DOI: 10.24996/ijs.2022.63.5.31

ISSN: 0067-2904

The Dynamics of the Aquatic Food Chain System in the Contaminated Environment

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Received: 19/6/2021 Accepted: 27/7/2021 Published: 30/5/2022

Abstract

In this paper, the aquatic food chain model, consisting of Phytoplankton, Zooplankton, and Fish, in the contaminated environment is proposed and studied. Modified Leslie–Gower model with Holling type IV functional response are used to describe the growth of Fish and the food transition throughout the food chain, respectively. The toxic substance affects directly the Phytoplankton and indirectly the other species. The local stability analysis of all possible equilibrium points is done. The persistence conditions of the model are established. The basin of attraction for each point is specified using the Lyapunov function. Bifurcation analysis near the coexistence equilibrium point is investigated. Detecting the existence of chaos is carried out using bifurcation diagrams. Numerical simulation shows that the food chain has rich dynamics including chaos. Moreover, the existence of toxic substances works as a stabilizing factor in the model.

Keywords: Aquatic food chain, Leslie–Gower model, Stability, bifurcation, chaos

ديناميكيه نظام السلسلة الغذائية المائية في البيئة الملوثة

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الخلاصه

في هذا البحث تم اقتراح ودراسة نموذج السلسلة الغذائية المائية المكون من العوالق النباتية والعوالق الحيوانية والأسماك في البيئة الملوثة. تم استخدام نموذج ليزلي–كور المعدل مع الاستجابة الوظيفية من النوع الرابع من هولينج لوصف نمو الأسماك والانتقال الغذائي في جميع أنحاء السلسلة الغذائية على التوالي. تؤثر المادة السامة بشكل مباشر على العوالق النباتية وعلى الأنواع الأخرى بشكل غير مباشر. تحليل الاستقرارية المحلية لجميع نقاط التوازن تم اجراءه. تم بناء شروط الثبات للنموذج. حددنا حوض التجاذب لكل نقطة من نقاط توازن النموذج باستخدام دالة ليابانوف. بحثنا التشعب المحلي بالقرب من نقطة الاتزان الموجبة. تم التحري عن وجود الفوضى في النموذج باستخدام مخطط التشعب. بينت المحاكاة العددية للنموذج وجود

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ديناميكية غنية للنموذج بما في ذلك الفوضى. بالاضافة لما تقدم فان وجود المواد السامة يعمل كعامل استقرار لديناميكية النموذج.

1. Introduction

Mathematical biology is the most important subject to study real-world systems in both ecology and epidemiology. Ecological systems are real-world systems in which the interaction between their compartments is nonlinear. Such systems have stimulated great interest in the development of mathematical models for several ecological systems so that a remarkable variety of dynamic behaviour's including periodic and chaos were discovered. The food chain, which represents one of the most important ecological systems, is a linear sequence of organisms where nutrients and energy is transferred from one organism to the other. A food chain explains which organism eats another organism in the environment.

Since the marine ecosystems cover around 70% of the Earth’s surface and account for nearly half of global primary production that supports human life. In the last few decades, many mathematical models have been developed and applied in aquatic ecosystems [1]. These models simulate the transport of nutrients or aquatic populations. The aquatic ecosystem provides an incubation area for the plankton population, fishes and invertebrates, and conserves the rest of the coastline by bounding wave action and controlling water [2]. The plankton population plays a vital role in the aquatic food chain system. The food chain demonstrates the feeding style or connection between living creatures. A trophic level indicates the successive stages in a food chain, starting with producers at the bottom, followed by a sequence of consumers. Every level in a food chain is recognized as a trophic level. Plankton species have defense mechanisms against predation through their production of toxins [3]. Such defensive behavior has a considerable impact on phytoplankton–zooplankton dynamics [4–7]. Upadhyay and Chattopadhyay [8] demonstrated that the defense mechanisms against predation by plankton may sometimes act as a biocontrol by the stabilizing effect towards the plankton population. The flowering of such algal and fish predation on zooplankton has a great negative effect on zooplankton and the marine ecosystem.

On the other hand, the study of the aquatic food chains in a polluted marine environment by external toxicity has been considered by many researchers. It is observed that external toxicity plays a pivotal role in the aquatic ecosystem, different dynamical behavior in such a food chain has been obtained, see for example [9-14]. Recently, Raw et al [2] have studied a three species plankton–fish system that incorporates external toxicity and nonlinear harvesting. They consider that the growth of species is affected by an external toxic substance, however, the predation rate is considered as Holling type II functional response. It is observed that there are complex dynamics in the system.

According to the above, a variety of aquatic mathematical models are proposed and studied. These models are considered different biological factors such as toxicity, harvesting, delay, etc. Raw and Mishra [15] proposed and studied a tri-trophic reaction-diffusion model that incorporates Holling type III and Monod–Haldane functional response for food chain consisting of phytoplankton, zooplankton, and fish, in the existence of toxic grouped phytoplankton on zooplankton and fish populations. They observed that the inhibitory effect is able to destabilize the homogeneous steady-state and also able to produce chaos in the plankton–fish system. Recently, Thakur and Ojha [16] proposed and studied delayed plankton–fish model with Monod–Haldane-type functional response. They observed that the system is rich in complex dynamics, and due to defense ability in prey and middle predator system shows extinction in top predator.

The main objective of this paper is to propose and studied a tri-trophic aquatic food chain model consisting of phytoplankton–zooplankton-Fish with the usage of Holling type IV (simplified Monod–Haldane-type) functional response to model the feeding process. It is assumed that the food chain system lived in a contaminated environment in which the toxic

substance affects directly the Phytoplankton, while it affects indirectly the other species. Moreover, a modified Leslie–Gower model is used to describe the growth of Fish.
The outline of the paper is: Section 2 gives the description of the model and how to reduce its parameters. Section 3 determines the equilibrium points and gives conditions for their local stability. Section 4 treats the persistence of the model. However, the investigation of global behavior is given in section 5. Section 6 investigates the possibility of the occurrence of local bifurcation. Section 7 interests in the numerical simulation of the model. Finally, section 8 gives the conclusion of this study.

2. Mathematical model formulation

In an aquatic food chain system, some species are affected by external toxic substances, for example, industrial wastage, which inhibits the growth of that species. Recently, Chakraborty and Das [13] studied the effect of an external toxic substance on the population dynamics of a system consisting of two zooplanktons and phytoplankton with constant harvesting. In this section, a food chain model for the interaction of phytoplankton, zooplankton, and fishing in contaminated aquatic areas is formulated. Let $X(T)$, $Y(T)$, and $Z(T)$ be the density at time T for phytoplankton, zooplankton, and fish, respectively. Now in order to formulate the above mathematical model of an aquatic food chain, the following assumptions have been adopted:

1. Assume that phytoplankton, zooplankton, and fish participate in an aquatic food chain system, wherein in the absence of zooplankton, the phytoplankton grows logistically with constant intrinsic growth rate and carrying capacity. The existence of toxic substances causes depletion in the density of phytoplankton. Because the phytoplankton directly depends on environmental resources a cubic term $b_1X^3(T)$ is used to describe the intensity of effectiveness at time$T$.
2. It is assumed that, both the species phytoplankton and zooplankton have the capability of group defense against any attack by a predator, therefore Holling type IV functional responses are used to describe the consumption process in the first and second level of food chain.
3. It is assumed that zooplankton is hurt due to natural death and indirect infection of external toxicity.
4. Because the fish at the upper-level consumes the preferred food, represented by the zooplankton, as well as additional food from the environment, the logistic growth rate is used to describe the fish growth in which the carrying capacity depends on the zooplankton. Moreover, the fish decreases due to indirect infection of external toxicity. Finally, because the fish grows by sexual reproduction and loses due to intra-species competition. The term c_1Z^2 signifies the fact that fish is sexually reproducing species. It shows that the mating frequency is directly proportional to the number of males as well as that of females present at any instant of time T.
5. It is assumed to use the quadratic term for describing the indirect infection of external toxicity in both the zooplankton and fish.

According to the above hypotheses, the dynamics of described aquatic food chain can be written in the following set of equations:

$$\begin{aligned}
\frac{dX}{dT} &= rX\left[1-\frac{X}{K}\right]-\frac{a_0XY}{1+m_0X^2}-b_1X^3,\\
\frac{dY}{dT} &= \frac{a_1XY}{1+m_0X^2}-\frac{a_2YZ}{1+m_1Y^2}-b_2Y^2-dY,\\
\frac{dZ}{dT} &= c_1\left[Z^2-\frac{Z^2}{c_2+c_3Y}\right]-b_3Z^2,
\end{aligned}\qquad (1)$$

where $X(0)\geq 0, Y(0)\geq 0$, and $Z(0)\geq 0$. All the parameters are assumed to be positive and described in the Table (1).

Table 1- Brief description of the system (1) parameters

Parameters	Description
r	The intrinsic growth rate of phytoplankton.
K	Environment carrying capacity of phytoplankton.
b_1	The coefficient of toxicity efficiency of the phytoplankton population.
b_2	The coefficient of toxicity efficiency of the zooplankton population.
b_3	The coefficient of toxicity efficiency of the fish population.
a_0	The maximum consumption rate of the phytoplankton population.
a_1	The maximum per capita growth rate of the zooplankton population.
a_2	The maximum consumption rate of the zooplankton population.
d	The natural death rate of the zooplankton population.
m_0	The defense efficiency of phytoplankton against zooplankton.
m_1	The defense efficiency of zooplankton against fish.
c_1	The growth rate of fish by sexual reproduction.
c_2	The protection rate of fish provided by the environment.
c_3	The fish's preference rate of zooplankton.

Now, in order to study the above system of equations more generally, we drop all the units from it by using the following dimensionless variables and constant.

$$\begin{aligned}&x = \frac{X}{K}, y = \frac{Y}{K}, z = \frac{Z}{K}, t = rT, u_1 = \frac{a_0 K}{r}, u_2 = m_0 K^2,\\ &u_3 = \frac{b_1 K^2}{r}, u_4 = \frac{a_1 K}{r}, u_5 = \frac{a_2 K}{r}, u_6 = m_1 K^2,\ u_7 = \frac{b_2 K}{r},\\ &u_8 = \frac{d}{r}, u_9 = \frac{c_1 K}{r}, u_{10} = \frac{1}{c_3 K},\ u_{11} = \frac{c_2}{c_3 K}, u_{12} = \frac{b_3 K}{r}.\end{aligned} \tag{2}$$

Accordingly, the dimensionless system corresponding to the system (1) can be written as:

$$\begin{aligned}\frac{dx}{dt} &= x\left[(1-x) - \frac{u_1 y}{1+u_2x^2} - u_3x^2\right] = xf_1,\\ \frac{dy}{dt} &= y\left[\frac{u_4x}{1+u_2x^2} - \frac{u_5z}{1+u_6y^2} - u_7\mathrm{y} - u_8\right] = yf_2,\\ \frac{dz}{dt} &= z\left[u_9\left(z - \frac{u_{10}z}{u_{11}+y}\right) - u_{12}\,\mathrm{z}\right] = zf_3,\end{aligned} \tag{3}$$

where $x(t) \geq 0, y(t) \geq 0,$ and $z(t) \geq 0.$ Note that the number of parameters has been reduced from fourteen in the system (1) to twelve in the system (3). Moreover, the functions $f_i; i = 1,2,3,$ in the right-hand side are continuous and have continuous partial derivatives on the following space:

$R_+{}^3 = \{(x,y,z) \in R^3 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0\}.$

Therefore, the solution of the system (3) exists and is unique.

