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FI-Extending Semimodule and Singularity

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Abstract
The main aim of this research is to present and to study several basic characteristics of the idea of FI-extending semimodules. The semimodule F is said to be an FI-extending semimodule if each fully invariant subsemimodule of F is essential in direct summand of F. The behavior of the FI-extending semimodule with respect to direct summands as well as the direct sum is considered. In addition, the relationship between the singularity and FI-extending semimodule has been studied and investigated. Finally extending propertywhich is stronger than FI extending, that has some results related to FI-extending and singularity is also investigated.

Keywords: Semimodules, Fully Invariant Subsemimodule, FI-Extending Semimodule, Extending Semimodule, Singular Semimodule, Nonsingular Semimodule.

شبه مقاس التوسع من النمط FI و خاصية الشذوذ

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الخلاصة
الهدف الرئيسي من البحث هو تقديم ودراسة عدة خصائص لمفهوم شبه مقاس التوسع من النمط FI . شبه المقاس F يسمى شبه مقاس التوسع من النمط FI اذا كان كل شبه مقاس جزئي تام الثبات يكون اساس في مركبة جداء مباشر ل F .كذلك تم دراسة سلوك شبه مقاس التوسع من النمط FI بالنسبة الى مركبة الجداء المباشر و الجداء المباشر . اضافه الى ذلك تم دراسة العلاقة بين الشذوذ و شبه مقاس التوسع من النمط FI اخيرا درسنا العلاقه بين شبه مقاس التوسع وهذا النوع من شبه المقاسات مع خاصية الشذوذ.

1. Introduction

The originality of CS-modules is given by Von Neumann in 1930 [1]. In [2], Utumi in 1960 had identified and studied modules with a C1 condition in his research on the continuous and self-injective rings. The C1 condition is a common generalization of the semisimple and injective conditions. In [3], authors developed a CS condition which is another aspect of C1 condition . In the last years, extended modules theory has been came to

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play a significant role and many researchers have been published and contributed to this hypothesis due to its interesting and widely available findings on expanding properties in the theoretical formulation of the module[4].

There are many generalization papers of extending property can be presented as follows: Wang and Wu [5] are studied the CLS-modules, as well as they provided a condition that makes the direct sum of CLS-modules to be CLS-module. In the other hand, in [6], authors introduced CLS-module ,and they developed the properties of y-closed submodules, by considering every y-closed submodule to be a direct summand. In [7], the concept of fully extending modules is introduced ,and proved that the class of fully extending modules is a proper subclass of the class of extending modules. Ungor and Halicioglu in [8] introduced a strongly extending module, and they investigated its properties as a particular extending module.

The idea of strongly extending modules is also defined by Atani, Hesari, and Khoramdel in [9] as a particular subclass of the class of extending modules, as well as they discussed some basic properties of this subclass of modules. The concept of generalized CS-module is defined by Zeng and Shaoguan in which homomorphic images of generalized CS-module, in addition the direct sum of semi-simple modules and singular modules are also generalized CS-module[10]. In[11], the concept of semi-extending modules, as a generalization of extending modules is studied by Ahmed and Abbas.

In [12], authors defined the concept of FI-CS in which a direct sum of FI-CS modules is FI-CS, while the authors in [13] studied some conditions that applied to make the direct summand of FI-CS module is FI-CS module. Y¨ucel in [14] also introduced the generalized FI-CS module, and they demonstrated that the class of FI-CS modules is not closed under direct summands as well as they proved that it is closed under direct sums.

Recently, a great attention in the field of semimodule has appeared via the study of many topics that are previously studied in the module converting to semimodules for more details see the following studies: Alhashemi and Alhossaini in [4] introduced and studied the extending semimodule over semiring, and they also studied some properties of the direct sum and direct summand of semimodule. In [15], the properties of singular and nonsingular semimodule are studied and investigated, as well as the relationship between singularity and extending semimodule is also proved in the same article.

In this paper, we generalize the concept of extending semimodule by studying and introducing the FI-CS semimodule, and we also investigate some conditions that make the direct summand of FI-CS to be FI-CS, as well as we prove that the direct sum of FI-CS is always FI-CS.

This paper is organized as follows: some preliminaries that are needed in this study are introduced in Section 2.

The main contributions have been presented in Section 3 and Section 4, respectively. In Section 5 the concluding remarks of this work are given.

