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nC- symmetric operators

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Abstract:
In this paper, we present a concept of nC- symmetric operator as follows: Let A be a bounded linear operator on separable complex Hilbert space $\mathcal{H}$, the operator A is said to be nC-symmetric if there exists a positive number n (n> 1) such that $CA^n = A^{*^n} C$ ($A^n = C A^{*^n} C$). We provide an example and study the basic properties of this class of operators. Finally, we attempt to describe the relation between nC-symmetric operator and some other operators such as Fredholm and self-adjoint operators.

Keywords: Separable Complete Hilbert Space, Conjugation operators, C-symmetric operators, nC-symmetric operators.

لمؤثرات المتماثلة من النمط nC

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الخلاصه
في هذا البحث ، قدمنا مفهوم المؤثر المتناظر من النمط nC على أنهُ : المؤثر المُقيد الخطي المُعرف على فضاء هلبرت العقدي القابل للفصل $\mathcal{H}$ أذا وجد عدد صحيح n ‏(n > 1) بحيث أن $CA^n = A^{*^n}C$. كذلك تم اعطاء مثال عن هذا النوع من المؤثرات و قمنا بدراسة الخواص المهمة وكذلك حاولنا وصف العلاقة بين المؤثر المتناظر من النمط nC و بعض المؤثرات الأخرى.

1. Introduction and Preliminaries:

Let $\mathcal{H}$ be a separable complex Hilbert space and $B(\mathcal{H})$ be an algebra of all bounded linear operators on H . A conjugation on $\mathcal{H}$ is an antilinear operator $C: \mathcal{H} \to \mathcal{H}$ which is both involution ($C^2= I$) and isometric operator which satisfies $\langle Cx , Cy\rangle = \langle y, x\rangle$ for all $x,y \in \mathcal{H}$. An operator $A\in B(\mathcal{H})$ is said to be C-symmetric operator if $CA = A^* C$ ($A = CA^*C$); it is complex symmetric if A is C-symmetric with respect to some C [1]. In particular, an $n\times n$ matrix A is symmetric if and only if $A = CA^*C$ where C denotes the standard conjugation $C(z_1,z_2,\ldots,z_n)= (\bar{z}_1 , \bar{z}_2 , \ldots , \bar{z}_n)$. Thus, Complex symmetric operators generalize of concepts of symmetric matrices of linear algebra. In fact, if C is a conjugation on $\mathcal{H}$, then there exists an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all n [1, lemma1] and since $\langle Cx , Cy\rangle = \langle y, x\rangle$ for all $x,y \in H$, then the matrix of a C-symmetric operator A with respect to $\{e_n\}$ is symmetric:

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$[A]_{i,j}$ = <A e_n, e_j> = <CA*Ce_j, e_i> = <Ce_i, A*Ce_j> = <Ae_i, e_j> = $[A]_{j,i}$.
The converse of this fact is also true. That is, if there is an orthonormal basis such that A has a symmetric matrix representation, then A is complex symmetric [1]. The class of complex symmetric operators includes all normal operators, Toeplitz operators (including finite Toeplitz matrices and the compressed shift) and Volterra integration operator [1], [2], [3].
The study of complex symmetric operators has an interaction between the fields of operator theory and complex analysis. Recently, many authors have been interested in non-Hermitian quantum mechanics and the spectral analysis of certain complex symmetric operators [4], [5]. In particular, several authors have studied an antilinear operator which is the only type of a nonlinear operator that is important in quantum mechanics [6]. If C and J are conjugation on a Hilbert space $\mathcal{H}$, then U = CJ is a unitary operator. Moreover, U is both C-symmetric and J-symmetric [2].
In this paper, the concept of nC-symmetric operator is introduced. We also investigate the basic properties of this kind of operators like if A is nC-symmetric operator and A^{-1} exists then A^{-1} is also nC-symmetric. Moreover, If A is nC-symmetric and Fredholm operator ,then indA=0.

2. Main Results:

In this section, we present the concept of nC-symmetric operators. We also discuss the basic properties of this class of operators.

Definition 2.1. An operator A ∈ B($\mathcal{H}$) is said to be nC-symmetric operator if there exists a positive number n (n> 1) such that $CA^n = A^{*^n} C$ ($A^n = C A^{*^n} C$).