Theorem (1): All solutions of the system (3) initiating in R^3_+ are uniformly bounded.

Proof: From the first equation of the system (3), we have

$\frac{dx}{dt} \leq x(1-x).$

Then, it is obtained that $sup\, x(t) \leq 1$. Consider $M(t) = x(t) + \frac{u_1}{u_4}y(t)$, then

$\frac{dM}{dt} = \frac{dx}{dt} + \frac{u_1}{u_4}\frac{dy}{dt} \leq x(1-x) + u_8x - u_8M,$

Therefore, for all $t > 0$, it is observed that $M(t) \leq \left(1 + \frac{1}{4u_8}\right) = \tau_1$.

Now, consider the function $N(t) = x(t) + \frac{u_1}{u_4}y(t) + \propto z(t)$, then the derivative of $N(t)$ can be written as:

$\frac{dN}{dt} \leq x(1-x) + u_8x - u_8N + u_8z(1-\Lambda z),$

where $\Lambda = \frac{1}{u_8}\left[\frac{u_9u_{10} + \left(u_{11} + \frac{u_4}{u_1}\tau_1\right)u_{12}}{\left(u_{11} + \frac{u_4}{u_1}\tau_1\right)} - u_9\right]$, which is positive provided that

$$u_9u_{10} + \left(u_{11} + \frac{u_4}{u_1}\tau_1\right)u_{12} > u_9\left(u_{11} + \frac{u_4}{u_1}\tau_1\right).$$

Therefore, by using the maximum values of the logistic terms and that for the variable x, it is obtained that

$$\frac{dN}{dt} + u_8 N \leq \frac{1}{4} + u_8 + \frac{u_8}{4\Lambda}.$$

Hence, for all $t > 0$, it is observed that $N(t) \leq \frac{\left(\frac{1}{4}+u_8+\frac{u_8}{4\Lambda}\right)}{u_8} = \tau_2$, thus the proof is done.

3. Existence of equilibrium points and local stability

In this section, the existence of equilibrium points of the system (3) is carried out. Then the linearization technique is used to investigate the local stability for each of them. Notes that, there are at most six non-negative equilibrium points of the system (3), these points are described as follows:

- The trivial equilibrium point $E_0 = (0, 0, 0)$ always exits.
- The first single-species equilibrium point $E_1 = (x_1, 0, 0)$ always exits, where

$$x_1 = \frac{-1+\sqrt{1+4u_3}}{2u_3}. \tag{4}$$

- The second single-species equilibrium point $E_2 = (0, 0, z_1)$, where $z_1 > 0$, exists provided that the following condition holds.

$$u_9 = \frac{u_9 u_{10} + u_{11} u_{12}}{u_{11}}. \tag{5}$$

- The fish-free equilibrium point that denoted by $E_3 = (\bar{x}, \bar{y}, 0)$, where

$$\bar{y} = \frac{1}{u_7}\left[\frac{u_4 \bar{x}}{1+u_2 \bar{x}^2} - u_8\right], \tag{6}$$

while, $\bar{x}$ is a positive root of the following six-order polynomial equation:

$$\begin{aligned} &-u_2^2 u_3 u_7 \bar{x}^6 - u_2^2 u_7 \bar{x}^5 + (u_2^2 u_7 - 2u_2 u_3 u_7)\bar{x}^4 - 2u_2 u_7 \bar{x}^3 \\ &\quad +(2u_2 u_7 + u_1 u_2 u_8 - u_3 u_7)\bar{x}^2 - (u_7 + u_1 u_4)\bar{x} + (u_7 + u_1 u_8) = 0. \end{aligned} \tag{7}$$

Clearly the fish-free equilibrium point exists uniquely in the positive quadrant of xy −plane provided that the following conditions hold:

$$u_8 < \frac{u_4 \bar{x}}{1+u_2 \bar{x}^2}, \tag{8a}$$

$$2u_2 + \frac{u_1 u_2 u_8}{u_7} < u_3. \tag{8b}$$

- The zooplankton-free equilibrium point that denoted by $E_4 = (x_1, 0, z_1)$, where x_1 is given by equation (4) and $z_1 > 0$, exists uniquely in the positive quadrant of xz −plane under condition (5).
- The coexistence equilibrium point $E_5 = (x^*, y^*, z^*)$, where

$$\boldsymbol{y^*} = \frac{\boldsymbol{u_9 u_{10} - u_{11}(u_9 - u_{12})}}{\boldsymbol{u_9 - u_{12}}}, \quad \boldsymbol{z^*} = \left(\frac{\boldsymbol{1+u_6 {y^*}^2}}{\boldsymbol{u_5}}\right)\left[\frac{\boldsymbol{u_4 x^*}}{\boldsymbol{1+u_2 {x^*}^2}} - (\boldsymbol{u_7 y^* + u_8})\right],$$

while x^* is a positive root to the following fourth order polynomial equation:

$$\begin{aligned} &-u_2 u_3 (u_9 - u_{12}){x^*}^4 - u_2(u_9 - u_{12}){x^*}^3 + (u_9 - u_{12})(u_2 - u_3){x^*}^2 \\ &\quad -(u_9 - u_{12})x^* + (u_9 - u_{12})(1 + u_1 u_{11}) - u_1 u_9 u_{10} = 0. \end{aligned} \tag{10}$$

Obviously, the coexistence equilibrium point exists uniquely in the interior of ${R_+}^3$, provided that the following conditions hold.

$$0 < u_{11}(u_9 - u_{12}) < u_9 u_{10} < (u_9 - u_{12})\left(\frac{1}{u_1} + u_{11}\right), \tag{11a}$$

$$u_2 < u_3, \tag{11b}$$

$$(u_7 y^* + u_8) < \frac{u_4 x^*}{1+u_2 {x^*}^2}. \tag{11c}$$

Now, to investigate the local stability at each of the above equilibrium points the Jacobian matrix is determined and then their eigenvalues are found. The equilibrium is said to be local asymptotically stable if and only if all the eigenvalues have negative real parts. However, it is unstable if there is a positive real part eigenvalue. Finally, it said to be non-hyperbolic equilibrium point if there exists zero real part eigenvalue and then the linearization does not applicable in this case.

For the trivial equilibrium point $E_0 = (0, 0, 0)$, the Jacobian matrix can be written as:

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -u_8 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12a}$$

Thus the eigenvalues of $J(E_0)$ are given by:

$$\lambda_{01} = 1 > 0,\ \lambda_{02} = -u_8 < 0, \text{ and } \lambda_{03} = 0. \tag{12b}$$

Clearly, the existence of positive eigenvalue ensures that the trivial equilibrium E_0 is unstable point.

The Jacobian matrix at the first single-species equilibrium point $E_1 = (x_1, 0,0)$ can be written as:

$$J(E_1) = \begin{bmatrix} -x_1(1 + 2u_3\, x_1) & -\frac{u_1 x_1}{1+ u_2 x_1^2} & 0 \\ 0 & \frac{u_4 x_1}{1+ u_2 x_1^2} - u_8 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{13a}$$

Thus the eigenvalues of $J(E_1)$ are written as:

$$\lambda_{11} = -x_1(1 + 2u_3\, x_1) < 0, \lambda_{12} = \frac{u_4 x_1}{1+ u_2 x_1^2} - u_8, \text{ and } \lambda_{13} = 0. \tag{13b}$$

The existence of zero eigenvalue of $J(E_1)$ leads to that, the first single-species equilibrium point E_1 is a non-hyperbolic point. Therefore, the stability of the equilibrium point E_1 can be studied using the Lyapunov method provided that the second eigenvalue is negative too.

The Jacobian matrix at the second single-species equilibrium point $E_2 = (0,0, z_1)$ can be written as:

$$J(E_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(u_5 z_1 + u_8) & 0 \\ 0 & \frac{u_9 u_{10}\, z_1{}^2}{u_{11}{}^2} & 0 \end{bmatrix}. \tag{14a}$$

Thus the eigenvalues of $J(E_2)$ are determined by:

$$\lambda_{21} = 1 > 0,\ \lambda_{22} = -(u_5 z_1 + u_8) < 0, \text{ and } \lambda_{23} = 0. \tag{14b}$$

Clearly, the existence of positive eigenvalue ensures that the second single-species equilibrium E_2 is unstable point.

For the fish-free equilibrium point $E_3 = (\bar{x}, \bar{y}, 0)$, the Jacobian matrix can be written in the form:

$$J(E_3) = \begin{bmatrix} \bar{x}\left[\frac{2u_1 u_2 \bar{x}\bar{y}}{(1+ u_2\bar{x}^2)^2} - (1 + 2u_3\,\bar{x})\right] & -\frac{u_1\bar{x}}{1+ u_2\bar{x}^2} & 0 \\ \frac{u_4\bar{y}(1- u_2\bar{x}^2)}{(1+ u_2\bar{x}^2)^2} & -u_7\bar{y} & -\frac{u_5\bar{y}}{1+ u_6 y_3{}^2} \\ 0 & 0 & 0 \end{bmatrix}. \tag{15a}$$

Clearly, the characteristic equation of $J(E_3)$ can be written as

$$\lambda(\lambda^2 + D_1\lambda + D_2) = 0, \tag{15b}$$

where

$$D_1 = \bar{x}\left[(1 + 2u_3\,\bar{x}) - \frac{2u_1 u_2 \bar{x}\bar{y}}{(1+ u_2\bar{x}^2)^2}\right] + u_7\bar{y}.$$

$$D_2 = u_7\bar{x}\bar{y}\left[(1 + 2u_3\,\bar{x}) - \frac{2u_1 u_2 \bar{x}\bar{y}}{(1+ u_2\bar{x}^2)^2}\right] + \left(\frac{u_1\bar{x}}{1+ u_2\bar{x}^2}\right)\left(\frac{u_4\bar{y}(1- u_2\bar{x}^2)}{(1+ u_2\bar{x}^2)^2}\right).$$

Clearly, there exists a zero eigenvalue given by $\lambda_{33} = 0$, which ensures that the fish-free equilibrium E_3 is a non-hyperbolic point. While the other eigenvalues are the roots of the second order polynomial equation that given in equation (15b), and can be written as

$$\lambda_{31} = \frac{-D_1 - \sqrt{D_1{}^2 - 4D_2}}{2}, \lambda_{32} = \frac{-D_1 + \sqrt{D_1{}^2 - 4D_2}}{2}. \tag{15c}$$

Obviously, the eigenvalues λ_{31}, and λ_{32} have negative real parts provided that the following sufficient conditions hold.

$$\frac{2u_1u_2\bar{x}\bar{y}}{(1+u_2\bar{x}^2)^2} < (1 + 2u_3\,\bar{x}). \tag{15d}$$
$$u_2\bar{x}^2 < 1. \tag{15e}$$

Note that, since the equilibrium point E_3 is a non-hyperbolic point with simple zero eigenvalue and two negative real parts eigenvalues under the conditions (15d)-(15e), then the stability of E_3 can be studied using other methods e.g. Lyapunov method.