2. Preliminaries

In order to study and to investigate for FI-CS semimodule over a semiring, R denotes a commutative *semiring* with identity, and $\mathcal{F}$ is a left *R–semimodule.*

Definition 2.1 [16]**:** Let $(\mathcal{F}, +)$ be an additive abelian monoid with additive identity $0_{\mathcal{F}}$, then $\mathcal{F}$ is called a left *R–semimodule* if there exists a scalar multiplication $R \times \mathcal{F} \longrightarrow \mathcal{F}$ which is denoted by $(\mathfrak{r}, \mathcal{s}) \mapsto \mathfrak{r}\,\mathcal{s}$, such that $(\mathfrak{r}\,\mathfrak{r}')\mathcal{s} = \mathfrak{r}\,(\mathfrak{r}'\,\mathcal{s})$; $\mathfrak{r}\,(\mathcal{s} + \mathcal{s}') = \mathfrak{r}\,\mathcal{s} + \mathfrak{r}\,\mathcal{s}'$; $(\mathfrak{r} + \mathfrak{r}')\mathcal{s} = \mathfrak{r}\,\mathcal{s} + \mathfrak{r}'\mathcal{s}$; $\mathfrak{r}\,0_{\mathcal{F}} = 0_R\,\mathcal{s} = 0_{\mathcal{F}}$ $for\ all\ \mathfrak{r}, \mathfrak{r}' \in R\ and\ all\ \mathcal{s}, \mathcal{s}' \in \mathcal{F}$.

Definition 2.2 [17]**:** A subset $\mathcal{T}$ of an R–semimodule $\mathcal{F}$ is called a *subsemimodule* of $\mathcal{F}$ if for $\mathcal{s}, \mathcal{s}' \in \mathcal{T}$ and $\mathfrak{r} \in R$, $\mathcal{s} + \mathcal{s}' \in \mathcal{T}$ and $\mathfrak{r}\,\mathcal{s} \in \mathcal{T}$ and write $(\mathcal{T} \leq \mathcal{F})$.

Definition 2.3 [18]**:** A subsemimodule $\mathcal{K}$ of $\mathcal{F}$ is said to be *fully invariant* if $g(\mathcal{K}) \subseteq \mathcal{K}$ for each R-endomorphism g on $\mathcal{F}$(denoted $\mathcal{K} \trianglerighteq \mathcal{F}$).

Definition 2.4 [19]**:** A nonzero *R*-subsemimodule $\mathcal{S}$ of $\mathcal{F}$ *is called essential* ,which is denoted by ($\mathcal{S} \leq^{e} \mathcal{F}$) if $\mathcal{S} \cap \mathcal{T} \neq 0$ for every $0 \neq \mathcal{T} \leq \mathcal{F}$.

Definition 2.5 [20]**:** A subsemimodule $Z(\mathcal{F})$ of $\mathcal{F}$ is defined by $Z(\mathcal{F}) = \{f \in \mathcal{F} |$ ann $f \leq^{e} R$ $\}$ is said to be *singular* subsemimodule of $\mathcal{F}$. If $Z(\mathcal{F}) = \mathcal{F}$ then $\mathcal{F}$ is called singular. If $Z(\mathcal{F}) = 0$, and $\mathcal{F}$ is called *nonsingular*.

Definition 2.6 [21]**:** The *second singular* subsemimodule $Z_2(\mathcal{F})$ of $\mathcal{F}$ is that subsemimodule of $\mathcal{F}$, containing $Z(\mathcal{F})$, such that $Z_2(\mathcal{F})/Z(\mathcal{F})$ is the singular subsemimodule of $\mathcal{F}/Z(\mathcal{F})$.

Definition 2.7 [22]**:** An *R*–semimodule $\mathcal{F}$ is called a *direct sum* of subsemimodules $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ of $\mathcal{F}$ if each $\not{f} \in \mathcal{F}$ can be written uniquely as $\mathcal{F} = \not{f}_1 + \not{f}_2 + \ldots + \not{f}_k$, where $\not{f}_i \in \mathcal{F}_i$. It is denoted by $\mathcal{F} = \not{f}_1 \oplus \not{f}_2 \oplus \ldots \oplus \not{f}_k$. In this case each $\not{f}_i$ is called a *direct summand* of $\mathcal{F}$(denoted by DS).

Definition 2.8[4]**:** An *R*–semimodule $\mathcal{F}$ is called *extending (CS–semimodule*) if every subsemimodule of $\mathcal{F}$ is essential in a direct summand of $\mathcal{F}$. This is equivalent to following: Every closed subsemimodule of $\mathcal{F}$ is a direct summand of $\mathcal{F}$.

Definition 2.9 [23]**:** A subsemimodule $\mathcal{S}$ of a semimodule $\mathcal{F}$ is said to be *closed* if $\mathcal{S} \leq^{e} \mathcal{S}' \leq \mathcal{F}$ implies $\mathcal{S} = \mathcal{S}'$ (denoted by $\mathcal{S} \leq^{C} \mathcal{F}$).

Definition 2.10 [24]**:** A semimodule $\mathcal{F}$ is said to be *semisimple* if it is a direct sum of its simple subsemimodule.

Definition 2.11[25]**:** A semimodule $\mathcal{F}$ is said to be *uniform* if any subsemimodule $\mathcal{N}$ of $\mathcal{F}$ is essential.

Definition 2.12 [23]**:** If $\mathbb{E}$ is an injective *R–semimodule*, and a minimal injective extension of the *R–semimodule* $\mathcal{F}$, then $\mathbb{E}$ is said to be an *injective hull* of $\mathcal{F}$ which is denoted by $\mathbb{E}(\mathcal{F})$.

