcal{M}_l(\alpha_i,d_j)$,where $\mathcal{M}_l(\alpha_i,d_j)$ is the linear operator that is introduced by Mohammed and Darus[4].

For $l\longrightarrow 1,\alpha_i=l^{\gamma_i},b_j=l^{\beta_j}$, where $\gamma_i,\beta_j\in\mathbb{C}$ and $\beta_j\neq 0(i=1,2,\dots,t\ and\ j=1,2,\dots,r)$, we have the following operators:

4- If $p=1,n=1,t=2,r=1,\gamma_1=\lambda+1,\gamma_2=\lambda+1,\beta_1=\nu+1$, then we obtain the operator that is considered by Prajapat and Bulboca[5].

5- If $p=1,n=1,t=2,r=1,\gamma_1=\lambda,\gamma_2=1,\beta_1=\nu+1$, then we have the operator that is considered by Noor and Bukhari[6].

6- If $p=0,n=1,c=0,\alpha=0,t=2,r=1,\gamma_1=\lambda,\gamma_2=1,\beta_1=\nu+1$, then we obtain the Choi-Saigo-Srivastava operator [7].

7- If $p=0,n=1,t=2,r=1,\gamma_1=\beta_1,\gamma_2=1$, then we obtain the Srivastava-Attiya operator[8].

8- If $p=0,n=1,c=-x,t=2,r=1,\gamma_1=\beta_1,\gamma_2=1$, then we obtain the Cho and Srivastava operator[9].

9- If $p=0,n=1,c=-k(k\in\mathbb{N}),\alpha=0,t=2,r=1,\gamma_1=\lambda=\beta_1,\gamma_2=1$, then we obtain the Salagean operator [10].

10- If $p=0,n=1,c=1,t=2,r=1,\gamma_1=\beta_1,\gamma_2=1,\alpha\geq -1$, then we obtain the Bernardi operator[11].

11- If $p=0,n=1,c=0,\alpha=0,t=2,r=1,\gamma_1=\lambda,\gamma_2=1,\beta_1=\nu$, then we obtain the Carlson –Shaffer operator[12].

12- If $c=0,t=r+1$, then we obtain the Dziok-Srivastava operator[13]. Some of these operators contained some other operators for more details see[14; 15]).

We extend the linear operator (1.2) on complex valued harmonic function $f=h+\bar{g}$, which is defined by (1.1) as follows:

$$H_l^{c,\alpha,n}(\alpha_i,d_j)f(z)=H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)+\overline{H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)}, \tag{1.4}$$

where $H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)=z^p+\sum_{w=p+n}^{\infty}\frac{(\alpha_1,l)_{w-1}\dots(\alpha_t,l)_{w-1}}{(l,l)_{w-1}\dots(d_1,l)_{w-1}\dots(d_r,l)_{w-1}}\left(\frac{1+\alpha}{w+\alpha}\right)^c a_w z^w$,

$$H_l^{c,\alpha,n}(\alpha_i,d_j)g(z)=\sum_{w=p+n-1}^{\infty}\frac{(\alpha_1,l)_{w-1}\dots(\alpha_t,l)_{w-1}}{(l,l)_{w-1}\dots(d_1,l)_{w-1}\dots(d_r,l)_{w-1}}\left(\frac{1+\alpha}{w+\alpha}\right)^c b_w z^w$$

where $z\in U,\alpha\in\mathbb{C}/\{0,-1,-2,\dots\},c\in\mathbb{C},\alpha_i,d_j\in\mathbb{C},d_j\in\mathbb{C}/\{0,-1,-2,\dots\},|n|<1$ and $t=r+1,r\in\mathbb{N}_0=\{0,1,2,\dots\}$. As a special notation for convenience, we make

$$\mathrm{F}_w(c,\alpha,d)=\frac{(\alpha_1,l)_{w-1}\dots(\alpha_t,l)_{w-1}}{(l,l)_{w-1}\dots(d_1,l)_{w-1}\dots(d_r,l)_{w-1}}\left(\frac{1+\alpha}{w+\alpha}\right)^c. \tag{1.5}$$