Now the Jacobian matrix of the system (3) at the zooplankton-free equilibrium point that denoted by $E_4 = (x_1, 0, z_1)$ can be written by:

$$J(E_4) = \begin{bmatrix} -x_1(1+2u_3\,x_1) & -\frac{u_1x_1}{1+u_2{x_1}^2} & 0 \\ 0 & \frac{u_4x_1}{1+u_2{x_1}^2} - (u_5z_1 + u_8) & 0 \\ 0 & \frac{u_9u_{10}{z_1}^2}{u_{11}^2} & 0 \end{bmatrix}. \tag{16a}$$

Direct computation gives that the eigenvalues of $J(E_4)$ are given by

$$\lambda_{41} = -x_1(1+2u_3\,x_1),\ \lambda_{42} = \frac{u_4x_1}{1+u_2{x_1}^2} - (u_5z_1 + u_8),\ \lambda_{43} = 0. \tag{16b}$$

Here, the eigenvalue λ_{42} is negative provided that the following condition holds.

$$\frac{u_4x_1}{1+u_2{x_1}^2} < (u_5z_1 + u_8). \tag{16c}$$

However, the existence of zero eigenvalue makes the zooplankton-free equilibrium point non-hyperbolic point. Similarly, point E_4can be studied using the Lyapunov method.

Finally, the local stability conditions of the coexistence equilibrium point are established in the following theorem.

Theorem 2: The coexistence equilibrium point $E_5 = (x^*, y^*, z^*)$ of the system (3) is locally asymptotically stable provided that the following conditions hold:

$$\frac{2u_1u_2x^*y^*}{\left(1+u_2{x^*}^2\right)^2} < (1 + 2u_3x^*), \tag{17a}$$
$$\frac{2u_5u_6y^*z^*}{\left(1+u_6{y^*}^2\right)^2} < u_7, \tag{17b}$$
$$u_2{x^*}^2 < 1. \tag{17c}$$

Proof. The Jacobian matrix at the coexistence equilibrium point can be written as:

$$J(E_5) = [k_{ij}]_{3\times 3}, \tag{18a}$$

where

$$k_{11} = x^*\left[\frac{2u_1u_2x^*y^*}{\left(1+u_2{x^*}^2\right)^2} - (1 + 2u_3\,x^*)\right],\ k_{12} = -\frac{u_1x^*}{1+u_2{x^*}^2},\ k_{13} = 0,$$
$$k_{21} = \frac{u_4y^*\left(1-u_2{x^*}^2\right)}{\left(1+u_2{x^*}^2\right)^2},\ k_{22} = y^*\left(\frac{2u_5u_6y^*z^*}{\left(1+u_6{y^*}^2\right)^2} - u_7\right),\ k_{23} = -\frac{u_5y^*}{1+u_6{y^*}^2},$$
$$k_{31} = 0,\ k_{32} = \frac{u_9u_{10}z^*}{(u_{11}+y^*)^2},\ k_{33} = 0.$$

The characteristic equation of the $J(E_5)$, is given by:

$$\lambda^3 + K_1\lambda^2 + K_2\lambda + K_3 = 0, \tag{18b}$$

where

$K_1 = -(k_{11} + k_{22})$,
$K_2 = k_{11}\,k_{22} - k_{23}k_{32} - k_{21}k_{12}$,
$K_3 = k_{11}k_{23}k_{32}$,

while, $\Delta = K_1K_2 - K_3 = -(k_{11} + k_{22})[k_{11}\,k_{22} - k_{21}k_{12}]$.

It is easy to verify that, the conditions (17a)-(17c) guarantee that $K_1 > 0$, $K_3 > 0$, and $\Delta > 0$. Hence, according to the Routh-Hurtwitz criterion all the eigenvalues of $J(E_5)$ have negative real parts. Therefore, the coexistence equilibrium point is locally asymptotically stable.

4. Persistence

In this section, the persistence of the system (3), which means the survival of all the species for all the time, is investigated. Before that the dynamics in the interior of boundary plane are studied using Bendixson - Dulac criterion [17].
Obviously, the system (3) has two subsystems belong to xy −plane and xz −plane respectively. These subsystems can be written respectively as following.

$$\begin{aligned}\frac{dx}{dt} &= x\left[(1-x) - \frac{u_1 y}{1+u_2x^2} - u_3x^2\right] = \rho_1(x,y),\\ \frac{dy}{dt} &= y\left[\frac{u_4x}{1+u_2x^2} - u_7y - u_8\right] = \rho_2(x,y).\end{aligned} \tag{19a}$$

And

$$\begin{aligned}\frac{dx}{dt} &= x\,[(1-x) - u_3x^2] = \rho_3(x,z),\\ \frac{dz}{dt} &= z\left[u_9\left(z - \frac{u_{10}z}{u_{11}}\right) - u_{12}\,\mathrm{z}\right] = \rho_4(x,z).\end{aligned} \tag{19b}$$

Now, in order to investigate the existence of periodic dynamics in the $Int.{R_+}^2$ of the xy −plane, define the Dulac function as $H_1(x,y) = \frac{1}{xy}$ that satisfies $H(x,y) > 0$ and C^1 function. Moreover, straightforward computation gives that:

$H_1\,\rho_1 = \frac{1}{y}\left[1 - x - \frac{u_1y}{1+u_2x^2} - u_3x^2\right]$, $H_1\rho_2 = \frac{1}{x}\left[\frac{u_4x}{1+u_2x^2} - u_7y - u_8\right]$.

Hence

$$\Delta(x,y) = \frac{\partial(H_1\,\rho_1)}{\partial x} + \frac{\partial(H_1\,\rho_2)}{\partial y} = \frac{1}{y}\left[-1 + \frac{2u_1u_2\,xy}{(1+u_2x^2)^2} - 2u_3\,x\right] - \frac{u_7}{x}$$

Accordingly, $\Delta(x,y)$ does not identically zero and does not change sign in the $Int.{R_+}^2$ of the xy −plane under the following condition:

$$\frac{2u_1u_2\,x}{(1+u_2x^2)^2} < \frac{1+2u_3\,x}{y} + \frac{u_7}{x}. \tag{20}$$

Therefore, by using Bendixson - Dulac criterion, there is no closed curve lying in the $Int.{R_+}^2$ of the xy −plane for all the trajectories satisfying condition (20). Hence, according to the Poincare Bendixon theorem [17], the unique equilibrium point in the $Int.{R_+}^2$ of the xy −plane that given by $(\bar{x},\bar{y})$ will be a globally asymptotically stable whenever it is locally asymptotically stable.
Similar argument can be obtained regarding to the nonexistence of closed curve in the $Int.{R_+}^2$ of the xz −plane for the second subsystem (19b) using the Dulac function as follows $H_2(x,z) = \frac{1}{xz}$. It is observed that, there is no periodic dynamics in the $Int.{R_+}^2$ of the xz −plane provided that

$$u_9 \le \frac{u_9u_{10}+u_{11}u_{12}}{u_{11}}. \tag{21}$$

Indeed, the zooplankton-free equilibrium point will be globally asymptotically stable when the equality occurs in the condition (21).

Theorem 3: Assume that there are no periodic dynamics in the boundary planes. Then the system (3) is uniformly persistent provided that the following conditions hold

$$u_8 < \frac{u_4x_1}{1+u_2{x_1}^2}, \tag{22a}$$

$$(1+2u_3\,\bar{x}) < \frac{2u_1u_2\bar{x}\bar{y}}{(1+u_2\bar{x}^2)^2}, \tag{22b}$$

$$1 < u_2\bar{x}^2, \tag{22c}$$

$$u_5z_1 + u_8 < \frac{u_4x_1}{1+u_2{x_1}^2}, \tag{22d}$$

Proof. Consider the point P in the $Int.R_+^3$ and $O(P)$ is the orbit through P. Let the omega limit set of the $O(P)$ is given by $\Omega(P)$. Clearly, $\Omega(P)$ is bounded, due to the boundedness of the solution of the system (3). The proof will follow if we can prove that all the boundary equilibrium points do not belong to the $\Omega(P)$.

Now, assume that $E_0 \in \Omega(P)$, then since E_0 is a saddle point, then by Butler-McGhee lemma [18], P cannot be the only point in $\Omega(P)$, and hence there are at least one other point, say k_1, such that $k_1 \in w^s(E_0) \cap \Omega(P)$, where $w^s(E_0)$ is the stable manifold of E_0. Since $w^s(E_0)$ is the yz −plane (or at least $y-$ axis) and the entire orbit through k_1, say $O(k_1)$, is contained in $\Omega(P)$. Then, if k_1 is on either boundary axes of yz −plane, it is obtained that the positive specific axis that containing k_1 is contained in $\Omega(P)$ which is contradicting to its boundedness. Otherwise, k_1 belongs to the interior of yz −plane and since there is no equilibrium point in the interior of yz −plane, the orbit through k_1 that contained in $\Omega(P)$ must be unbounded which leads to contradiction too. Thus we obtain that $E_0 \notin \Omega(P)$.
Now, since the conditions (22a) guarantees that E_1 is a saddle point, whiles E_2 is already saddle point, and the conditions (22b) and (22c) guarantee that E_3 is a saddle point, however condition (22d) guarantees that E_4 be a saddle point. Hence using similar arguments as that used for the point E_0, it is observed that all the boundary equilibrium points do not belong to the $\Omega(P)$. Therefore, the proof is done.

5. Globally stability

In this section, the dynamics of the system (3) is further investigated with the help of Lyapunov function. The objective is to specify the basin of attraction for the locally asymptotically stable equilibrium points and the non-hyperbolic point.

Theorem 4: The first single-species equilibrium point $E_1 = (x_1, 0, 0)$ of system (3) is a global asymptotically stable if the following condition hold:

$$u_1 x_1 < u_8, \tag{23a}$$
$$u_9 < u_{12}. \tag{23b}$$

Proof. Consider the following scalar function $N_1 = (x - x_1 - x_1 \ln\frac{x}{x_1}) + y + z$.