Definition 2.13 [26]**:** A semimodule $\mathcal{F}$ is *additively cancellative* if $n + n' = n + n''$ for all $n, n'$and $n'' \in \mathcal{F}$ implies $n' = n''$.

Definition 2.14 [23]**:** A non-zero semimodule $\mathcal{F}$ is said to be *indecomposable* if its direct summands are $\{0\}$ and itself only.

Definition 2.15 [23]**:** A subsemimodule $\mathcal{C}$ of a semimodule $\mathcal{F}$ is called *complement* of a subsemimodule $\mathcal{T}$ of $\mathcal{F}$ if $\mathcal{C} \cap \mathcal{T} = 0$ and $\mathcal{C}$ is a maximal with this property.

Definition 2.16 [23]**:** A subsemimodule $\mathcal{T}$ of a semimodule $\mathcal{F}$ is said to be *closure* of a subsemimodule $\mathcal{S}$ in $\mathcal{F}$ if $\mathcal{T}$ is closed and $\mathcal{S}$ essential in $\mathcal{T}$. This equivalents to the following: The closure of $\mathcal{S}$ is the smallest closed subsemimodule containing $\mathcal{S}$.

Definition 2.17[26]**:** An *R*–semimodule $\mathcal{F}$ is said to be *$\mathcal{A}$-injective*, if for each subsemimodule $\mathcal{T}$ of $\mathcal{A}$, any homomorphism from $\mathcal{T}$ into $\mathcal{F}$ can be extended to an *R*-homomorphism from $\mathcal{A}$ into $\mathcal{F}$. The *R*–semimodule $\mathcal{F}$ is injective if it is injective relative to every *R*–semimodule.

Definition 2.18[27]**:** A semimodule $\mathcal{F}$ is said to be duo if each subsemimodule of $\mathcal{F}$ is fully invariant.

Definition 2.19 [28]**:** The *Socle radical* of a semimodule $\mathcal{F}$ is denoted by $Soc(\mathcal{F})$ and it is defined as $Soc(\mathcal{F}) = \Sigma\{\mathcal{L} : \mathcal{L}$ is a simple $R - subsemimodule$ of $\mathcal{F}\}$.

Lemma (2.20) [18]**:** Let $\mathcal{F}$ be an *R*–semimodule. If $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ and $\mathcal{T} \trianglerighteq \mathcal{F}$, then $\mathcal{T} = (\mathcal{T} \cap \mathcal{F}_1) \oplus (\mathcal{T} \cap \mathcal{F}_2)$.

Lemma (2.21) [23]**:** If $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, then $\mathbb{E}(\mathcal{F}) = \mathbb{E}(\mathcal{F}_1) \oplus \mathbb{E}(\mathcal{F}_2)$**.**

Lemma (2.22) [23]**:** If $\mathcal{F}$ is an *R–semimodule* with injective hull, $\mathcal{K} \leq^{e} \mathcal{F}$, then $\mathbb{E}(\mathcal{K}) = \mathbb{E}(\mathcal{F})$.

3. FI-CS Semimodule

The properties of the FI-CS semimodule are introduced and investigated in this section. It can be seen by analyzing the structure of FI-CS semimodule the following : There are many properties of fully invariant subsemimodules that are also useful. Next we will give some properties of a fully invariant submodule, these properties will be converted to subsemimodule.

Lemma (3.1): Let $\mathcal{F}$ be an *R–semimodule*. If $\mathcal{N} \trianglerighteq \mathcal{F}$ and $\mathcal{M} \trianglerighteq \mathcal{F}$, then $(\mathcal{N}+\mathcal{M}) \trianglerighteq \mathcal{F}$.

Proof: It is clear so that it is omitted.

Lemma (3.2): Let $\mathcal{F}$ be an *R–semimodule*. If $\mathcal{N} \trianglerighteq \mathcal{F}$ and $\mathcal{M} \trianglerighteq \mathcal{F}$, then$(\mathcal{N} \cap \mathcal{M}) \trianglerighteq \mathcal{F}$.

Proof: It is clear so that it is omitted.

Lemma (3.3): Let $\mathcal{F}$ be an *R–semimodule*. If $\mathcal{A} \leq \mathcal{B}$, such that $\mathcal{B} \trianglerighteq \mathcal{F}$ and $\mathcal{A} \trianglerighteq \mathcal{B}$, then $\mathcal{A} \trianglerighteq \mathcal{F}$.

*Proof***:** Let $h \in \mathrm{End}(\mathcal{F})$, then $g = h|_{\mathcal{B}} \in \mathrm{End}(\mathcal{B})$, since $g(\mathcal{B}) \subseteq \mathcal{B}$. Now $h(\mathcal{A}) = g(\mathcal{A}) \subseteq \mathcal{A}$ (since$\mathcal{A} \trianglerighteq \mathcal{B}$), therefore $\mathcal{A} \trianglerighteq \mathcal{F}$. □

Definition (3.4): An *R–semimodule* $\mathcal{F}$ is called FI-CS if each fully invariant subsemimodule of $\mathcal{F}$ is essential in a DS of $\mathcal{F}$.