Definition(1.1): A function $f \in H(p)$ is said to be a member of the class $H_w(p,\mu,n,q)$ if the following condition holds:

$$Re\left(\frac{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^{q+1}-\overline{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^{q+1}}}{z^q\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^q+\overline{z^q\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^q}}\right)>p\mu, \qquad (1.6)$$

wherever $h(z)$ and $g(z)$ are given by (1.1), for $p \geq 1, q \geq 0, q<p, 0 \leq \mu=\frac{1}{\tau}<1, \tau>p, |z|=r<1$, where

$$\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^q = \frac{p!}{(p-q)!}z^{p-q}+\sum_{w=p+n}^{\infty}\frac{w!}{(w-q)!}\, \mathrm{F}_w(c,\alpha;d)a_w z^{w-q}\ , \qquad (1.7)$$

$$\left(H_l^{c,\alpha,n}(\alpha_i,d_j)g(z)\right)^q = \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q)!}\, \mathrm{F}_w(c,\alpha;d)b_w z^{w-q}. \qquad (1.8)$$

Further, let $\overline{H_w}(p,\mu,n,q)$ be the subclass of $H_w(p,\mu,n,q)$ which includes harmonic function $(z)=h(z)+\overline{g(z)}$, where

$$h(z)=z^p-\sum_{w=p+n}^{\infty}a_w z^w,\ g(z)=\sum_{w=p+n-1}^{\infty}b_w z^w. \qquad (1.9)$$

Remark(1.2): Using the values $n=1$ and $q=0$ in the class $H_w(p,\mu,n,q)$, we have the class that is presented by Abdulamer et al. , [2].

Several authors studied a class of multivalent harmonic functions for other conditions, like Atshan et al. [16].

2. Coefficient bounds

In this section, the main important results are stated and proved with sufficient coefficient conditions to functions of harmonic multivalent classes.

Theorem(2.1): Let $f(z)=h(z)+\overline{g(z)}$ with $h(z)$ and $g(z)$ defined as in equation (1.1). If

$$\sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)a_w + \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)b_w \leq \frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right) \quad (2.1)$$

, where $a_1=1, p\geq 1, q\geq 0, q<p, 0\leq\mu=\frac{1}{\tau}<1, \tau>p, |z|=r<1,\ w\in\mathbb{N}$, then $f(z)$ is harmonic, multivalent in U and $f(z)\in H_w(p,\mu,n,q)$with $\mathrm{F}_w(c,\alpha;d)$ defined by (1.3).

Proof: If (2.1) is true, we have to show that

$$Re\left(\frac{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^{q+1}-\overline{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^{q+1}}}{z^q\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^q+\overline{z^q\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^q}}\right)=Re\left(\frac{\mathbb{A}(z)}{\mathbb{B}(z)}\right)>p\mu$$

Using $Re(\mathbb{V})>\alpha$ if and only if $|\mathbb{V}-(1+\alpha)|\leq|\mathbb{V}+(1-\alpha)|$, it suffices to indicate that $|\mathbb{A}(z)-p(1+\mu)\mathbb{B}(z)|-|\mathbb{A}(z)+p(1-\mu)\mathbb{B}(z)|\leq 0$.

In order to compensate for $\mathbb{A}(z)$ and $\mathbb{B}(z)$, we use $|\mathbb{A}(z)-p(1+\mu)\mathbb{B}(z)|=$

$$\left|z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^{q+1}-\overline{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^{q+1}}-p(1+\mu)\left[z^q\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^q+\overline{z^q\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^q}\right]\right|$$

$$=\left|\left(\frac{p!}{(p-q-1)!}-p(1+\mu)\frac{p!}{(p-q)!}\right)z^p+\sum_{w=p+n}^{\infty}\left(\frac{w!}{(w-q-1)!}-p(1+\mu)\frac{w!}{(w-q)!}\right)\mathrm{F}_w(c,\alpha;d)a_w z^w-\sum_{w=p+n-1}^{\infty}\left(\frac{w!}{(w-q-1)!}+p(1+\mu)\frac{w!}{(w-q)!}\right)\mathrm{F}_w(c,\alpha;d)b_w\bar{z}^w\right|$$

$$\leq \left|\frac{p!}{(p-q-1)!} - p(1+\mu)\frac{p!}{(p-q)!}\right| |z|^p + \sum_{w=p+n}^{\infty} \left|\frac{w!}{(w-q-1)!} - p(1+\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)a_w|z|^w + \sum_{w=p+n-1}^{\infty} \left|\frac{w!}{(w-q-1)!} + p(1+\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)b_w|\bar{z}|^w . \quad (2.2)$$

