nC- symmetric operators

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Abstract:
In this paper, we present a concept of nC- symmetric operator as follows: Let $A$ be a bounded linear operator on separable complex Hilbert space $\mathcal{H}$, the operator $A$ is said to be nC-symmetric if there exists a positive number $n$ such that $CA^n = A^* C (A^n = C A^* C)$. We provide an example and study the basic properties of this class of operators. Finally, we attempt to describe the relation between nC-symmetric operator and some other operators such as Fredholm and self-adjoint operators.

Keywords: Separable Complete Hilbert Space, Conjugation operators, C-symmetric operators, nC-symmetric operators.

1. Introduction and Preliminaries:
Let $\mathcal{H}$ be a separable complex Hilbert space and $B(\mathcal{H})$ be an algebra of all bounded linear operators on $\mathcal{H}$. A conjugation on $\mathcal{H}$ is an antilinear operator $C: \mathcal{H} \to \mathcal{H}$ which is both involution ($C^2 = I$) and isometric operator which satisfies $<Cx , Cy> = <y , x>$ for all $x,y \in \mathcal{H}$. An operator $A \in B(\mathcal{H})$ is said to be C-symmetric operator if $CA = A^* C (A = CA^* C)$; it is complex symmetric if $A$ is C-symmetric with respect to some $C$ [1]. In particular, an $n \times n$ matrix $A$ is symmetric if and only if $A = CA^* C$ where $C$ denotes the standard conjugation $C(z_1,z_2,\ldots,z_n) = (\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_n)$. Thus, Complex symmetric operators generalize of concepts of symmetric matrices of linear algebra. In fact, if $C$ is a conjugation on $\mathcal{H}$, then there exists an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all $n$ [1, lemma1] and since $<Cx , Cy> = <y , x>$ for all $x,y \in \mathcal{H}$, then the matrix of a C-symmetric operator $A$ with respect to $\{e_n\}$ is symmetric.

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The converse of this fact is also true. That is, if there is an orthonormal basis such that $A$ has a symmetric matrix representation, then $A$ is complex symmetric [1]. The class of complex symmetric operators includes all normal operators, Toeplitz operators (including finite Toeplitz matrices and the compressed shift) and Volterra integration operator [1], [2], [3].

The study of complex symmetric operators has an interaction between the fields of operator theory and complex analysis. Recently, many authors have been interested in non-Hermitian quantum mechanics and the spectral analysis of certain complex symmetric operators [4], [5]. In particular, several authors have studied an antilinear operator which is the only type of a nonlinear operator that is important in quantum mechanics [6]. If $C$ and $J$ are conjugation on a Hilbert space $\mathcal{H}$, then $U = CJ$ is a unitary operator. Moreover, $U$ is both $C$-symmetric and $J$-symmetric [2].

In this paper, the concept of nC-symmetric operator is introduced. We also investigate the basic properties of this kind of operators like if $A$ is nC-symmetric operator and $A^{-1}$ exists then $A^{-1}$ is also nC-symmetric. Moreover, if $A$ is nC-symmetric and Fredholm operator, then $\text{ind}A = 0$.

2. Main Results:

In this section, we present the concept of nC-symmetric operators. We also discuss the basic properties of this class of operators.

**Definition 2.1.** An operator $A \in B(\mathcal{H})$ is said to be nC-symmetric operator if there exists a positive number $n$ (n > 1) such that $CA^a = A^*C(A^a = CA^*)$.

In some cases, an operator $A$ is not C-symmetric, while the following example shows that $A^n$ is a C-symmetric for some $n$:

**Example 2.2.** Let $A: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ be an operator defined by the matrix

$$
\begin{bmatrix}
0 & x & 0 \\
0 & 0 & y \\
0 & 0 & 0
\end{bmatrix}
$$

With $xy \neq 0$ or $|x| \neq |y|$. It follows from [7, Ex.1] that $A$ is not C-symmetric operator.

However, $A^2 = \begin{bmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has rank one so that by [7, Corl. 5] $A^2$ is C-symmetric operator.