Remark (3.5): It is clear that any *CS–semimodule* is FI-CS, however the converse is not true. For example if $\mathcal{F} = \mathbb{Z}_2 \oplus \mathbb{Z}_8$, $R = \mathbb{Z}$, then $\mathcal{F}$ is not CS[4], but $\mathcal{F}$ is FI-CS since $[\mathcal{A} = \langle(\bar{1}, \bar{2})\rangle]$, the only subsemimodule which is not essential in a DS this implies that it is not fully invariant of $\mathcal{F}$.

Proposition (3.6): Let $\mathcal{F}$ be an *R*–semimodule and $\mathcal{N} \trianglerighteq \mathcal{F}$. If $\mathcal{F}$ is FI-CS then $\mathcal{N}$ is FI-CS.

*Proof***:** Assume that $\mathcal{F}$ is FI-CS, and $\mathcal{K} \trianglerighteq N$, by Lemma (3.3), $\mathcal{K} \trianglerighteq \mathcal{F}$, hence $\mathcal{K} \leq^e D_1$, where D_1 is a DS subsemimodule of $\mathcal{F}$, say $\mathcal{F} = \mathcal{D}_1 \oplus \mathcal{D}_2$, for some $\mathcal{D}_2 \leq \mathcal{F}$, since $\mathcal{N} \trianglerighteq \mathcal{F}$, hence by Lemma (2.20), $\mathcal{N} = (\mathcal{N} \cap \mathcal{D}_1) \oplus (\mathcal{N} \cap \mathcal{D}_2)$, since $\mathcal{K} \leq^e \mathcal{D}_1$, then $\mathcal{K} \leq^e (N \cap \mathcal{D}_1)$, where $(N \cap \mathcal{D}_1)$ is a DS of $\mathcal{N}$, therefore $\mathcal{N}$ is FI-CS. □

Proposition (3.7): Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$. If $\mathcal{F}_1$ and F_2 are FI-CS then $\mathcal{F}$ is FI-CS.

*Proof***:** Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are FI-CS, and let $\mathcal{N} \trianglerighteq \mathcal{F}$. Let $\pi_i : \mathcal{F} \to \mathcal{F}i$, be the natural projections of $\mathcal{F}$onto $\mathcal{F}_i$ (i = 1,2), then $\mathcal{N} = \pi_1(\mathcal{N}) \oplus \pi_2(\mathcal{N})$, where $\pi_i(\mathcal{N}) \leq \mathcal{F}_i$ (i = 1,2), since $\mathcal{F}_i$ is FI-CS, then there exist DS subsemimodules $\mathcal{D}_i$ of $\mathcal{F}_i$ such that $\pi_i(\mathcal{N}) \leq^e \mathcal{D}_i$, then $\mathcal{F}_i = \mathcal{D}_i \oplus \mathcal{D}_i{}'$, and $\pi_1(\mathcal{N}) \oplus \pi_2(\mathcal{N}) \leq^e \mathcal{D}_1 \oplus \mathcal{D}_2$, so $\mathcal{N} \leq^e \mathcal{D}_1 \oplus \mathcal{D}_2$, where $\mathcal{F} = \mathcal{D}_1 \oplus \mathcal{D}_1{}' \oplus \mathcal{D}_2 \oplus \mathcal{D}_2$, and $\mathcal{D}_1 \oplus \mathcal{D}_2$ is a DS of $\mathcal{F}$, therefore $\mathcal{F}$ is FI-CS. □

Corollary (3.8): If $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, where $\mathcal{F}_1$and $\mathcal{F}_2$ are CS *R–semimodule* then $\mathcal{F}$ is FI-CS .

Proof: By Remark (3.5) $\mathcal{F}_1$ and $\mathcal{F}_2$ are FI-CS, hence by Proposition (3.7) $\mathcal{F}$ is FI-CS. □

Recall that, every uniform or semisimple *R–semimodule* is CS [15].

Corollary (3.9): If $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are uniform or semisimple *R–semimodule*, then $\mathcal{F}$ is FI-CS.

Proof: Assume $\mathcal{F}_1$and $\mathcal{F}_2$ are uniform or semisimple *R–semimodules* then $\mathcal{F}_1$ and $\mathcal{F}_2$ are CS *R–semimodules* by Remark (3.5) and Corollary (3.8) $\mathcal{F}$ is FI-CS. □

Lemma (3.10): If e and $\hbar$ are idempotent endomorphism's of a cancellative semimodule $\mathcal{F}$, with$e + \hbar = I_{\mathcal{F}}$, then$\mathcal{F} = e(\mathcal{F}) \oplus \hbar(\mathcal{F})$.