Now, in order to compensate for $\mathbb{A}(z)$ and $\mathbb{B}(z)$, we use $|\mathbb{A}(z) + p(1-\mu)\mathbb{B}(z)|$, then we get

$$|\mathbb{A}(z) + p(1-\mu)\mathbb{B}(z)| = \left| z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^{q+1} - \overline{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_\iota,d_J)g(z)\right)^{q+1}} + p(1-\mu)\left[z^q\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^q + \overline{z^q\left(H_l^{c,\alpha,n}(\alpha_\iota,d_J)g(z)\right)^q}\right]\right|$$

$$= \left|\left(\frac{p!}{(p-q-1)!} + p(1-\mu)\frac{p!}{(p-q)!}\right)z^p + \sum_{w=p+n}^{\infty}\left(\frac{w!}{(w-q-1)!} + p(1-\mu)\frac{w!}{(w-q)!}\right) \mathrm{F}_w(c,\alpha;d)a_w z^w - \sum_{w=p+n-1}^{\infty}\left(\frac{w!}{(w-q-1)!} - p(1-\mu)\frac{w!}{(w-q)!}\right) \mathrm{F}_w(c,\alpha;d)b_w\bar{z}^w\right|$$

$$\geq - \left|\frac{p!}{(p-q-1)!} + p(1-\mu)\frac{p!}{(p-q)!}\right| |z|^p - \sum_{w=p+n}^{\infty}\left|\frac{w!}{(w-q-1)!} + p(1-\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)a_w|z|^w - \sum_{w=p+n-1}^{\infty}\left|\frac{w!}{(w-q-1)!} - p(1-\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)b_w|\bar{z}|^w. \quad (2.3)$$

Then we compensate for equation (2.2) and (2.3), and we get

$$|\mathbb{A}(z) - p(1+\mu)\mathbb{B}(z)| - |\mathbb{A}(z) + p(1-\mu)\mathbb{B}(z)|$$
$$= \left|\frac{p!}{(p-q-1)!} - p(1+\mu)\frac{p!}{(p-q)!}\right| |z|^p + \sum_{w=p+n}^{\infty}\left|\frac{w!}{(w-q-1)!} - p(1+\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)a_w|z|^w + \sum_{w=p+n-1}^{\infty}\left|\frac{w!}{(w-q-1)!} + p(1+\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)b_w|\bar{z}|^w + \left|\frac{p!}{(p-q-1)!} + p(1-\mu)\frac{p!}{(p-q)!}\right| |z|^p + \sum_{w=p+n}^{\infty}\left|\frac{w!}{(w-q-1)!} + p(1-\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)a_w|z|^w + \sum_{w=p+n-1}^{\infty}\left|\frac{w!}{(w-q-1)!} - p(1-\mu)\frac{w!}{(w-q)!}\right| \mathrm{F}_w(c,\alpha;d)b_w|\bar{z}|^w$$
$$= \sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\left(1 - \frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)a_w + \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\left(1 + \frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)b_w - \frac{p!}{(p-q-1)!}\left(1 - \frac{p}{(p-q)}\mu\right) \leq 0.$$

Then we get

$$\sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\left(1 - \frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)a_w + \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\left(1 + \frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)b_w \leq \frac{p!}{(p-q-1)!}\left(1 - \frac{p}{(p-q)}\mu\right)$$

This completes the proof of Theorem 1.

The function is harmonic multivalent

$f(z) =$

$$z^p + \sum_{w=p+n}^{\infty} \frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)} \mathbb{X}_w z^w + \sum_{w=p+n-1}^{\infty} \frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)} \overline{\mathbb{Y}_w z^w}, \quad (2.4)$$

where $\sum_{w=p+n}^{\infty}|\mathbb{X}_w| + \sum_{w=p}^{\infty}|\mathbb{Y}_w| = p$, we indicate that the coefficient bound which is defined by (2.1) is true.