It is clear that $N_1 : \mathbb{R}_+{}^3 \to \mathbb{R}$, so that $N_1(E_1) = 0$ and $E_1(x,y,z) > 0$ for all $\{(x,y,z) \in \mathbb{R}_+{}^3 : x > 0, y \geq 0, z \geq 0, (x,y,z) \neq E_1\}$. Hence, N_1 is a positive definite function. Now, by differential N_1 with respect to time and simplify the result it is obtained that

$$\begin{aligned}\frac{dN_1}{dt} = &-[1 + u_3(x + x_1)](x - x_1)^2 - \frac{xy(u_1 - u_4)}{1 + u_2 x^2} \\ &+ \frac{u_1 x_1 y}{1 + u_2 x^2} - \frac{u_5 z y}{1 + u_6 y^2} - u_7 y^2 - u_8 y \\ &+ u_9 z^2 - \frac{u_9 u_{10} z^2}{u_{11} + y} - u_{12} z^2.\end{aligned}$$

Hence,

$$\frac{dN_1}{dt} \leq -[1 + u_3(x + x_1)](x - x_1)^2 - (u_8 - u_1 x_1) y - (u_{12} - u_9) z^2.$$

Therefore, using the conditions (23a)-(23b) leads to $\frac{dN_1}{dt}$ is a negative definite. Accordingly, the function N_1 is strong Lyapunov function, hence the first single-species equilibrium point E_1 is globally asymptotically stable.

Theorem 5: The fish-free equilibrium point $E_3 = (\bar{x}, \bar{y}, 0)$ is an asymptotically stable in the interior of sub-region of $\mathbb{R}_+{}^3$, that satisfies the following conditions:

$$\frac{u_1 u_2 \bar{y}(x + \bar{x})}{(1 + u_2 x^2)(1 + u_2 \bar{x}^2)} < 1 + u_3(x + \bar{x}), \tag{24a}$$
$$u_9 < u_{12} + \frac{u_9 u_{10}}{u_{11} + y}, \tag{24b}$$
$$(\alpha_{12})^2 < 2\alpha_{11} u_7, \tag{24c}$$
$$(\alpha_{23})^2 < 2\alpha_{33} u_7, \tag{24d}$$

where all the symbols $\alpha_{ij}; i,j = 1,2,3$ are given in the proof.

Proof. Consider the scalar function $N_3 = \left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + \left(y - \bar{y} - \bar{y}\ln\frac{y}{\bar{y}}\right) + z$.

It is clear that $N_3: \mathbb{R}_+^3 \to \mathbb{R}$, so that $N_3(E_3) = 0$, and $N_3(x, y, z) > 0$ for all $\{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y > 0, z \geq 0, (x, y, z) \neq E_3\}$. Hence, the function N_3 is positive definite function.
Now by differentiate N_3 with respect to time, and then simplify the result, it is obtained that

$$\frac{dN_3}{dt} = -\alpha_{11}(x - \bar{x})^2 + \alpha_{12}(x - \bar{x})(y - \bar{y}) - \frac{u_7}{2}(y - \bar{y})^2$$
$$-\frac{u_7}{2}(y - \bar{y})^2 - \alpha_{23}\, z(y - \bar{y}) - \alpha_{33} z^2,$$

where $\alpha_{11} = 1 + u_3(x + \bar{x}) - \frac{u_1 u_2 \bar{y}\,(x+\bar{x})}{(1+u_2 x^2)(1+u_2\bar{x}^2)}$, $\alpha_{12} = \frac{u_4\,(1-u_2 x\bar{x}) - u_1(1+u_2\bar{x}^2)}{(1+u_2 x^2)(1+u_2\bar{x}^2)}$,
$\alpha_{23} = \frac{u_5}{1+u_6 y^2}$, and $\alpha_{33} = u_{12} - u_9 + \frac{u_9 u_{10}}{u_{11}+y}$.
Hence, using the above conditions (24a)-(24d) gives

$$\frac{dN_3}{dt} \leq -\left[\sqrt{\alpha_{11}}\,(x - \bar{x}) - \sqrt{\frac{u_7}{2}}\,(y - \bar{y})\right]^2 - \left[\sqrt{\frac{u_7}{2}}\,(y - \bar{y}) + \sqrt{\alpha_{33}}\, z\right]^2.$$

Clearly, $\frac{dN_3}{dt}$ is negative definite and hence N_3 is a strong Lyapunov function. Therefore, the fish-free equilibrium point if an asymptotically stable in the region that satisfies the above set of conditions.

Theorem 6: The zooplankton-free equilibrium point $E_4 = (x_1, 0, z_1)$ is an asymptotically stable in the interior of the sub-region of R_+^3, that satisfies the following conditions:

$$u_1 x_1 < u_8, \tag{25a}$$
$$\frac{u_9 u_{10} z_1}{u_{11}^2} < \frac{u_5}{1+u_6\left(\frac{u_4}{u_1}\tau_1\right)^2}, \tag{25b}$$
$$\sigma_{22}(z - z_1)^2 < \sigma_{11}(x - x_1)^2. \tag{25c}$$

where all the symbols σ_{11} and σ_{22} are given in the proof.

Proof. Consider the function $N_4 = (x - x_1 - x_1 \ln\frac{x}{x_1}) + y + (z - z_1 - z_1 \ln\frac{z}{z_1})$.
It is clear that $N_4: \mathbb{R}_+^3 \to \mathbb{R}$, so that $N_4(E_4) = 0$, and $N_4(x, y, z) > 0$ for all $\{(x, y, z) \in \mathbb{R}_+^3 : x > 0, y \geq 0, z > 0, (x, y, z) \neq E_4\}$. Hence, the function N_4 is positive definite function. Now by differentiate N_4 with respect to time, and then simplify the result, it is obtained that.

$$\frac{d\,N_4}{dt} \leq -[1 + u_3(x + x_1)](x - x_1)^2 - (u_8 - u_1 x_1)y$$
$$-\left[u_{12} - u_9 + \frac{u_9 u_{10}}{(u_{11}+y)}\right](z - z_1)^2$$
$$-\left[\frac{u_5}{1+u_6 y^2} - \frac{u_9 u_{10} z_1}{(u_{11}+y)u_{11}}\right] yz.$$

Obviously, the last term is negative under the condition (25b), hence it is obtained that:

$$\frac{d\,N_4}{dt} \leq -\sigma_{11}(x - x_1)^2 - (u_8 - u_1 x_1)y + \sigma_{22}(z - z_1)^2,$$

where $\sigma_{11} = [1 + u_3(x + x_1)] > 0$, and $\sigma_{22} = u_9 - \frac{u_9 u_{10}}{(u_{11}+y)} - u_{12}$. Clearly, $\sigma_{22} > 0$ under the existence condition (5). Therefore, the derivative $\frac{d\,N_4}{dt}$ is negative definite under the conditions (25a) and (25c). Hence N_4 is a strong Lyapunov function. Therefore, the zooplankton-free equilibrium point if an asymptotically stable in the sub-region that satisfies the above set of conditions.
asymptotically stable then it has a basin of attraction in the interior of R_+^3 that satisfies the conditions:

$$\frac{u_1 u_2 y^*}{1+u_2 x^{*2}} < u_3, \tag{26a}$$
$$\frac{u_5 u_6 z^*(y+y^*)}{1+u_6 y^{*2}} < u_7, \tag{26b}$$

$$u_9 < u_{12} + \frac{u_9 u_{10} y^*}{(u_{11}+y)(u_{11}+y^*)}, \tag{26c}$$
$$(\mu_{12})^2 < 2\mu_{11}\mu_{22}, \tag{26d}$$
$$(\mu_{23})^2 < 2\mu_{22}\mu_{33}, \tag{26e}$$
where all the symbols $\mu_{ij}; i,j = 1,2,3$ are given in the proof.

Proof. Consider the following scalar function

$$N_5 = \left(x - x^* - x^* \ln\frac{x}{x^*}\right) + \left(y - y^* - y^* \ln\frac{y}{y^*}\right) + \left(z - z^* - z^* \ln\frac{z}{z^*}\right).$$

It is clear that $N_5: \mathbb{R}_+{}^3 \to \mathbb{R}$, so that $N_5(E_5) = 0$, and $N_5(x,y,z) > 0$ for all $\{(x,y,z) \in \mathbb{R}_+{}^3 : x > 0, y > 0, z > 0, (x,y,z) \neq E_5\}$. Hence, the function N_5 is positive definite function. Now by differentiate N_5 with respect to time, and then simplify the result, it is obtained that:

$$\frac{dN_5}{dt} \leq -\mu_{11}(x - x^*)^2 - \mu_{12}(x - x^*)(y - y^*) - \mu_{22}\frac{(y-y^*)^2}{2}$$
$$-\mu_{22}\frac{(y-y^*)^2}{2} - \mu_{23}(z - z^*)(y - y^*) - \mu_{33}(z - z^*)^2,$$

where $\mu_{11} = 1 + \left(u_3 - \frac{u_1 u_2 y^*}{1+u_2 x^{*2}}\right)(x + x^*)$, $\mu_{22} = u_7 - \frac{u_5 u_6 z^*(y+y^*)}{1+u_6 y^{*2}}$,
$\mu_{12} = \frac{u_2 x^*}{1+u_2 x^{*2}}(u_1 x^* + u_4 x)$, $\mu_{22} = u_7 - \frac{u_5 u_6 z^*(y+y^*)}{1+u_6 y^{*2}}$,
$\mu_{33} = u_{12} - u_9 + \frac{u_9 u_{10} y^*}{(u_{11}+y)(u_{11}+y^*)}$, $\mu_{23} = \frac{u_5}{1+u_6 y^{*2}} + \frac{u_5 u_6 y^{*2}}{1+u_6 y^{*2}} - \frac{u_9 u_{10} z}{(u_{11}+y)(u_{11}+y^*)}$.

Hence, using the above conditions (26a)-(26e) , it gives

$$\frac{dN_5}{dt} \leq -\left[\sqrt{\mu_{11}}(x - x^*) + \sqrt{\frac{\mu_{22}}{2}}(y - y^*)\right]^2 - \left[\sqrt{\frac{\mu_{22}}{2}}(y - y^*) + \sqrt{\mu_{33}}(z - z^*)\right]^2.$$

Clearly, $\frac{dN_5}{dt}$ is negative definite and hence N_5 is a strong Lyapunov function in the sub-region of $\mathbb{R}_+{}^3$ that satisfy the conditions (26a)-(26e). Therefore, the coexistence equilibrium point is an asymptotically stable for any trajectory starting from a point in the region satisfies the above set of conditions.