Proof: It is enough to prove that $e(\mathcal{F}) \cap \hbar(\mathcal{F}) = 0$. Let $x \in e(\mathcal{F}) \cap \hbar(\mathcal{F})$, then $x = e(t) = \hbar(s)$, for some $t, s \in \mathcal{F}$, *but* $x = e(x) + \hbar(x) = e(e(t)) + \hbar(\hbar(s)) = x + x$, by cancellative property, $x = 0$, hence $\mathcal{F} = e(\mathcal{F}) \oplus \hbar(\mathcal{F})$. □

Proposition (3.11): Let $\mathcal{F}$ be an *R*–semimodule with injective hull, then $\mathcal{F}$ is FI-CS if and only if for each fully invariant subsemimodule $\mathcal{N}$ in $\mathcal{F}$, there exist e, $\hbar$ idempotent endomorphisms of $\mathbb{E}(\mathcal{F})$, with $e + \hbar = I_{E(\mathcal{F})}$, such that $\mathcal{N} \leq^e e(\mathbb{E}(\mathcal{F}))$ and $e(\mathcal{F}) \subseteq \mathcal{F}$.

*Proof***:** Assume that $\mathcal{F}$ is FI-CS, and$\mathcal{N} \trianglerighteq \mathcal{F}$, then there exists a DS subsemimodule D of $\mathcal{F}$ such that $\mathcal{N} \leq^e \mathcal{D}$, say $\mathcal{F} = \mathcal{D} \oplus \mathcal{D}'$ for some $\mathcal{D}' \leq \mathcal{F}$**,** since $\mathcal{F}$ has injective hull, then by Lemma (2.21), $\mathbb{E}(\mathcal{F}) = \mathbb{E}(\mathcal{D}) \oplus \mathbb{E}(\mathcal{D}')$. Now let e, $\hbar$ be the natural projections of $\mathbb{E}(\mathcal{F})$ onto $\mathbb{E}(\mathcal{D})$ and $\mathbb{E}(\mathcal{D}')$ respectively, then e can be considered as an idempotent endomorphism's of $\mathbb{E}(\mathcal{F})$. On other hand, $\mathcal{N} \leq^e \mathcal{D}$ and $\mathcal{D} \leq^e \mathbb{E}(\mathcal{D})$ imply $\mathcal{N} \leq^e \mathbb{E}(\mathcal{D})$, since $\mathcal{F} = \mathcal{D} \oplus \mathcal{D}'$, hence $e(\mathcal{F}) = e(\mathcal{D}) \oplus e(\mathcal{D}')$, but $\mathcal{D}' \leq \mathbb{E}(\mathcal{D}')$, hence $e(\mathcal{D}') = 0$. Therefore $e(\mathcal{F}) = e(\mathcal{D})$, but

$\mathcal{D} \leq \mathbb{E}(\mathcal{D})$, then $e(\mathcal{D}) = \mathcal{D}$, so $e(\mathcal{F}) = \mathcal{D} \leq \mathcal{F}$. On other hand, $\mathcal{N} \leq^e \mathcal{D} \leq^e \mathbb{E}(\mathcal{D}) = e(\mathbb{E}(\mathcal{F}))$, hence $\mathcal{N} \leq^e e(\mathbb{E}(\mathcal{F}))$.

Now we assume that N⊳F, and N≤^e e(E (F)) and e(F) ⊆F by assumption, where e is an idempotent endomorphism's of E(F), then N≤^e F∩e(E (F))⊆ F, and N≤^e e(F). By Lemma (3.10), e(F) is a DS subsemimodule of F, hence F is FI-CS semimodule. □

Proposition (3.12): Let $\mathcal{F}$ be FI-CS semimodule with injective hull, and $\mathcal{S} = \mathcal{F} \cap \mathcal{P}$ where $\mathcal{P} \trianglerighteq^{\mathrm{DS}} \mathbb{E}(\mathcal{F})$, then $\mathcal{S} \trianglerighteq^{\mathrm{DS}} \mathcal{F}$.

Proof: Assume that $h \in \mathrm{End}(\mathcal{F})$, since $\mathbb{E}(\mathcal{F})$ is injective, then there exists $h' \in \mathrm{End}(\mathbb{E}(\mathcal{F}))$, such that h' extends h. Let $s \in \mathcal{S}$, since $\mathcal{S} \subseteq \mathcal{F}$, then $s \in \mathcal{F}$ and $h(s) \in h(\mathcal{F}) \subseteq \mathcal{F}$, therefore $h(s) \in \mathcal{F}$. Since $\mathcal{S} \subseteq \mathcal{P}$, hence $h'(s) = h(s) \in \mathcal{F} \cap \mathcal{P} = \mathcal{S}$, so $\mathcal{S} \trianglerighteq \mathcal{F}$. While $\mathcal{P}$ is injective (since it is a DS of $\mathbb{E}(\mathcal{F})$), and $\mathcal{S} \leq \mathcal{F}$, then $\mathbb{E}(\mathcal{S}) \leq \mathbb{E}(\mathcal{F}) = \mathcal{P}$. Since $\mathcal{F}$ is FI-CS, then there exists a DS subsemimodule $\mathcal{X}$ of $\mathcal{F}$ such that $\mathcal{S} \leq^e \mathcal{X}$. Therefore by Lemma (2.22), $\mathbb{E}(\mathcal{S}) = \mathbb{E}(\mathcal{X})$, so $\mathcal{X} \leq \mathcal{F} \cap \mathbb{E}(\mathcal{X}) = \mathcal{F} \cap \mathbb{E}(\mathcal{S}) \leq \mathcal{F} \cap \mathcal{P} = \mathcal{S}$. Hence $\mathcal{S} = \mathcal{X}$, since $\mathcal{X}$ is a DS of $\mathcal{F}$, then we get that $\mathcal{S}$ is a DS of $\mathcal{F}$. □