Because

$$\sum_{w=p+n}^{\infty} \frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)} |a_w| + \sum_{w=p+n-1}^{\infty} \frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)} |b_w| \leq 1,$$

$$\sum_{w=p+n}^{\infty} \frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)} \times \frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)} |\mathbb{X}_w| +$$

$$\sum_{w=p+n-1}^{\infty} \frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)} \times \frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right) \mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)} |\mathbb{Y}_w| = \sum_{w=p+n}^{\infty}|\mathbb{X}_w| +$$

$$\sum_{w=p+n-1}^{\infty}|\mathbb{Y}_w| = p. \blacksquare$$

Here, we need to show that the condition of (2.1) is as well necessary for the function $f = h + \bar{g}$, wherever h and g are defined by (1.9).

Theorem(2.2): Suppose that $f = h + \bar{g}$ is given by (1.9) consequently, $f \in \overline{H_w}(p,\mu,n,q)$ if and only if the coefficient in condition (2.1) holds.

Proof: We want to prove that "only if "part of the theorem since $\overline{H_w}(p,\mu,n,q) \subset H_w(p,\mu,n,q)$.

Consequently by(1.6), we get $Re\left(\frac{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^{q+1} - \overline{z^{q+1}\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^{q+1}}}{z^q\left(H_l^{c,\alpha,n}(\alpha_i,d_j)h(z)\right)^q + \overline{z^q\left(H_l^{c,\alpha,n}(\alpha_\iota,d_j)g(z)\right)^q}}\right) > p\mu.$

Or, equally

$$Re\left(\frac{\left[\frac{p!}{(p-q-1)!}z^p - \sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\mathrm{F}_w(c,\alpha;d)a_w z^w - \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\mathrm{F}_w(c,\alpha;d)b_w\bar{z}^w\right] - p\mu\left[\frac{p!}{(p-q)!}z^p - \sum_{w=p+n}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)a_w z^w + \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)b_w\bar{z}^w\right]}{\frac{p!}{(p-q)!}z^p - \sum_{w=p+n}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)a_w z^w + \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)b_w\bar{z}^w}\right) \geq 0$$

$$Re\left(\frac{\frac{p!}{(p-q-1)!}\left[1-\frac{p\mu}{(p-q)}\right]z^p - \sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\left[1+\frac{p\mu}{(w-q)}\right]\mathrm{F}_w(c,\alpha;d)a_w z^w - \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\left[1+\frac{p\mu}{(w-q)}\right]\mathrm{F}_w(c,\alpha;d)b_w\bar{z}^w}{\frac{p!}{(p-q)!}z^p - \sum_{w=p+n}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)a_w z^w + \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)b_w\bar{z}^w}\right) \geq 0. \quad (2.5)$$

The above condition (2.4) must hold for all values of z, where z is a positive number and $0 \leq z < \ell < 1$, we must have

$$\left[\frac{\frac{p!}{(p-q-1)!}\left[1-\frac{p\mu}{(p-q)}\right]\ell^p - \sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\left[1+\frac{p\mu}{(w-q)}\right]\mathrm{F}_w(c,\alpha;d)a_w\ell^w - \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\left[1+\frac{p\mu}{(w-q)}\right]\mathrm{F}_w(c,\alpha;d)b_w\bar{\ell}^w}{\frac{p!}{(p-q)!}\ell^p - \sum_{w=p+n}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)a_w\ell^w + \sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q)!}\mathrm{F}_w(c,\alpha;d)b_w\bar{\ell}^w}\right] \geq 0. \quad (2.6)$$

We note that if the condition (2.1) does not true then the numerator of (2.6) when ℓ goes to 1 is negative. This contradicts the condition for $f \in \overline{H_w}(p,\mu,n,q)$. Therefore, the proof is complete

3. Convolution (Hadamard product)

In this section, we prove that the class $\overline{H_w}(p,\mu,n,q)$ is closed under convolution. In the case of harmonic functions ,
$f(z)=z^p-\sum_{w=p+n}^{\infty}|a_w|\,z^w+\sum_{w=p+n-1}^{\infty}|b_w|\bar{z}^w$ And $F(z)=z^p-\sum_{w=p+n}^{\infty}|\mathbb{A}_w|\,z^w+\sum_{w=p+n-1}^{\infty}|\mathbb{B}_w|\bar{z}^w$.