6. Local bifurcation analysis

In this section, the sensitivity of the dynamical behavior near the locally asymptotically stable equilibrium points of the system (3), in which a specific parameter is varying, is investigated using the Sotomayor's theorem for the local bifurcation [17]. The necessary but not sufficient condition for the local bifurcation to occur is the existence of a non-hyperbolic equilibrium point. Therefore, the candidate bifurcation parameter is selected so that the equilibrium point will be non-hyperbolic at a specific value of that parameter.

Now, we rewrite the system (3) in the matrix form as follows :

$$\frac{dY}{dt} = F(Y),\ Y = (x,y,z)^T, \text{ and } F = (xf_1, yf_2, zf_3)^T. \tag{27}$$

Then the second derivative of F with respect to Y can be written as:

$$D^2F(Y,\theta)(V,V) = [b_{i1}]_{3\times 1}, \tag{28}$$

where $V = (v_1, v_2, v_3)^T$ any vector and θ is any parameter, with

$$b_{11} = \left[2 - \frac{2u_1 u_2{}^2 x^3 y - 6u_1 u_2 xy}{(1+u_2 x^2)^3} - 6\,u_3\,x\right] v_1{}^2 - 2\left[\frac{u_1 - u_1 u_2 x^2}{(1+u_2 x^2)^2}\right] v_1 v_2,$$
$$b_{21} = \left[\frac{2u_2{}^2 u_4 x^3 y - 6u_2 u_4 xy}{(1+u_2 x^2)^3}\right] v_1{}^2 + 2\left[\frac{u_4 - u_2 u_4 x^2}{(1+u_2 x^2)^2}\right] v_1 v_2$$
$$-\left[\frac{2u_5 u_6{}^2 y^3 z - 6u_5 u_6 yz}{(1+u_6 y^2)^3} + 2u_7\right] v_2{}^2 - 2\left[\frac{u_5 - u_5 u_6 y^2}{(1+u_6 y^2)^2}\right] v_2 v_3,$$
$$b_{31} = \left[-\frac{2u_9 u_{10} z^2}{(u_{11}+y)^3}\right] v_2{}^2 + \left[4\frac{u_9 u_{10} z}{(u_{11}+y)^2}\right] v_2 v_3 + \left[2u_9 - 2\frac{u_9 u_{10}}{u_{11}+y} - 2u_{12}\right] v_3{}^2.$$

Theorem 8. Assume that the conditions (17b) and (17c) hold, Then the system (3) undergoes a saddle node bifurcation near the coexistence equilibrium point as the parameter u_3 passes through the value $u_3{}^* = \frac{u_1 u_2 y^*}{(1+u_2 x^{*2})^2} - \frac{1}{2x^*}$, provided that

$$\frac{8u_1 u_2{}^2 x^{*3} y^*}{(1+u_2 x^{*2})^3} \neq 5. \tag{29}$$

Proof. According to the form of the determinant of the Jacobian matrix of the system (3) at E_5 that is given by $K_3 = k_{11}k_{23}k_{32}$, in equation (15b). It is easy to verify that $K_3 = 0$, when $u_3 = u_3{}^*$. Therefore, the characteristic equation of the $J(E_5)$ that is given in (15b) has a zero eigenvalue, and hence the coexistence equilibrium point becomes a non-hyperbolic point and the Jacobian matrix at $(E_5, u_3{}^*)$ can be written as:

$J^* = [k_{ij}]_{3\times 3}$,

where all elements of J^* are the same that are given in equation (15a) except $k_{11}(u_3{}^*) = 0$.

Let the vector $V_1 = (v_{11}, v_{21}, v_{31})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_5{}^* = 0$ of the matrix J^*. Then, direct computation gives that $V_1 = \left(v_{11}, 0, \left(-\frac{k_{21}}{k_{23}}\right) v_{11}\right)^T$, where v_{11} be any non-zero real number.

Let the vector $V_2 = (v_{12}, v_{22}, v_{32})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_5{}^* = 0$ of the matrix J^{*T}. Then, direct computation gives that $V_2 = \left(v_{12}, 0, \left(-\frac{k_{12}}{k_{32}}\right) v_{12}\right)^T$, where v_{12} be any non-zero real number.

Now since $\frac{\partial F}{\partial u_3} = F_{u_3} = (-x^3 \ 0 \ 0\ )$. Then it is obtained that:

$$V_2{}^T F_{u_3}(E_5, u_3{}^*) = -x^{*3} v_{12} \neq 0.$$

Moreover, substituting the value of $(E_5, u_3{}^*)$ and V_1 in the equation (28) gives that:

$$D^2F(E_5, u_3{}^*)(V_1, V_1) = [b_{i1}{}^*]_{3\times 1}$$

where $b_{11}{}^* = \left[2 - \frac{2u_1 u_2{}^2 x^{*3} y^* - 6u_1 u_2 x^* y^*}{(1+u_2 x^{*2})^3} - 6\, u_3{}^* x^*\right] v_{11}{}^2 = \left[5 - \frac{8u_1 u_2{}^2 x^{*3} y^*}{(1+u_2 x^{*2})^3}\right] v_{11}{}^2$,

$b_{21}{}^* = \left[\frac{2u_2{}^2 u_4 x^{*3} y^* - 6u_2 u_4 x^* y^*}{(1+u_2 x^{*2})^3}\right] v_{11}{}^2$,

$b_{31}{}^* = \left[2u_9 - 2\frac{u_9 u_{10}}{u_{11}+y^*} - 2u_{12}\right]\left(\left(-\frac{k_{21}}{k_{23}}\right) v_{11}\right)^2 = 0.$

Therefore, it is obtained that:

$$V_2{}^T D^2F(E_5, u_3{}^*)(V_1, V_1) = \left[5 - \frac{8u_1 u_2{}^2 x^{*3} y^*}{(1+u_2 x^{*2})^3}\right] v_{11}{}^2 v_{12}.$$

Clearly, $V_2{}^T D^2F(E_5, u_3{}^*)(V_1, V_1) \neq 0$ under the condition (29), and hence the system (3) undergoes a saddle node bifurcation in the sense of Sotomayor.

7. Numerical simulation

In this section, the global dynamics of the system (3) is studied numerically. The system (3) is numerically solved using four step Predictor-Corrector methods for different sets of parameters and different sets of initial conditions. The objective is to complete the vision of the dynamic behavior of the system (3) especially when the parameter values are varying. It is observed that, for the following set of hypothetical parameter values, the trajectory of the system (3) approaches asymptotically to the coexistence equilibrium point, starting from different initial conditions. This is shown in figure (1).

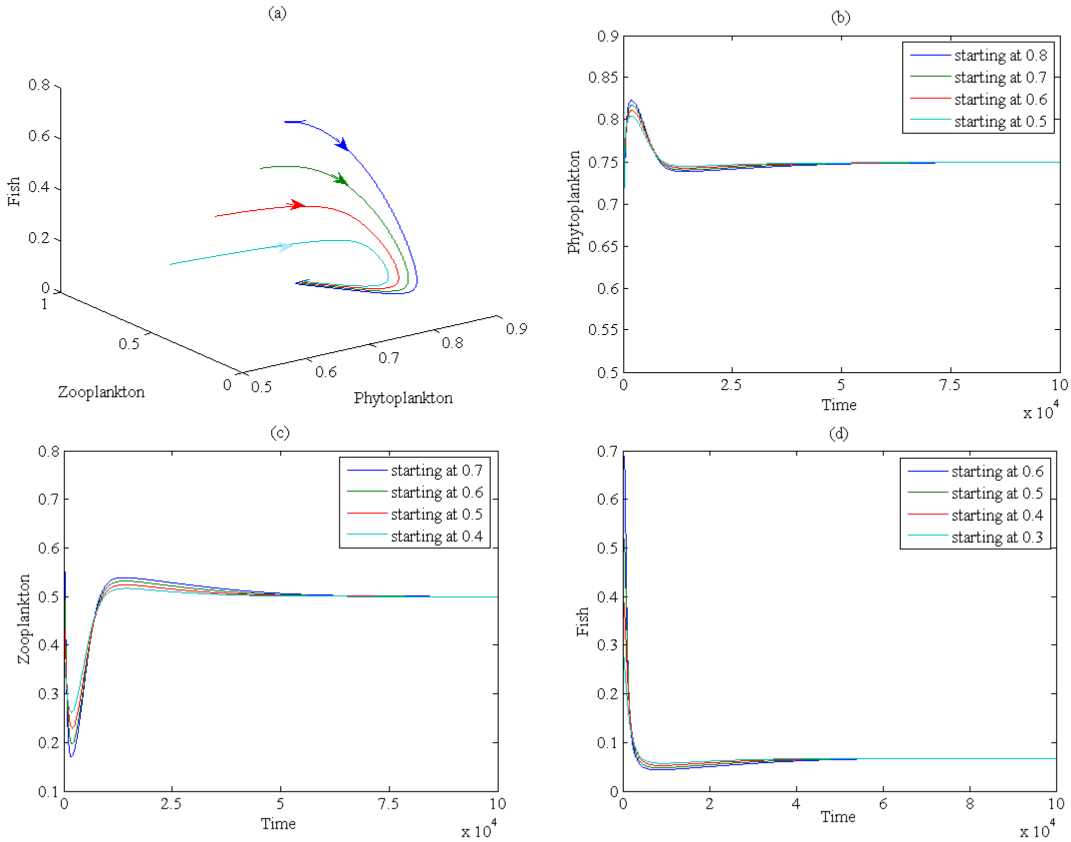

Figure 1-The trajectory of the system (3) approaches asymptotically to a global stable coexistence equilibrium point $E_5 = (0.74, 0.5, 0.06)$ for the data given by (30) starting from different points. (a) 3D attractor. (b) Trajectories of Phytoplankton versus time. (c) Trajectories of Zooplankton versus time. (d) Trajectories of Fish versus time.

$$\begin{gathered} u_1 = 0.9\,, u_2 = 4, u_3 = 0.2, u_4 = 0.6, u_5 = 0.5, u_6 = 6, u_7 = 0.15, \\ u_8 = 0.05, u_9 = 0.6, u_{10} = 0.5, u_{11} = 0.1,\; u_{12} = 0.1. \end{gathered} \tag{30}$$

However, for the following hypothetical set of parameters in which the toxicity efficiency is decreasing throughout the food chain levels, the system (3) undergoes a chaotic attractor. This is shown in figure (2).

$$\begin{gathered} u_1 = 0.9\,, u_2 = 4, u_3 = 0.1, u_4 = 0.6, u_5 = 0.5, u_6 = 6, u_7 = 0.05, \\ u_8 = 0.08, u_9 = 0.6, u_{10} = 0.5, u_{11} = 0.1,\; u_{12} = 0.01. \end{gathered} \tag{31}$$

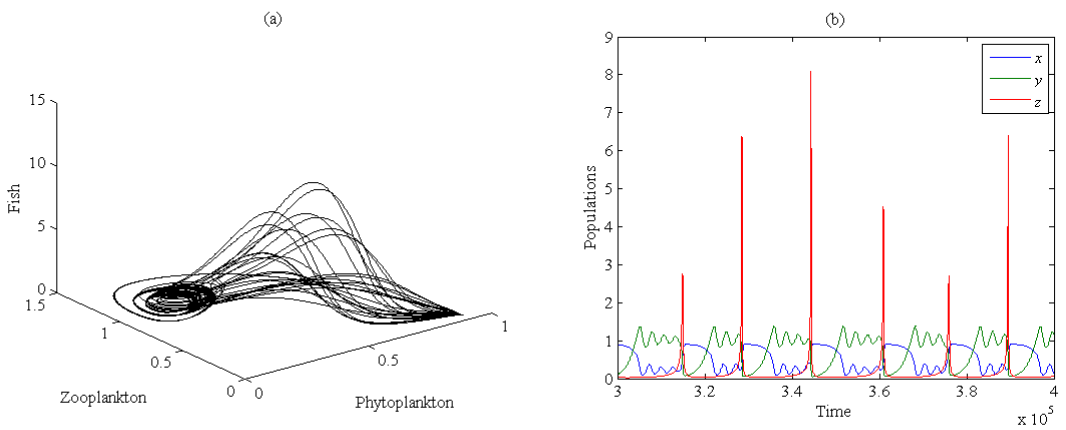

Figure 2-(a) The trajectory of the system (3) approaches a chaotic attractor for the data given by (31) with $u_8 = 0.05$. (b) Trajectories of Populations versus time.