Proposition (3.13): Let $\mathcal{F}$ be an *R*–semimodule. Then $\mathcal{F}$ is FI-CS if and only if every fully invariant subsemimodule of $\mathcal{F}$ has a complement which is a DS of $\mathcal{F}$.

*Proof***:** Assume that $\mathcal{F}$ is FI-CS, let $\mathcal{S} \trianglerighteq \mathcal{F}$, then there exists a DS subsemimodule $\mathcal{K}$ of $\mathcal{F}$ such that $\mathcal{S} \leq^e \mathcal{K}$, say $\mathcal{F} = \mathcal{K} \oplus \mathcal{K}'$, for some $\mathcal{K}' \leq \mathcal{F}$. Let $e = \pi_{\mathcal{K}}$ and $h = \pi_{\mathcal{K}}$ be the natural projections of $\mathcal{F}$ onto $\mathcal{K}$ *and* $\mathcal{K}'$, respectively. It is clear that $e = e^2$, $h = h^2$, and $e + h = I_{\mathcal{F}}$, so $\mathcal{S} \leq^e e\,\mathcal{F}$, since $\mathcal{K}'$ is a complement of $\mathcal{K}$, so $\mathcal{K}'$ is the desired complement.

Conversely, assume that $\mathcal{S} \trianglerighteq \mathcal{F}$ and $\mathcal{K}'$ is a complement of $\mathcal{S}$, say $\mathcal{F} = \mathcal{K} \oplus \mathcal{K}'$ for some $\mathcal{K} \leq \mathcal{F}$. Since $\mathcal{S} \trianglerighteq \mathcal{F}$ then $\mathcal{S} = (\mathcal{S} \cap \mathcal{K}) \oplus (\mathcal{S} \cap \mathcal{K}') = (\mathcal{S} \cap \mathcal{K})$. Therefore $\mathcal{S} \subseteq \mathcal{K}$. If $\mathcal{Z} \subseteq \mathcal{K}$ *s*uch that $\mathcal{S} \cap \mathcal{Z} = 0$, so $\mathcal{S} \cap [\mathcal{Z} + \mathcal{K}'] = 0$, (if $x \in \mathcal{S}$, and $x = z + u'$, where $z \in \mathcal{Z}$ and $u' \in \mathcal{K}'$ so $u' \in \mathcal{K} \cap \mathcal{K}' = 0$, hence $x = z \in \mathcal{S} \cap \mathcal{Z} = 0$), so that $\mathcal{Z} + \mathcal{K}' = \mathcal{K}'$, and $\mathcal{Z} = 0$, *then* $\mathcal{S} \leq^e \mathcal{K}$. □

Corollary (3.14): A *semimodule* $\mathcal{F}$ is FI-CS if and only if for any $\mathcal{B} \trianglerighteq \mathcal{F}$ there exists a DS subsemimodule $\mathcal{N}$ of $\mathcal{F}$ such that $\mathcal{B} \cap \mathcal{N} = 0$, and $\mathcal{B} \oplus \mathcal{N} \leq^e \mathcal{F}$.

*Proof***:** It is clear by Proposition (3.13) and definition of complement. □

Proposition (3.15): A *semimodule* $\mathcal{F}$ is FI-CS if and only if the closure of any fully invariant subsemimodule in $\mathcal{F}$ is a DS of $\mathcal{F}$.

*Proof***:** Assume that $\mathcal{F}$ is FI-CS, and $\mathcal{N} \trianglerighteq \mathcal{F}$ with $\mathcal{N}'$ is closure of $\mathcal{N}$ by definition $\mathcal{N} \leq^e \mathcal{K}$, and $\mathcal{K}$ is a DS of $\mathcal{F}$ then $\mathcal{N}' \leq \mathcal{K}$. In fact $\mathcal{N}' \leq^e \mathcal{K}$, therefore $\mathcal{N}' = \mathcal{K}$, which is a DS subsemimodule of $\mathcal{F}$. Conversely we assume the closure of any fully invariant subsemimodule in $\mathcal{F}$ is a DS of $\mathcal{F}$. Let $\mathcal{N} \trianglerighteq \mathcal{F}$ and $\mathcal{N}'$ be closure of $\mathcal{N}$, then $\mathcal{N}'$ is a DS of $\mathcal{F}$, and $\mathcal{N} \leq^e \mathcal{N}'$. Therefore $\mathcal{F}$ is FI-CS. □

Corollary (3.16): If $\mathcal{F}$ is FI-CS then any fully invariant closed subsemimodule of $\mathcal{F}$ is a DS of $\mathcal{F}$.