The convolution of $f(z)$ and $F(z)$ is given by

$$(f*F)(z)=f(z)*F(z)=z^p-\sum_{w=p+n}^{\infty}|a_w\mathbb{A}_w|\,z^w+\sum_{w=p+n-1}^{\infty}|b_w\mathbb{B}_w|\bar{z}^w. \tag{3.1}$$

Theorem(3.1): Let $f(z)\in\overline{H_w}(p,\mu',n,q)$ and $F(z)\in\overline{H_w}(p,\mu'',n,q)$. Then $f*F\in\overline{H_w}(p,\mu'',n,q)\subset\overline{H_w}(p,\mu',n,q)$ for $0\le\mu'\le\mu''<1$.

Proof: We want to show that the coefficient $f*F$ satisfies the required condition that is given in Theorem(2.2). For $F(z)\in\overline{H_w}(p,\mu'',n,q)$, we note that $|\mathbb{A}_w|\le 1$ and $|\mathbb{B}_w|\le 1$. Now, for the convolution function $f*F$, we obtain

$$\sum_{w=p+n}^{\infty}\frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu''\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu''\right)}|a_w||\mathbb{A}_w|+$$
$$\sum_{w=p+n-1}^{\infty}\frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu''\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu''\right)}|b_w||\mathbb{B}_w|\le$$
$$\sum_{w=p+n}^{\infty}\frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu'\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu'\right)}|a_w|+\sum_{w=p+n-1}^{\infty}\frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu'\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu'\right)}|b_w|\le 1.$$

Since $0\le\mu'\le\mu''<1$ and $f(z)\in\overline{H_w}(p,\mu',n,q)$. Therefore,

$$f*F\in\overline{H_w}(p,\mu'',n,q)\subset\overline{H_w}(p,\mu',n,q).$$

4. Convex combination

In this section, we show that $\overline{H_w}(p,\mu,n,q)$ is closed under the convex combination of its member.

Consider that the function $f_i(z)$ is defined for every $i\in\mathbb{N}$, by

$$f_i(z)=z^p+\sum_{w=p+n}^{\infty}\left|a_{w,i}\right|z^w-\sum_{w=p+n-1}^{\infty}\left|b_{w,i}\right|\bar{z}^w \tag{4.1}$$

Theorem(4.1): If the function $f_i(z)$ that is defined by (4.1) in the class $\overline{H_w}(p,\mu,n,q)$ for every $i\in\mathbb{N}$. Then the function $t_i(z)$ that are defined by $t_i(z)=\sum_{i=1}^{\infty}C_i\,f_i(z)$, $0\le C_i\le 1$ are also in the class $\overline{H_w}(p,\mu,n,q)$, where $\sum_{i=1}^{\infty}C_i=1$.

Proof: According to the definition of $t_i(z)$, we can write

$$t_i(z)=z^p+\sum_{w=p+n}^{\infty}\left(\sum_{i=1}^{\infty}C_i\left|a_{w,i}\right|\right)z^w-\sum_{w=p+n-1}^{\infty}\left(\sum_{i=1}^{\infty}C_i\left|b_{w,i}\right|\right)\bar{z}^w.$$

Further, since $f_i(z)$ are in $\overline{H_w}(p,\mu,n,q)$, for every $i\in\mathbb{N}$, then

$$\sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)\left(\sum_{i=1}^{\infty}C_i\left|a_{w,i}\right|\right)+$$
$$\sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)\left(\sum_{i=1}^{\infty}C_i\left|b_{w,i}\right|\right)=$$
$$\sum_{i=1}^{\infty}C_i\left(\sum_{w=p+n}^{\infty}\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)\left|a_{w,i}\right|+\sum_{w=p+n-1}^{\infty}\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)\left|b_{w,i}\right|\right)$$
$$\le\sum_{i=1}^{\infty}C_i\left(\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)\right)\le\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right).$$

Therefore, the proof is completed.