According to the chaotic attractor in figure (2), the system (3) is sensitive to varying in the parameters therefore in the following the bifurcation diagrams as a function of some parameters are drawn in order to specify those parameters how have vital effects on the dynamical behavior of the system (3).

It is observed that, the system (3) is sensitive to varying in the u_1 so that the system approaches to different attractors as varying including periodic, chaotic, and then return to asymptotic stable point, see the bifurcation diagram given in figure (3). However, figure (4) shows clearly the rout to chaos through periodic and then periodic doubling after that the system (3) approaches to chaotic. Finally, figure (5) shows that the as the parameter u_1 increases the system approaches to the coexistence equilibrium point and then the extinction in the Fish population occurs and the system approaches to the fish-free equilibrium point.

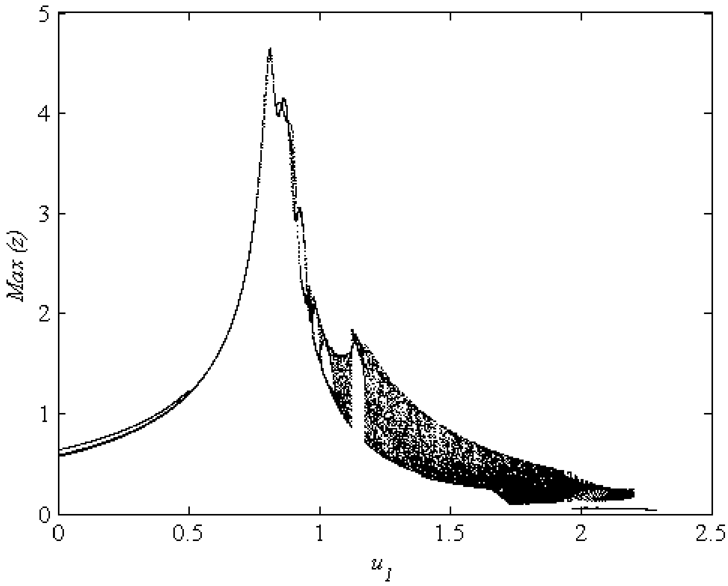

Figure 3- Bifurcation diagram of the system (3) using data (31) in which the maximum value of z is drawn as a function of u_1.

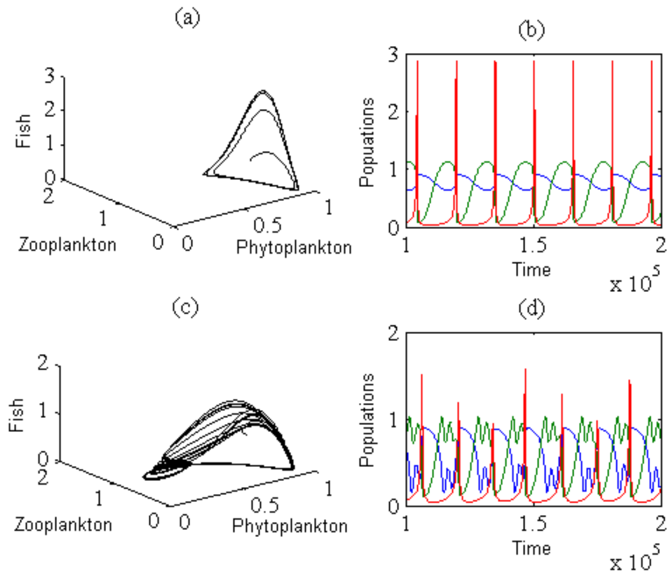

Figure 4-Transition of the trajectory from periodic to chaotic using data (31). (a) Periodic attractor when $u_1 = 0.75$. (b) Trajectories of Populations versus time for Fig. (4a). (c) Chaotic attractor when $u_1 = 1.1$. (d) Trajectories of Populations versus time for Fig. (4c).

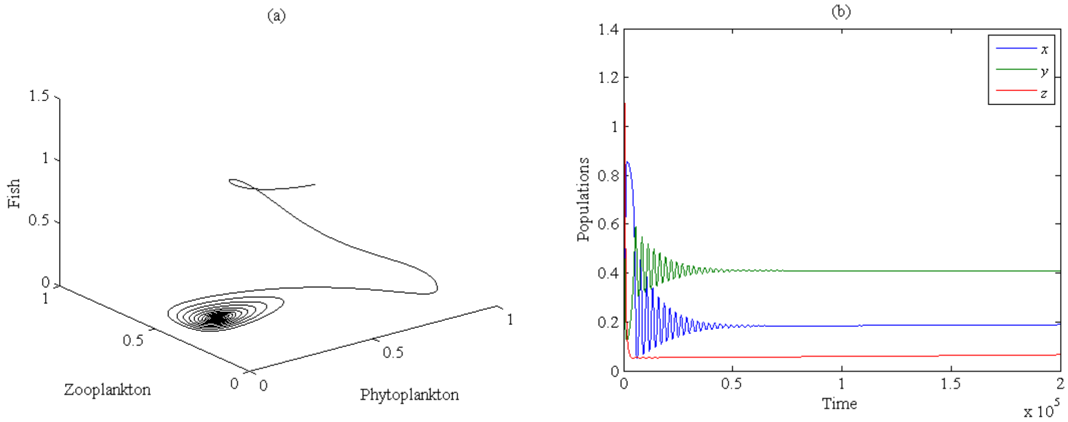

Figure 5-(a) The system (3) approaches to asymptotically stable coexistence point when $u_1 = 2.25$ with rest of data as in (31). (b) Trajectories of Populations versus time for Fig. (5a).

Now, the bifurcation diagram of the system (3) as a function of varying the parameter u_2 is drawn in figure (6). However, figure (7) shows for the typical values of u_2 within the range of bifurcation diagram given in figure (6) the existence of exchange between the chaotic and periodic dynamics as the parameter varying. While increasing the parameter $u_2 > 6.4$ leads first to approach the system to coexistence equilibrium point and then extinction of Fish population.

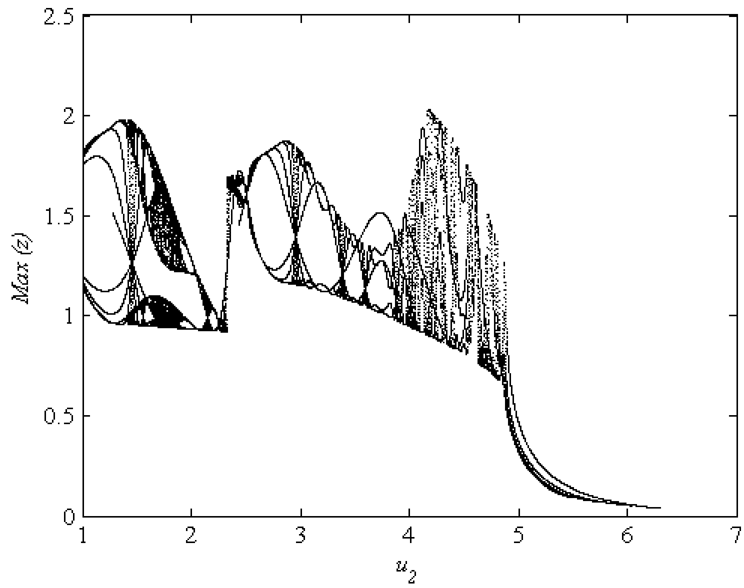

Figure 6- Bifurcation diagram of the system (3) using data (31) in which the maximum value of z is drawn as a function of u_2.

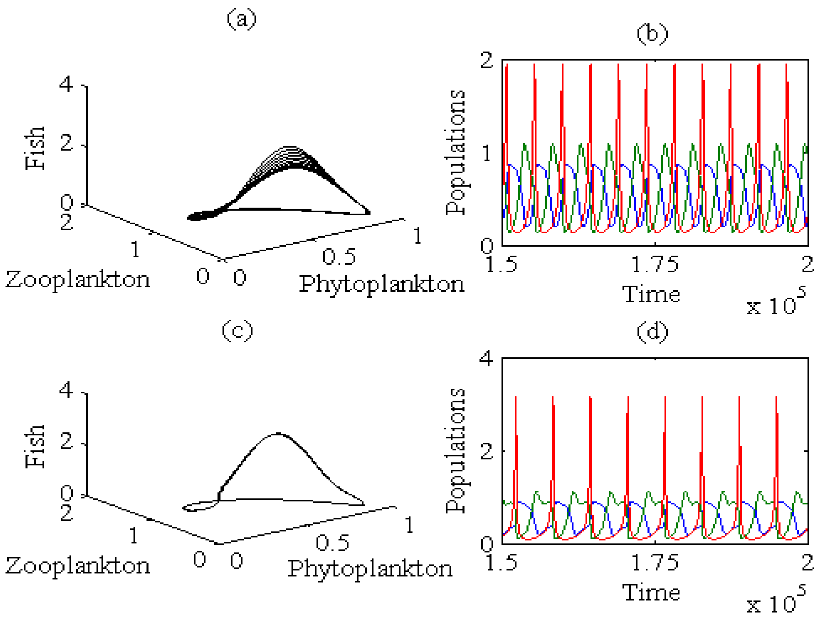

Figure 7- Transition of the trajectory from chaotic to periodic using data (31). (a) Chaotic attractor when $u_2 = 1.74$. (b) Trajectories of Populations versus time for Fig. (7a). (c) Periodic attractor when $u_2 = 2$. (d) Trajectories of Populations versus time for Fig. (7c).