Hence, $A$ is 2C-symmetric operator.

**Proposition 2.3.** Let $A \in B(\mathcal{H})$, then $A$ is nC-symmetric operator for a conjugation $C$ if and only if there exists an orthonormal basis of $\mathcal{H}$ with respect to which $A$ has a symmetric matrix representation.

**Proof:** If $A$ is nC-symmetric operator that is $CA^n = A^{*n}C$ and $\{e_n\}$ orthonormal basis of $\mathcal{H}$ then:

$[A^n]_{ij} = <a^n e_i, e_j> = <CA^{*n}Ce_i, e_j> = <Ce_i, A^{*n}Ce_j> = <A^n e_i, e_j> = [A^n]_{ji}$

Conversely, let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$ such that $A^n$ has a symmetric matrix representation such that $[A^n]_{ij} = [A^n]_{ji}$.

Now, define a conjugation $C$ by $C(\sum_{n}a_ne_n) = \overline{a_n}e_n$ for $a_n \in \mathbb{C}$ and $i = 1,\ldots,n$.

Then, $<CA^{*n}Ce_j, e_i> = <Ce_j, A^{*n}Ce_i> = <e_j, A^{*n}e_i> = <A^n e_i, e_j> = <A^n e_i, e_j>$.

Hence, we obtain that $A^n = CA^{*n}C$.

**Proposition 2.4.** Let $A$ be $n \times n$ matrix of complex entries. If $A^n = CA^{*n}C$ for some conjugation $C$ on $\mathbb{C}^n$, then $A$ is unitarily equivalent to a complex symmetric matrix.

**Proof:** Let $A$ be $n \times n$ matrix of complex entries such that $A^n = CA^{*n}C$ for some conjugation $C$ on $\mathbb{C}^n$. Then there exists an orthonormal basis $\{ei\}_{i=1}^n$ such that $Ce_i = e_i$, for all $i=1,2,\ldots,n$.

Let $R = (e_1 | e_2 | \ldots | e_n)$ be the unitary matrix where columns are these basis vectors.

Because of $[W]_{ij} = <A^n e_j, e_i> = <CA^{*n}Ce_j, e_i> = <e_i, A^{*n}e_j> = <A^n e_i, e_j> = [W]_{ji}$, we have the matrix $W = R^*A^nR$ is complex matrix.
Proposition 2.5. If $A$ is nC-symmetric operator, then
1. $A^m$ is also nC-symmetric operator for $m > n$.
2. $p(A^n)$ is C-symmetric for any polynomial $p(z)$.

Proof: 1. Let $A$ be nC-symmetric operator that is $CA^n = A^{n^*}C$ for $n > 1$. Hence, $C(A^n)^m = C(A)^{nm} = (A^{n^*})^m C = (A^m)^* C$.
2. $C p(A^n) = C \left[ a_0 I + a_1 A^n + a_2 (A^n)^2 + \ldots + a_m (A^n)^m \right]$
   $= \bar{a}_0 C + \bar{a}_1 A^{n^*} C + \bar{a}_2 (A^{n^*})^2 C + \ldots + \bar{a}_m (A^{n^*})^m C$
   $= [\bar{a}_0 + \bar{a}_1 A^{n^*} + \bar{a}_2 (A^{n^*})^2 + \ldots + \bar{a}_m (A^{n^*})^m] C
   = p(A^n)^* C$.

Proposition 2.6. If $A$ is nC-symmetric operator, then
1. If $A^{-1}$ exists, then $A^{-1}$ is also nC-symmetric operator.
2. $A^n$ is left invertible if and only if $A^{-1}$ is right invertible.