*Proof***:** It is clear by Proposition (3.15). □

Corollary (3.17): Assume that $\mathcal{F}$ is a duo semimodule then $\mathcal{F}$ is CS if and only if $\mathcal{F}$ is FI-CS.

*Proof***:** It is clear by definition of duo, Remark (3.5) and Proposition (3.15). □

Lemma (3.18): Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$. If $\mathcal{F}_1 \trianglerighteq \mathcal{F}$ and $\mathcal{P} \trianglerighteq \mathcal{F}_2$ then $(\mathcal{F}_1 \oplus \mathcal{P}) \trianglerighteq \mathcal{F}$.

Proof: Assume $\mathcal{P} \trianglerighteq \mathcal{F}_2$, and $h \in \mathrm{End}\,\mathcal{F}$, then $h(\mathcal{F}_1 \oplus \mathcal{P}) = h(\mathcal{F}_1) \oplus h(\mathcal{P})$, where $h(\mathcal{F}_1) \leq \mathcal{F}_1$ (since $\mathcal{F}_1 \trianglerighteq \mathcal{F}$), and $h(\mathcal{P}) = \pi_1(h(\mathcal{P})) + \pi_2(h(\mathcal{P})) = \pi_1(h(\mathcal{P})) + \pi_2(h(i(\mathcal{P})))$, and i is the inclusion map from $\mathcal{F}_2$ into $\mathcal{F}$. Now $\pi_2 h i \in \mathrm{End}\,\mathcal{F}_2$, and $\mathcal{P} \trianglerighteq \mathcal{F}_2$, hence $\pi_2 h i(\mathcal{P}) \leq \mathcal{P})$, on other hand $\pi_1(h(\mathcal{P})) \leq \mathcal{F}_1$. Hence $h(\mathcal{F}_1 \oplus \mathcal{P}) \leq (\mathcal{F}_1 \oplus \mathcal{P})$, and $(\mathcal{F}_1 \oplus \mathcal{P}) \trianglerighteq \mathcal{F}$. □

Proposition (3.19): Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ be FI-CS. If $\mathcal{F}_1 \trianglerighteq \mathcal{F}$ and $\mathcal{P} \trianglerighteq \mathcal{F}_2$ then both $\mathcal{F}_1$ and $(\mathcal{F}_1 \oplus \mathcal{P})$ are FI-CS.

*Proof***:** It is clear by Lemma (3.18) and Proposition (3.6) . □

Proposition (3.20): Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ be FI-CS and $\mathcal{F}_1 \trianglerighteq \mathcal{F}$ then both $\mathcal{F}_1$ and $\mathcal{F}_2$ are FI-CS.

Proof: By proposition (3.6) $\mathcal{F}_1$ is FI-CS. Let $\mathcal{P} \trianglerighteq \mathcal{F}_2$, by Lemma (3.18), $(\mathcal{F}_1 \oplus \mathcal{P}) \trianglerighteq \mathcal{F}$, since $\mathcal{F}$ is FI-CS, then there exists a DS subsemimodule $\mathcal{K}$ of $\mathcal{F}$, such that $\mathcal{F}_1 \oplus \mathcal{P} \leq^e \mathcal{K}$, say $\mathcal{F} = \mathcal{K} \oplus \mathcal{K}'$, for some $\mathcal{K}' \leq F$. By modular law[30], $\mathcal{K} = \mathcal{F}_1 \oplus (\mathcal{K} \cap \mathcal{F}_2)$ [since $\mathcal{F}_1 \leq \mathcal{F}_1 \oplus \mathcal{P} \leq \mathcal{K}$]. Now $\mathcal{P} \leq \mathcal{F}_1 \oplus \mathcal{P} \leq \mathcal{K}$, and $\mathcal{P} \leq \mathcal{F}_2$ imply $\mathcal{P} \leq \mathcal{K} \cap \mathcal{F}_2$. Furthermore, we have $\mathcal{F}_1 \oplus \mathcal{P} \leq^e \mathcal{K}$, this implies $(\mathcal{F}_1 \oplus \mathcal{P}) \cap \mathcal{F}_2 \leq^e \mathcal{K} \cap \mathcal{F}_2$,however $(\mathcal{F}_1 \oplus \mathcal{P}) \cap \mathcal{F}_2 = \mathcal{P}$ [by modular law], so that $\mathcal{P} \leq^e \mathcal{K} \cap \mathcal{F}_2$. Now $\mathcal{F} = \mathcal{F}_1 \oplus (\mathcal{K} \cap \mathcal{F}_2) \oplus \mathcal{K}'$ implies $\mathcal{F}_2 = \mathcal{K} \cap \mathcal{F}_2 \oplus (\mathcal{F}_1 \oplus \mathcal{K}') \cap \mathcal{F}_2$. So $that\ \mathcal{P} \leq^e \mathcal{K} \cap \mathcal{F}_2$, where $\mathcal{K} \cap \mathcal{F}_2$ is a DS of $\mathcal{F}_2$. Therefore $\mathcal{F}_2$is FI-CS. □

Proposition (3.21): Let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$be an *R–semimodule*. If $\mathcal{F}_1$ is FI-CS and $\mathcal{F}_2$ is $\mathcal{F}_1$ injective then every fully invariant closed $\mathcal{S}$ in $\mathcal{F}$ with $\mathcal{S} \cap \mathcal{F}_2$=0 is a DS of $\mathcal{F}$.