In some cases, an operator A is not C-symmetric, while the following example shows that A^n is a C-symmetric for some n:

Example 2.2. Let A: $\mathbb{C}^3 \to \mathbb{C}^3$ be an operator defined by the matrix

$$\begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}.$$

With xy ≠ 0 or |x| ≠ |y|. It follows from [7, Ex.1] that A is not C-symmetric operator.
However, $A^2 = \begin{bmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has rank one so that by [7, Corl. 5] A^2 is C-symmetric operator.
Hence, A is 2C-symmetric operator.

Proposition 2.3. Let A ∈ B($\mathcal{H}$), then A is nC-symmetric operator for a conjugation C if and only if there exists an orthonormal basis of $\mathcal{H}$ with respect to which A has a symmetric matrix representation.

Proof: If A is nC-symmetric operator that is $CA^n = A^{*^n}$ C and {e_n} orthonormal basis of $\mathcal{H}$ then:
$[A^n]_{i,j}$ = <A^n e_j, e_j> = <CA^{*^n} Ce_j, e_i> = <Ce_i, A^{*^n} Ce_j> = <A^n e_i, e_j> = $[A^n]_{j,i}$
Conversely, let {e_n} be an orthonormal basis of $\mathcal{H}$ such that A^n has a symmetric matrix representation such that <A^n e_i , e_j> = <A^n e_j , e_i>.
Now, define a conjugation C by C $(\sum_n a_n e_n) = \overline{a_n}\, e_n$ for $a_i \in \mathbb{C}$ and i=1,…,n.
Then, < C A^{*^n} C e_i, e_j> = <Ce_j, A^{*^n} Ce_i> = < e_j, A^{*^n} e_i> = <A^n e_j, e_i> = <A^n e_i, e_j>.
Hence, we obtain that $A^n = C A^{*^n} C$.

Proposition 2.4. Let A be n × n matrix of complex entries. If $A^n = CA^{*^n}C$ for some conjugation C on $\mathbb{C}^n$, then A is unitarily equivalent to a complex symmetric matrix.

Proof: Let A be n × n matrix of complex entries such that $A^n = CA^{*^n}C$ for some conjugation C on $\mathbb{C}^n$. Then there exists an orthonormal basis $\{ei\}_{i=1}^n$ such that Ce_i = e_i, for all i=1,2,..,n.
Let R= (e_1|e_2|…|e_n) be the unitary matrix where columns are these basis vectors.
Because of $[W]_{i,j}$ = <A^n e_j, e_i> = < CA^{*^n}C e_j, e_i> = < e_i, $A^{*^n}$$e_j$> = <$A^n$ e_i, e_j> = $[W]_{j,}$, we have the matrix W= R* A^nR is complex matrix.

Proposition 2.5. If A is nC-symmetric operator, then

1. A^m is also nC-symmetric operator for $m > n$.
2. $p(A^n)$ is C-symmetric for any polynomial p(z).

Proof: 1. Let A be nC-symmetric operator that is $CA^n = A^{*^n} C$ for $n > 1$.Hence,
$C(A^m)^n = C(A^n)^m = (A^{*^n})^m C = (A^m)^{*^n} C$.

2. $C\,p(A^n) = C\,[\,a_0 I + a_1 A^n + a_2 (A_n)^2 + \ldots + a_m (A^n)^m]$
$= \overline{a_0}\, C + \overline{a_1}\, A^{*^n} C + \overline{a_2}\, (A^*)^2 C + \ldots + \overline{a_m}\, (A^{*^n})^m C$
$= [\overline{a_0} + \overline{a_1}\, A^{*^n} + \overline{a_2}\, (A^*)^2 + \ldots + \overline{a_m}\, (A^{*^n})^m]\, C$
$= p(A^n)^* C$.

Proposition 2.6. If A is nC-symmetric operator, then

1. If A^{-1} exists, then A^{-1} is also nC-symmetric operator.

2. A^n is left invertible if and only if A^n is right invertible.

Proof: 1. To verify that A^{-1} is also nC-symmetric operator, we need only to show that $C\,(A^{-1})^n = (A^{-1})^{*^n} C$:
$C\,(A^{-1})^n = C\,(A^n)^{-1} = C\,(C\,A^{*^n} C)^{-1} = CC\,A^{-n^*} C = (A^{-1})^{*^n} C$.