Proof: 1. To verify that $A^{-1}$ is also nC-symmetric operator, we need only to show that $A(A^{-1})^n = (A^{-1})^{n^*} C$.
2. Let $A^n$ be left invertible operator, to show that $A^n$ is right invertible, thus
   $(A^{-1})^n A^n = I = C^2$
   $(A^{-1})^n A^{n^*} C = C$
   $(A^{-1})^n A^{n^*} = C C^{-1} = I$
   Conversely, suppose that $A^{-1}$ is right invertible. To show that $A^{-1}$ is left invertible, we have
   $A^n (A^{-1})^n = I$, it follows $(A^n (A^{-1})^n = I)$, thus $(A^n)^{-1} A^{-1} n = I = C^2$.
   Also, $C(A^{n^*})^{-1} A^{n^*} C = C I = C$, then by (1) we obtain $(A^{n^*})^{-1} C A^{n^*} = C$ and $(A^{n^*})^{-1} C A^{n^*} C = C^2 = I$. Then, we get $(A^{-1})^{-1} A^{-1} = I$.

Proposition 2.7. If $A$ is nC-symmetric operator, then $A^n$ is one to one if and only if $\text{Ran} A^n$ is dense in $H$.

Proof: Let $A^n$ be one to one operator. Since $C$ is isometric operator (hence one to one), then $C A^n C$ is also one to one. But, $A A^n C A^n C$ thus we obtain $A A^n$ is also one to one ($\text{Ker} A A^n = 0$).

Now, $\text{Ran} A^n = (\text{Ker} A A^n)^\perp = \{0\}^\perp = H$. Hence, $A A^n$ is dense in $H$.

Conversely, $\text{Ran} A^n = (\text{Ker} A A^n)^\perp = H$, then $(\text{Ker} A A^n)^\perp H^\perp = H^\perp = 0$.

Since $A A^n$ is a closed linear subspace of $H$, $\text{Ker} A A^n = 0$ and then we obtain $A A^n$ is one to one and so is $A A^n C$.

But, $C A^n = A A^n C$ hence $A^n$ is one to one.

Proposition 2.8. Let $A$ be nC-symmetric operator. If $A$ is Fredholm operator, then $\text{ind} A = 0$.

Proof: Since $A$ is Fredholm operator, then $A^n$ is Fredholm operator for nonnegative integer $n$, $\text{ind} A^n = n \text{ind} A$ and $\text{ind} (A^n) = - \text{ind} A$ ([8],[9]).

Now, we have $\text{Ker} C = \text{Ker} A A^n = 0$ and $\text{Ran} C$ is closed linear subspace of $H$, $\text{Ker} A A^n$ is closed linear subspace of $H$, $\text{Ker} A A^n = 0$ and then we obtain $\text{ind} A = 0$.

Since $C A^n A^n C$, then it follows that:

\begin{align*}
\text{ind} C A^n &= \text{ind} A A^n C \\
\text{ind} C + n \text{ind} A &= -n \text{ind} A + \text{ind} C.
\end{align*}

Then, we obtain $2n \text{ind} A = 0$ and thus $\text{ind} A = 0$.

Proposition 2.9. Let $A$ be nC-symmetric operator. If $M$ is invariant subspace of $H$ under $C$ and $A$, then $M$ reduces $A^n$.

Proof: Let $M$ be an invariant subspace of $H$ under $C$ and $A$, so we have $C(M) \subset M$ and $A(M) \subset M$ for some $n > 1$ and hence $C A^n C \subset M$. But, $A A^n C = C A^n$, it follows that $A A^n (M) \subset M$ which implies that $M$ is reduced to $A^n$.

Proposition 2.10. Let $A$ be nC-symmetric operator. Then, $M$ reduces $A^n$ if and only if $CM$ reduces $A^n$. 
Proof: Let M be reduces $A^n$ such that $A^n(M) \subset M$ and $A^{*n}(M) \subset M$. A short computation reveals that $C A^n(M) = A^{*n} C(M) \subset CM$ and $CA^{*n}(M) = A^n C(M) \subset CM$, thus we obtain CM reduces $A^n$.

Conversely, if CM reduces $A^n$, then we have $A^n(CM) \subset CM$ and $A^{*n}(CM) \subset CM$. Then $C A^n(CM) = C A^n C(M) = A^{*n}(M) \subset C(CM) = M$. In a similar way, we can obtain $A^n(M) \subset M$.