5. Extreme points

We get the extreme points for the class $\overline{H_w}(p,\mu,n,q)$ in this section.

Theorem(5.1): Let f be a function that is defined by (1.3). Then $f\in\overline{H_w}(p,\mu,n,q)$ if and only if

$$f(z)=\sum_{w=p+n-1}^{\infty}\left(\mathbb{X}_w h_w(z)+\mathbb{Y}_w g_w(z)\right), \tag{5.1}$$

where $h_p(z)=z^p, h_w(z)=z^p-\left(\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\right)z^w, w=p+n,p+n+1,\ldots$

$g_w(z)=z^p+\left(\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\right)\bar{z}^w, w=p+n-1,p+n,\ldots$

and $\sum_{w=p+n-1}^{\infty}(\mathbb{X}_w+\mathbb{Y}_w)=1, \mathbb{X}_w\geq 0, \mathbb{Y}_w\geq 0.$

Specifically, the extreme points of $\overline{H_w}(p,\mu,n,q)$ are $\{h_w\}$ and $\{g_w\}$.

Proof: For the function $f(z)$ of the form (5.1), we have

$f(z)=\sum_{w=p+n-1}^{\infty}\left(\mathbb{X}_w h_w(z)+\mathbb{Y}_w g_w(z)\right)$

$=\sum_{w=p+n-1}^{\infty}X_w\left(z^p-\left(\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\right)z^w\right)+\quad \sum_{w=p+n-1}^{\infty}Y_w\left(z^p+\left(\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\right)\bar{z}^w\right)$

$=z^p-\sum_{w=p+n}^{\infty}\left(\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\right)\mathbb{X}_w z^w+\sum_{w=p+n-1}^{\infty}\left(\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\right)\mathbb{Y}_w\bar{z}^w.$

Therefore,

$\sum_{w=p+n}^{\infty}\left(\frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}\right)|a_w|+\sum_{w=p+n-1}^{\infty}\left(\frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}\right)|b_w|$

$=\sum_{w=p+n}^{\infty}\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\times\frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}|\mathbb{X}_w|+$
$\sum_{w=p+n-1}^{\infty}\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\times\frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}|\mathbb{Y}_w|$

$=\sum_{w=p+n}^{\infty}|\mathbb{X}_w|+\sum_{w=p}^{\infty}|\mathbb{Y}_w|=1-X_{p+n-1}\leq 1$ and so $f\in\overline{H_w}(p,\mu,n,q)$.

Conversely, suppose that $f\in\overline{H_w}(p,\mu,n,q)$.

Setting $X_w=\frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}|a_w|, 0\leq\mathbb{X}_w\leq 1, w=p+n,p+n+1,\ldots$

$Y_w=\frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}|b_w|, 0\leq\mathbb{Y}_w\leq 1, w=p+n-1,p+n,\ldots$, and

$X_{p+n-1}=1-\left(\sum_{w=p+n}^{\infty}\mathbb{X}_w+\sum_{w=p+n-1}^{\infty}\mathbb{Y}_w\right)$. As a result, we can write f as follows,

$f(z)=z^p-\sum_{w=p+n}^{\infty}|a_w|z^w+\sum_{w=p+n-1}^{\infty}|b_w|\bar{z}^w$

$=z^p-\sum_{w=p+n}^{\infty}\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\mathbb{X}_w z^w+\sum_{w=p+n-1}^{\infty}\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}\mathbb{Y}_w\bar{z}^w$

$=h_{p+n-1}(z)\mathbb{X}_{p+n-1}-\sum_{w=p+n}^{\infty}h_w(z)X_w+\sum_{w=p+n-1}^{\infty}g_w(z)\,\mathbb{Y}_w$

$=\sum_{w=p+n-1}^{\infty}(h_w(z)\mathbb{X}_w+g_w(z)\mathbb{Y}_w).$

Therefore, the proof is completed.