Further investigation for the effect of system's parameters on the dynamical behavior of the system (3) is done using bifurcation diagrams as shown in the figures (8), figure (9), figure (10), and figure (11) for the varying the parameters u_3, u_6, u_7, and u_{12} , respectively.

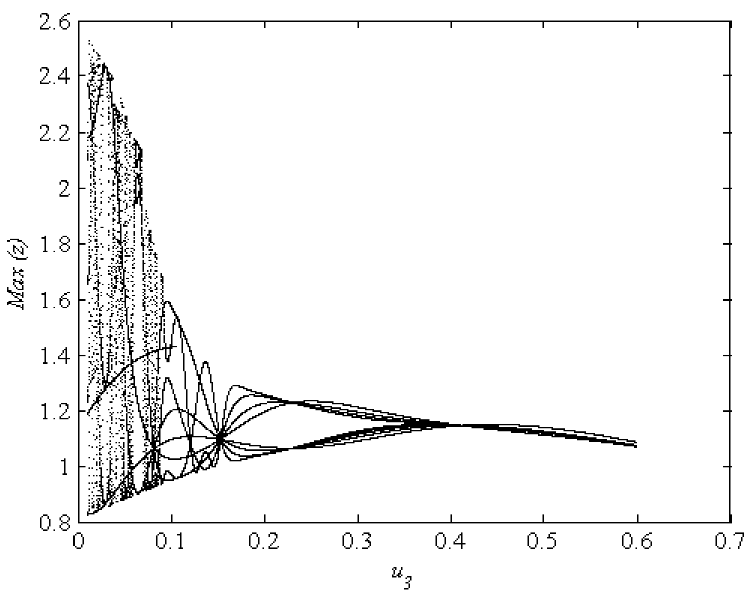

Figure 8- Bifurcation diagram of the system (3) using data (31) in which the maximum value of z is drawn as a function of u_3.

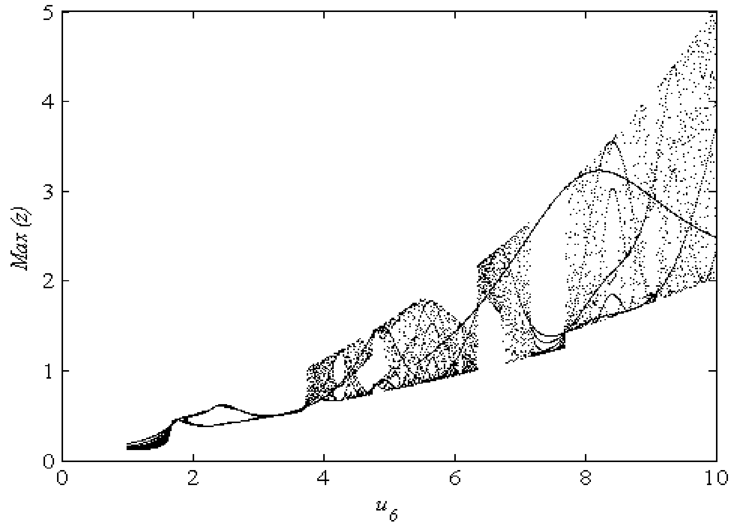

Figure 9-Bifurcation diagram of the system (3) using data (31) in which the maximum value of z is drawn as a function of u_6.

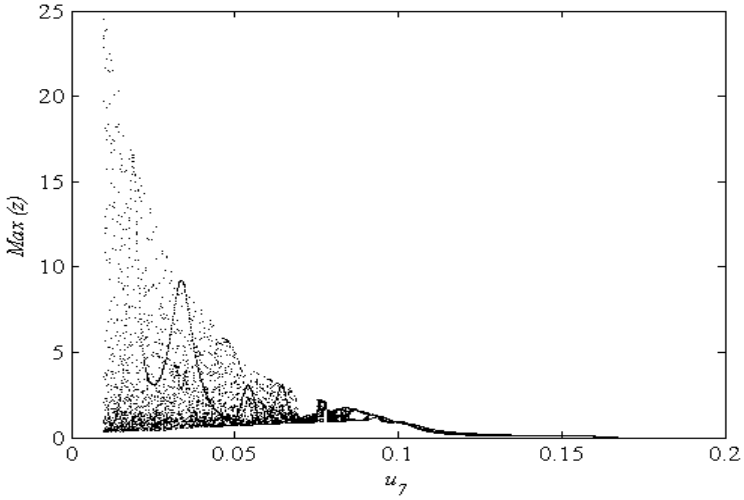

Figure 10- Bifurcation diagram of the system (3) using data (31) in which the maximum value of z is drawn as a function of u_7.

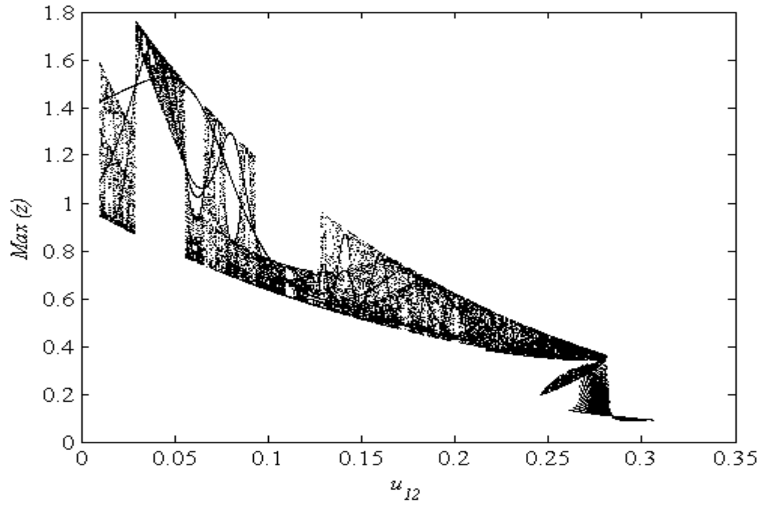

Figure 11-Bifurcation diagram of the system (3) using data (31) in which the maximum value of z is drawn as a function of u_{12}.

According to the bifurcation diagrams (8-11), it is observed that increasing the parameters u_3, u_7, and u_{12}, those stand for the toxicity efficiency of the Phytoplankton, Zooplankton, and Fish population respectively, reduces the chaotic and then leads to stabilizing the system (3) at the coexistence equilibrium point for the small range, after that the system faces extinction. On the other hand, from the bifurcation diagram given in figure (9), increasing the parameter u_6 that stands for defense efficiency of Zooplankton against Fish, makes the system more chaotic and the system still persistent. In the following, the varying of the other parameters of the system (3) is investigated by solving the system (3) numerically for the set of parameters given by equation (31) and then drawing the obtained trajectory at a typical value of these parameters to understand their effects on the dynamical behavior of the system (3), see figures (12), (13), (14), and (15) for the parameters u_4, u_8, u_9, and u_{10} respectively.

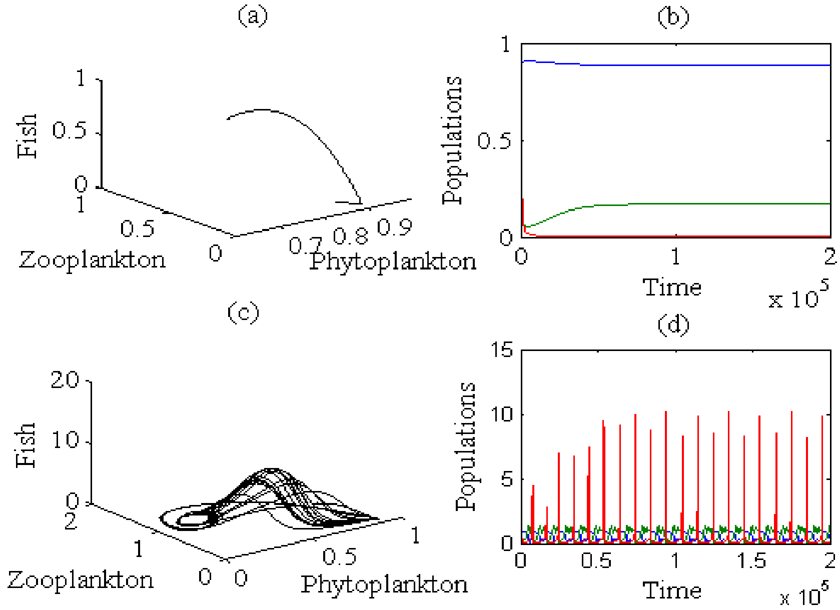

Figure 12-The trajectory of the system (3) for the data (31) with different values of u_4. (a) The system (3) approaches to fish-free equilibrium point when $u_4 = 0.3$. (b) Trajectories of the Populations versus time for Fig. (12a). (c) The system (3) approaches to chaotic attractor when $u_4 = 0.8$. (d) Trajectories of the Populations versus time for Fig. (12c).

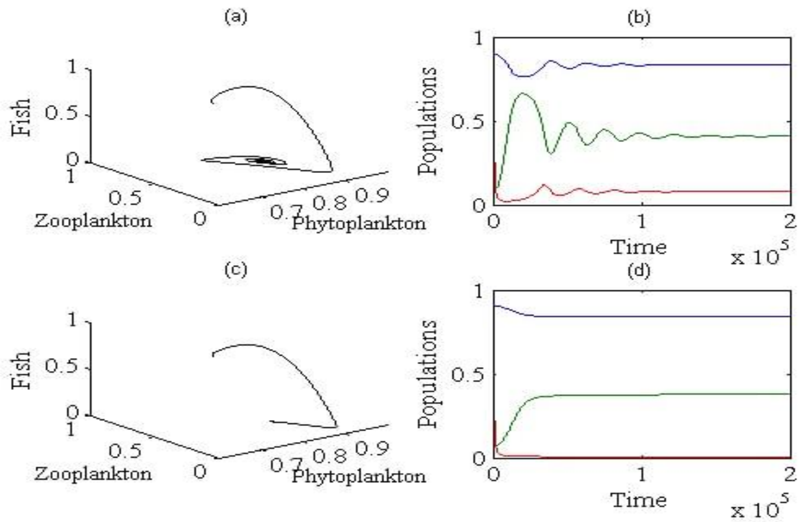

Figure 13- The trajectory of the system (3) for the data (31) with different values of u_8. (a) The system (3) approaches to coexistence equilibrium point when $u_8 = 0.08$. (b) Trajectories

of the Populations versus time for Fig. (13a). (c) The system (3) approaches to fish-free equilibrium point when $u_8 = 0.1$. (d) Trajectories of the Populations versus time for Fig. (13c).