Proposition 2.11. Let $A$ be nC-symmetric operator. If $M$ is an invariant subspace of $\mathcal{H}$ under $C$ and $P$ orthogonal projection on to $M$, then the compression $B^n = P A^n P$ of $A$ to $M$ which satisfies $CB^n = B^{*C}$.

Proof: Let $A$ be nC-symmetric operator.


Proposition 2.12. If $A$ is nC-symmetric operator, then we have the following:

1. $C A^n$ commutes with $A^{*n}$.
2. $A^n C$ commutes with $A^{*n}$.

Proof: 1. Since $(CA^n)^2 = CA^n CA^n = A^{*n} C A^n A^n = A^{*n} A^n$, then this would implies that:

$C A^n A^{*n} A^n = C (C A^{*n}) A^{*n} A^n$

$= A^{*n} C A^{*n} A^n$

$= A^{*n} A^n C A^n$.

2. Similarly, $(A^n C)^2 = A^n C A^n C = A^n A^{*n} C C = A^n A^{*n}$.

Now, $A^n C A^n A^{*n} = A^n C (C A^{*n} C) A^{*n}$

$= A^n A^{*n} C A^{*n}$

$= A^n A^{*n} A^n C$.

Proposition 2.13. Let $B$ be invertible and nC- symmetric operator on $\mathcal{H}$ such that $AB = BA$, then $A$ is nC-symmetric operator.

Proof: The proof is based on the equations $C A^n = A^{*n} C$, $C B^n = B^{*n} C$ and $AB = BA$. Hence, $C (A B)^n = C A^n B^n$

$= A^{*n} C B^n$

$= A^{*n} B^{*n} C$

$= (B^n A^n)^{*n} C$

$= (A^n B^n)^{*n} C$

$= (A B)^{*n} C$.

Conversely, let $AB$ be nC-symmetric operator. To show that $A$ is nC-symmetric operator, we set

$C A^n = C A^n I$

$= C A^n B^n (B^{-1})^n$

$= C (AB)^n (B^{-1})^n$ (AB is nC-symmetric)

$= (AB)^{*n} C (B^{-1})^n$ (B^{-1} is nC-symmetric by Prop. (2.6 (1)).

$= (AB)^{*n} (B^{-1})^{*n} C$

$= (AB)^{*n} (B^{*n})^{-1} C$

$= A^{*n} B^{*n} (B^{*n})^{-1} C$

$= A^{*n} C$.

Proposition 2.14. Let $A \in B(\mathcal{H})$ and $C$ be conjugation on $\mathcal{H}$. If $C A^n = A^{*n} C$, then $A$ is nC-symmetric operator if and only if $A^n$ is self-adjoint.

Proof: Let $A$ be nC-symmetric operator such that $C A^n = A^{*n} C$, then we have $A^n C = A^{*n} C$ which yields that $A^n C = A^{*n} C$ and so we get $A^n = (A^n)^*$. Conversely, suppose that $A^n$ is self-adjoint such that $A^n = (A^n)^*$, then it follows $A^n C = A^{*n} C$ and $CA^n = A^{*n} C$ which implies that $A$ is nC-symmetric operator.

Proposition 2.15. If $A$ is both nC-symmetric and nJ-symmetric, then $A$ is both n(CJC)-symmetric and n(JJC)-symmetric operator.

Proof: Since $C A^n = A^{*n}$ and $J A^n = A^{*n}$, then
(CJC) $A^n = CJ (CA^n) = CJ A^* C = CA^n JC = A^* (CJC)$, so that $A$ is n(CJC)-symmetric operator. Analogously, we can prove that $A$ is n(JCJ)-symmetric.

**Proposition 2.16.** If $A$ is both nC-symmetric and nJ-symmetric, then $A^n U$ is C-symmetric where $U = CJ$ is unitary operator.

**Proof:** Since $A$ is both nC-symmetric and nJ-symmetric, then by the previous proposition, $A$ is n(CJC)-symmetric, so we have


**References**