6. Distortion Bounds

The distortion bounds for functions in $\overline{H_w}(p,\mu,n,q)$are given by the following theorem , which yields a covering result for this class.

Theorem(6.1): If $f = h + \bar{g} \in \overline{H_w}(p,\mu,n,q)$, provided by (1.9) and $|z| = r < 1$, then

$$\left(1-|b_{n+p-1}|r^{n-1}\right)r^p - \frac{1}{\frac{(p+n)!}{((p+n)-q-1)!}\left(1-\frac{p}{((p+n)-q)}\mu\right)\mathrm{F}_{(p+n)}(c,\alpha;d)}\mathbb{H} \le |f(z)| \le$$
$$\left(1+|b_{n+p-1}|r^{n-1}\right)r^p + \frac{1}{\frac{(p+n)!}{((p+n)-q-1)!}\left(1-\frac{p}{((p+n)-q)}\mu\right)\mathrm{F}_{(p+n)}(c,\alpha;d)}\mathbb{H}, \qquad (\mathbf{6.1})$$

where

$$\mathbb{H} = \left(\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right) - \frac{(p+n-1)!}{((p+n-1)-q-1)!}\left(1+\frac{p}{((p+n-1)-q)}\mu\right)\mathrm{F}_{(p+n-1)}(c,\alpha;d)|b_{p+n-1}|r^{p+n}\right).$$

Proof: First, we shall prove the left hand side of inequality (6.1). Let $f \in \overline{H_w}(p,\mu,n,q)$. Hence,

$$|f(z)| = \left|z^p - \sum_{w=p+n}^{\infty} a_w z^w + \sum_{w=p+n-1}^{\infty} b_w \bar{z}^w\right|$$
$$\ge \left(1-|b_{n+p-1}||z|^{n-1}\right)|z|^p - \sum_{w=p+n}^{\infty}(|a_w|+|b_w|)|z|^w$$
$$\ge \left(1-|b_{n+p-1}|r^{n-1}\right)r^p - \sum_{w=p+n}^{\infty}(|a_w|+|b_w|)r^{p+n}$$
$$=\left(1-|b_{n+p-1}|r^{n-1}\right)r^p - \frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{(p+n)!}{((p+n)-q-1)!}\left(1-\frac{p}{((p+n)-q)}\mu\right)\mathrm{F}_{(p+n)}(c,\alpha;d)} \times$$
$$\sum_{w=p+n}^{\infty}\frac{\frac{(p+n)!}{((p+n)-q-1)!}\left(1-\frac{p}{((p+n)-q)}\mu\right)\mathrm{F}_{(p+n)}(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}(|a_w|+|b_w|)r^{p+n}$$
$$\ge$$
$$\left(1-|b_{n+p-1}|r^{n-1}\right)r^p - \frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{(p+n)!}{((p+n)-q-1)!}\left(1-\frac{p}{((p+n)-q)}\mu\right)\mathrm{F}_{(p+n)}(c,\alpha;d)} \times$$
$$\sum_{w=p+n}^{\infty}\left(\left|\frac{\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}\right| a_w + \sum_{w=p+n-1}^{\infty}\frac{\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\mathrm{F}_w(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}|b_w| - \right.$$
$$\left.\frac{\frac{(p+n-1)!}{((p+n-1)-q-1)!}\left(1+\frac{p}{((p+n-1)-q)}\mu\right)\mathrm{F}_{(p+n-1)}(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}|b_{p+n-1}|\right)r^{p+n}$$