2. Let A^n be left invertible operator, to show that A^n is right invertible, thus
$(A^n)^{-1} A^n = I = C^2$
$(A^n)^{-1} A^n C = C$
$C\,(A^n)^{-1} C\,A^{*^n} = CC$
$C\,(A^n)^{-1} CA^{*^n} = I$, then by (1) we obtain $(A^{*^n})^{-1} A^{*^n} = I$ .Therefore, $(A^n (A^n)^{-1})^* = I$. Hence, $A^n (A^n)^{-1} = I$.

Conversely, suppose that A^n is right invertible. To show that A^n is left invertible, we have $A^n\,(A^n)^{-1} = I$, it follows $(A^n\,(A^n)^{-1} = I)^*$, thus $(A^{*^n})^{-1} A^{*n} = I = C^2$.
Also, $C(A^{*^n})^{-1} A^{*^n} = C^3 = C$, then by (1) we obtain $(A^n)^{-1} C\,A^{*^n} = C$ and $(A^n)^{-1} C\,A^{*^n} C = C^2 = I$. Then, we get $(A^n)^{-1} A^n = I$.

Proposition 2.7. If A is nC-symmetric operator, then A^n is one to one if and only if Ran A^n is dense in $\mathcal{H}$.

Proof: Let A^n be one to one operator. Since C is isometric operator (hence one to one), then $C\,A^n C$ is also one to one. But, $A^{*^n} = C\,A^n C$ thus we obtain A^{*n} is also one to one ($\text{Ker } A^{*^n} = 0$).

Now, $\overline{\text{Ran } A^n} = (\text{Ker } A^{*^n})^{\perp} = \{0\}^{\perp} = H$. Hence, Ran A^n is dense in H.
Conversely, $\overline{\text{Ran } A^n} = (\text{Ker } A^{*^n})^{\perp} = H$, then $(\text{Ker } A^{*^n})^{\perp\perp} = H^{\perp} = 0$. Since $\text{Ker } A^{*^n}$ is a closed linear subspace of H, so $\text{Ker } A^{*^n} = 0$ and then we obtain A^{*^n} is one to one and so is $A^{*^n} C$. But, $CA^n = A^{*^n} C$ hence A^n is one to one.

Proposition 2.8. Let A be nC-symmetric operator. If A is Fredholm operator, then ind A = 0.

Proof: Since A is Fredholm operator, then A^n is Fredholm operator for nonnegative integer n , $\text{ind}A^n = n\,\text{ind}A$ and $\text{ind}(A^*) = -\,\text{ind}A$ ([8],[9]).

Now, we have Ker C = Ker C* = 0 and Ran C is closed linear subspace of H, thus C is Fredholm operator and indC = 0. We conclude the proof by showing that indA = 0.
Since $CA^n = A^{*^n} C$, then it follows that:
$\text{ind } CA^n = \text{ind } A^{*^n} C$
$\text{ind } C + n\,\text{ind } A = -n\,\text{ind } A + \text{ind } C$.
Then, we obtain $2n\,\text{ind}A = 0$ and thus indA = 0.

Proposition 2.9. Let A be nC-symmetric operator. If M is invariant subspace of $\mathcal{H}$ under C and A, then M reduces A^n.

Proof: Let M be an invariant subspace of $\mathcal{H}$ under C and A, so we have $C(M) \subset M$ and $A(M) \subset M$ so $A^n(M) \subset M$ for some $n > 1$ and hence $C\,A^n\,C \subset M$. But, $A^{*^n} = CA^n C$, it follows that $A^{*^n}(M) \subset M$ which implies that M is reduced to A^n.

Proposition 2.10. Let A be nC-symmetric operator. Then, M reduces A^n if and only if CM reduces A^n.