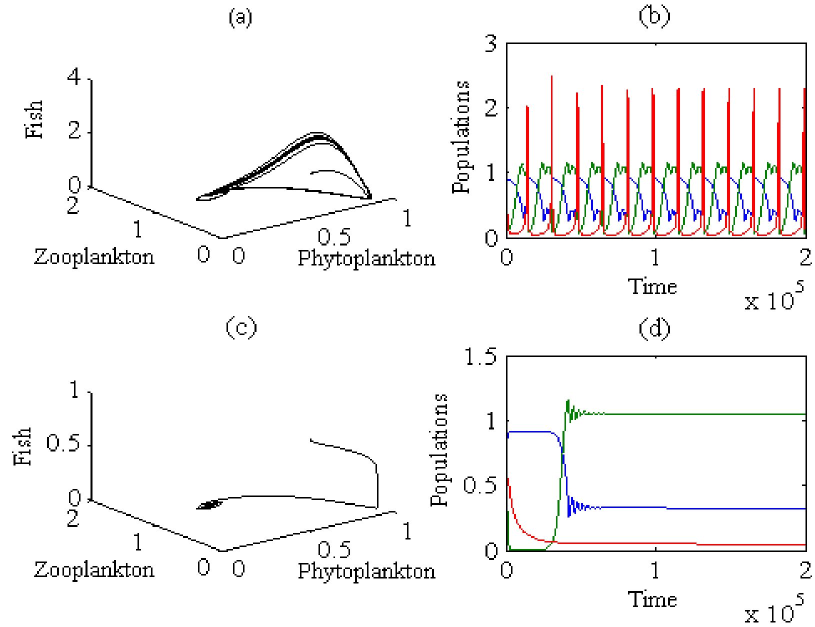

Figure 14-The trajectory of the system (3) for the data (31) with different values of u_9. (a) The system (3) approaches to chaotic attractor when $u_9 = 0.5$. (b) Trajectories of the Populations versus time for Fig. (14a). (c) The system (3) approaches to coexistence equilibrium point when $u_9 = 0.01$. (d) Trajectories of the Populations versus time for Fig. (14c).

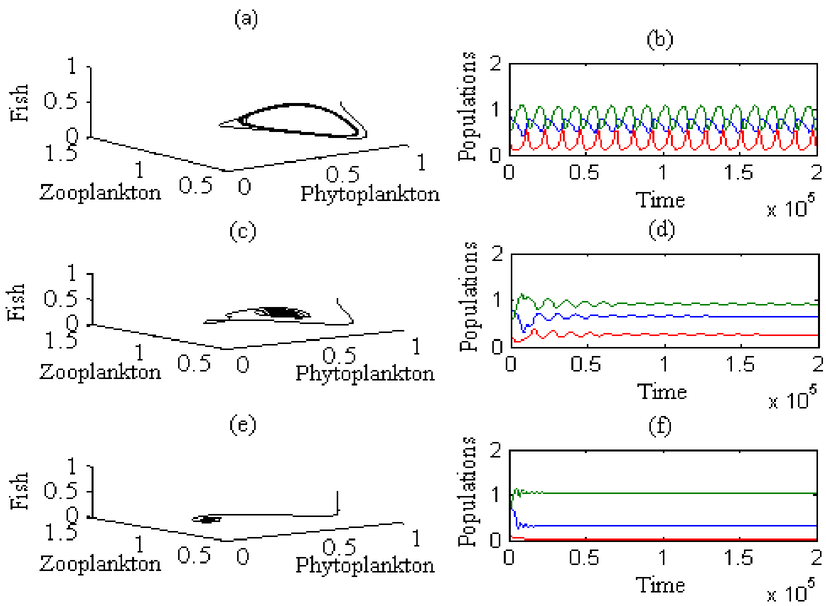

Figure 15- The trajectory of the system (3) for the data (31) with different values of u_{10}. (a) The system (3) approaches to periodic attractor when $u_{10} = 0.9$. (b) Trajectories of the Populations versus time for Fig. (15a). (c) The system (3) approaches to coexistence equilibrium point when $u_{10} = 1$. (d) Trajectories of the Populations versus time for Fig. (15c). (e) The system (3) approaches to fish-free equilibrium point when $u_{10} = 1.5$. (f) Trajectories of the Populations versus time for Fig. (15e).

According to the figures (12)-(15), it is observed that the dynamics of the system (3) are highly affected by varying parameters and different types of attractors can be obtained.
Finally the parameters u_5 and u_{11} have a quantitative effect on the dynamics of the system (3), but the behavior is still chaotic.

8. Conclusions

In this paper, an aquatic food chain model within a contaminated environment is suggested. Since the species at the first and second levels have a capability of group defense, the consumption of food through the predation process is considered as the Holling type-IV functional response. The pollution affects directly the phytoplankton individuals, while it affects indirectly the zooplankton and fish individuals. Finally, it is assumed that the fish at the upper level grow logistically and reproduce sexually using a modified Leslie–Gower type. It is observed that system (3) has at most six equilibrium points, some of them are the non-hyperbolic point. The conditions of local stability for hyperbolic equilibrium points are determined. The basin of attractions for each equilibrium point is specified using a suitable Lyapunov function. The persistence of the system (3) is also investigated. It is observed that the system (3) undergoes a saddle-node bifurcation near the coexistence equilibrium point too. Finally, with the help of numerical simulation, it is observed that there are rich dynamics in the proposed food chain model including periodic and chaos. Now, the obtained numerical simulation results are summarized as follows.

Although the system (3) has a globally asymptotically coexistence equilibrium point for different sets of parameters, it has rich dynamics as those parameter values vary including periodic and chaos, especially for low values of the toxicity efficiency. According to the bifurcation diagrams, system (3) approaches chaotic attractors through the cascade of periodic doubling. It is observed that decreasing the maximum consumption rate of the phytoplankton population leads to periodic dynamics in the interior of positive octant while increasing this parameter leads first to chaotic attractor for a specific range and then the system (3) stilled at coexistence equilibrium point before losing the persistence through extinction in the fish population. The varying of the defense efficiency of phytoplankton against zooplankton has a clear effect on the dynamics so that the system alternates their dynamics between periodic and chaotic for a large range while increasing it leads to persistent at a periodic dynamics in the positive octant. Moreover, the bifurcation diagrams as a function of the coefficients of toxicity efficiency show clearly that increasing the values of these parameters above specific values leads first to stabilizing the system (3) and then extinction in the fish population. However, increasing the defense efficiency of zooplankton against fish leads to destabilizing of the system (3) and the solution approaches chaotic dynamics.

On the other hand, it is observed that decreasing the maximum per capita growth rate of the zooplankton population below a specific value makes the system (3) face extinction in the Fish population and the solution approaches the Fish-free equilibrium point. However, system (3) is still chaotic otherwise. Also, increasing the natural death rate of the zooplankton population gradually makes the system (3) approaches asymptotically to a stable coexistence equilibrium point, then it faces extinction in the Fish population and the solution approaches the Fish-free equilibrium point. However, system (3) is still chaotic otherwise. On the other hand, it is observed that the behavior of the system (3) is transferred from chaotic to asymptotically stable at the coexistence equilibrium point as the growth rate of Fish by sexual reproduction decreases. Finally, increasing the parameter u_{10}, which stands for the inverse of the fish's preference rate of zooplankton, makes the solution of the system (3) transfer from chaotic to the periodic, asymptotically stable at the coexistence point and then extinction in the fish population and the solution approaches Fish-free equilibrium point.

References

[1] M. U. Belgrano, J. Dunne and R. E. Ulanowicz," Aquatic Food Webs: An Ecosystem Approach", Oxford University Press, USA. 2005.

[2] N. S. Raw, B. Tiwari, and P. Mishra, " Analysis of a plankton–fish model with external toxicity and nonlinear harvesting", Ricerche Di Matematica. , 2019,doi:10.1007/s11587-019-00478-4.

[3] V.E. Donk, A. Ianora, and M. Vos "Induced defences in marine and freshwater phytoplankton: a review ",Hydrobiologia. 668(1), pp3–19, 2011.

[4] S. Pal, S. Chatterjee, and J. Chattopadhyay,"Role of toxin and nutrient for the occurrence and termination of plankton bloomresults drawn from field observations and a mathematical mode"l, Biosystem. 90(1),pp87–100, 2007.

[5] S. Chakarborty, S. Roy, and J. Chattopadhyay,"Nutrient-limiting toxin producing and the dynamics of two phytoplankton in culture media: a mathematical model". Ecol Model. 213(2), pp.191–201. 2008.

[6] K. R. Naji and A. A. Aaid, "The dynamics of two harmful phytoplankton and herbivorous zooplankton system", Iraqi Journal of Science. 54(3), pp.676-695, 2013.

[7] K. Chakraborty, K. Das " Modeling and analysis of a twozooplankton one-phytoplankton system in the presence of toxicity", Appl Math Model. 39(3–4), pp.1241–1265, 2015.

[8] K. R, Upadhyay,J. Chattopadhyay , "Chaos to order: role of toxin producing phytoplankton in aquatic systems". Nonlinear Anal: Modell Control. 10(4), pp.383–396, 2005.

[9] B. Dubey, J. Hussain, "Modelling the interaction of two biological species in the polluted environment", J. Math. Anal. Appl. 246, pp. 58–79, 2000.

[10] J. R. Jang, J. Baglama, and J. Rick, " Nutrient–phytoplankton–zooplankton models with a toxin". Math. Comput. Model. 43, pp. 105–118, 2006.

[11] T. Saha, M. Bandopadhaya, "Dynamical analysis of toxin-producing phytoplankton–zooplankton interactions". Nonlinear Anal. Real World Appl. 10, pp.314–332, 2009.

[12] Q. Huang, L. Parshotham, and L. Wang, et al . "A model for the impact of contaminants on fish population dynamics", J. Theor. Biol. 334, pp.71–79, 2013.

[13] K. Chakraborty, K. Das, " Modeling and analysis of a two-zooplankton one-phytoplankton system in the presence of toxicity", App. Math. Model. 39,pp. 1241–1665, 2015

[14] Q. Huang, G. Seo, and C. Shan, " Bifurcations and global dynamics in a toxin-dependent aquatic population model", Math. Biol. 296, pp.26–35, 2018.

[15] N. S. Raw, and P. Mishra, "Modeling and analysis of inhibitory effect in plankton–fish model: Application to the hypertrophic Swarzedzkie Lake in Western Poland". Nonlinear Analysis: Real World Applications. 46, pp.465–492. doi:10.1016/j.nonrwa.2018.09.026, 2019.

[16] K. N. Thakur, and A, Ojha "Complex dynamics of delay-induced plankton–fish interaction exhibiting defense". SN Applied Sciences,2:1114, https://doi.org/10.1007/s42452-020-2860-7, 2020

[17] L. Perko , "Differential Equations and Dynamical System", Third edition, Springer- Verlag, New York, Inc.., 2001.

[18] H. Freedman and P. Waltman "Persistence in models of three interaction predator- prey populations", Mathematical Biosciences. 68:213-231, 1984.