$$\ge$$
$$\left(1-|b_{n+p-1}|r^{n-1}\right)r^p - \frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}{\frac{(p+n)!}{((p+n)-q-1)!}\left(1-\frac{p}{((p+n)-q)}\mu\right)\mathrm{F}_{(p+n)}(c,\alpha;d)} \times$$
$$\left(\frac{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right) - \frac{(p+n-1)!}{((p+n-1)-q-1)!}\left(1+\frac{p}{((p+n-1)-q)}\mu\right)\mathrm{F}_{(p+n-1)}(c,\alpha;d)}{\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right)}|b_{p+n-1}|r^{p+n}\right)$$
$$\left(1-|b_{n+p-1}|r^{n-1}\right)r^p - \frac{1}{\frac{(p+n)!}{((p+n)-q-1)!}\left(1-\frac{p}{((p+n)-q)}\mu\right)\mathrm{F}_{(p+n)}(c,\alpha;d)} \times \left(\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right) - \right.$$
$$\left.\frac{(p+n-1)!}{((p+n-1)-q-1)!}\left(1+\frac{p}{((p+n-1)-q)}\mu\right)\mathrm{F}_{(p+n-1)}(c,\alpha;d)|b_{p+n-1}|r^{p+n}\right).$$

In the same way, the right-hand inequality in (6.1) is easy to prove .

7. Integral operator

In this section, we will show the class $\overline{H_w}(p,\mu,n,q)$ is closed under the generalized Bernardi-Libera-Livingstone integral operator $\mathcal{L}_\Lambda$. The generalized Bernardi-Libera-Livingstone integral operator for an analytic function f is defined by

$$\mathcal{L}_\Lambda\big(f(z)\big)=\frac{\Lambda+1}{z^\Lambda}\int_0^z t^{\Lambda-1}f(t)\,dt,(\Lambda>-1).$$

For harmonic functions $f=h+\bar{g}$, however it is defined by

$$\mathcal{L}_\Lambda\big(f(z)\big)=\frac{\Lambda+1}{z^\Lambda}\int_0^z t^{\Lambda-1}h(t)\,dt+\overline{\frac{\Lambda+1}{z^\Lambda}\int_0^z t^{\Lambda-1}g(t)\,dt},(\Lambda>-1). \qquad (7.1)$$

Theorem(7.1): If $f\in\overline{H_w}(p,\mu,n,q)$, then $\mathcal{L}_\Lambda\big(f(z)\big)\in\overline{H_w}(p,\mu,n,q)$.

Proof: By definition of $\mathcal{L}_\Lambda\big(f_m(z)\big)$ that is given in (7.1), it follows that

$$\mathcal{L}_\Lambda\big(f(z)\big)=\frac{\Lambda+1}{z^\Lambda}\int_0^z t^{\Lambda-1}\Big(t-\sum_{w=2}^\infty|a_w|t^w+(-1)^m\sum_{w=1}^\infty|b_w|\bar{t}^w\Big)\,dt$$

$$=z-\sum_{w=2}^\infty\left(\frac{\Lambda+1}{\Lambda+w}\right)|a_w|z^w+(-1)^m\sum_{w=1}^\infty\left(\frac{\Lambda+1}{\Lambda+w}\right)|b_w|\bar{z}^w.$$

Hence,

$$\sum_{w=p+n}^\infty\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\,\mathrm{F}_w(c,\alpha;d)\left(\frac{\Lambda+1}{\Lambda+w}\right)|a_w|$$
$$+\sum_{w=p+n-1}^\infty\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\,\mathrm{F}_w(c,\alpha;d)\left(\frac{\Lambda+1}{\Lambda+w}\right)|b_w|$$
$$\le\sum_{w=p+n}^\infty\frac{w!}{(w-q-1)!}\left(1-\frac{p}{(w-q)}\mu\right)\,\mathrm{F}_w(c,\alpha;d)|a_w|$$
$$+\sum_{w=p+n-1}^\infty\frac{w!}{(w-q-1)!}\left(1+\frac{p}{(w-q)}\mu\right)\,\mathrm{F}_w(c,\alpha;d)|b_w|\ \le\frac{p!}{(p-q-1)!}\left(1-\frac{p}{(p-q)}\mu\right).$$

Therefore, by Theorem(2.2), we have $\mathcal{L}_\Lambda\big(f(z)\big)\in\overline{H_w}(p,\mu,n,q)$.

Conclusion

In this work, the higher derivatives multivalent harmonic function class which is defined by the general linear operator has been introduced. Some geometric properties, namely coefficient estimation, convex combination, extreme point, distortion theorem and convolution property have been given. The invariant property of this class under the Bernandi-Libera-Livingston integral for harmonic functions has been shown

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