Proof: Let M be reduces A^n such that $A^n (M) \subset M$ and $A^{*^n} (M) \subset M$. A short computation reveals that $C A^n (M) = A^{*^n} C (M) \subset CM$ and $CA^{*^n} (M) = A^n C (M) \subset CM$, thus we obtain CM reduces A^n.
Conversely, if CM reduces A^n , then we have $A^n (CM) \subset CM$ and $A^{*n} (CM) \subset CM$. Then $C A^n (CM) = C A^n C (M) = A^{*n} (M) \subset C (CM) = M$. In a similar way, we can obtain $A^n (M) \subset M$.
Proposition 2.11. Let A be nC-symmetric operator. If M is an invariant subspace of $\mathcal{H}$ under C and P orthogonal projection on to M, then the compression $B^n = P A^n P$ of A to M which satisfies $CB^n = B^{*^n}C$.
Proof: Let A be nC-symmetric operator.
$C B^n = C (P A^n P) = C P A^n P = P C A^n P = P A^{*^n} C P = P A^{*^n} P C = B^{*^n} C$.
Proposition 2.12. If A is nC-symmetric operator, then we have the folloing:
1. $C A^n$ commutes with $A^{*^n} A^n$.
2. $A^n C$ commutes with $A^n A^{*^n}$.
Proof: 1. Since $(CA^n)^2 = CA^n CA^n = A^{*^n} CC A^n = A^{*^n}A^n$, then this would implies that:

$$CA^n A^{*^n} A^n = C (C A^{*^n} C) A^{*^n} A^n$$
$$= A^{*^n} C A^{*^n} A^n$$
$$= A^{*^n} A^n CA^n.$$

2.Similaraly, $(A^n C)^2 = A^n C A^n C = A^n A^{*^n} CC = A^n A^{*^n}$.
Now,

$$A^n C A^n A^{*^n} = A^n C (C A^{*^n} C) A^{*^n}$$
$$= A^n A^{*^n} C A^{*^n}$$
$$= A^n A^{*^n} A^n C.$$

Proposition 2.13. Let B be invertible and nC- symmetric operator on $\mathcal{H}$ such that AB=BA, then A is nC-symmetric operator.
Proof: The proof is based on the equations $CA^n = A^{*^n} C$, $CB^n= B^{*^n} C$ and AB =BA. Hence,

$$C (A B)^n = C A^n B^n$$
$$= A^{*^n} C B^n$$
$$= A^{*^n} B^{*^n} C$$
$$= (B^n A^n)^* C$$
$$= (A^n B^n)^* C$$
$$= (A B)^{*^n} C.$$

Conversely, let AB be nC-symmetric operator . To show that A is nC-symmetric operator, we set

$$C A^n = C A^n I$$
$$= C A^n B^n (B^{-1})^n$$
$$= C (AB)^n (B^{-1})^n \qquad \text{(AB is nC-symmetric)}$$
$$= (AB)^{*^n} C (B^{-1})^n \qquad \text{(}B^{-1}\text{ is nC-symmetric by Prop. (2.6 (1)).}$$
$$= (AB)^{*^n} (B^{-1})^{*^n} C$$
$$= (AB)^{*^n} (B^{*^n})^{-1} C$$
$$= A^{*^n} B^{*^n} (B^{*^n})^{-1} C$$
$$= A^{*^n} C.$$

Proposition 2.14. Let $A \in B(\mathcal{H})$ and C be conjugation on $\mathcal{H}$. If $CA^n = A^nC$, then A is nC-symmetric operator if and only if A^n is self-adjoint.
Proof: Let A be nC-symmetric operator such that $CA^n = A^{*^n} C$, then we have $A^n C = A^{*^n} C$ which yields that $A^n CC = A^{*^n} CC$ and so we get $A^n = (A^n)^*$.
Conversely, suppose that A^n is self- adjoint such that $A^n = (A^n)^*$, then it follows $A^n C = A^{*^n} C$ and $CA^n = A^{*^n} C$ which implies that A is nC-symmetric operator.
Proposition 2. 15. If A is both nC-symmetric and nJ-symmetric, then A is both n(CJC)-symmetric and n(JCJ)- symmetric operator.
Proof: Since $CA^n = A^{*^n} C$ and $J A^n = A^{*^n} J$, then

(CJC) A^n = CJ (CA^n) = CJ A^{*^n} C = CA^n JC = A^{*^n} (CJC) , so that A is n(CJC)-symmetric operator. Analogously, we can prove that A is n(JCJ)- symmetric .

Proposition 2. 16. If A is both nC-symmetric and nJ-symmetric, then A^n U is C-symmetric where U = CJ is unitary operator.

Proof: Since A is both nC-symmetric and nJ-symmetric then by the previous proposition, A is n(CJC)-symmetric, so we have

$(A^n U) C = A^n (CJC) = (CJC) A^{*^n} = C U^* A^{*^n} = C (A^n U)^*$.

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