Proof: Let $\mathcal{S} \trianglerighteq^{C} \mathcal{F}$ with $\mathcal{S} \cap \mathcal{F}_2$ = 0, since $\mathcal{F}_2$ is $\mathcal{F}_1$ injective then by [4], there exists $\mathcal{F}' \leq \mathcal{F}$, such that $\mathcal{S} \subseteq \mathcal{F}'$, and $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}_2$. Therefore $\mathcal{F}' \cong \mathcal{F}_1$, since $\mathcal{F}_1$ is FI-CS, then $\mathcal{F}'$ is FI-CS, and $\mathcal{S}$ is a DS of $\mathcal{F}'$, we say $\mathcal{F}' = \mathcal{S} \oplus \mathcal{F}''$ for some, $\mathcal{F}'' \leq \mathcal{F}'$, so $\mathcal{F} = \mathcal{S} \oplus \mathcal{F}'' \oplus \mathcal{F}_2$, that means $\mathcal{S}$ is a DS of $\mathcal{F}$. □

Proposition (3.22): Let $\mathcal{F}_1$ be semisimple *R–semimodule*, then $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ is FI-CS for any FI-CS $\mathcal{F}_2$.

Proof: Assume $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ and $\mathcal{F}_2$ is FI-CS. Since $\mathcal{F}_1$ is semisimple so $\mathcal{F}_1$ is CS. Now by Remark (3.5) $\mathcal{F}_1$ is FI-CS, hence by Proposition (3.7) $\mathcal{F}$ is FI-CS. □

4. FI-CS with Singularity

In this section, the relationship between the singularity and FI-CS semimodule for direct summands as well as the direct sum is studied and investigated.

Proposition (4.1): A *semimodule* $\mathcal{F}$ is FI-CS if and only if $\mathcal{F} = Z_2(\mathcal{F}) \oplus K$, where $Z_2(\mathcal{F})$ and K are FI-CS.

Proof: By[15] we have $Z_2(\mathcal{F}) \trianglerighteq^{C} \mathcal{F}$, and $\mathcal{F}$ is FI-CS, then by Corollary (3.16) $Z_2(\mathcal{F})$ is a DS subsemimodule of $\mathcal{F}$, that $means\ \mathcal{F} = Z_2(\mathcal{F}) \oplus K$ for some $\mathcal{K} \leq \mathcal{F}$, since $\mathcal{F}$ is FI-CS, then by Proposition (3.20) both $Z_2(\mathcal{F})$ and $\mathcal{K}$ are FI-CS .

Conversely, it follows from Proposition (3.7). □

Proposition (4.2): Let $\mathcal{F}_1$ be nonsingular semisimple *R–semimodule* with injective hull $\mathbb{E}(\mathcal{F}_1)$, and $\mathcal{F}_2$ be an *R–semimodule* with $\mathcal{S}oc(\mathcal{F}_2) = 0$, if $\mathcal{F}_2$ is FI-CS and $\hbar \in$ Hom($\mathcal{F}_2$, $\mathbb{E}$ $(\mathcal{F}_1)$) such that ker$\hbar \trianglerighteq \mathcal{F}_2$, then $\hbar = 0$.

Proof: Assume that $\hbar \in$ Hom($\mathcal{F}_2$, $\mathbb{E}$ $(\mathcal{F}_1)$), and ker$\hbar \leq^e \mathcal{F}_2{}'$, where $\mathcal{F}_2{}' \leq \mathcal{F}_2$. Let $t_2 \in \mathcal{F}_2{}'$ then by [15] there exists $\mathcal{I} \leq^e R$, such that $\mathcal{I}t_2 \leq$ ker $\hbar$, hence $\mathcal{I}\hbar(t_2) = \hbar(\mathcal{I}t_2) = 0$. Since $\mathbb{E}(\mathcal{F}_1)$ is nonsingular by [15] we have $\hbar(t_2)$ = 0, therefore $t_2 \in$ ker$\hbar$, hence ker $\hbar$ = $\mathcal{F}_2{}'$, so ker $\hbar$ has no proper essential extension, then ker $\hbar$ is closed in $\mathcal{F}_2$, but by hypotheses ker$\hbar \trianglerighteq \mathcal{F}_2$, hence ker$\hbar \trianglerighteq^{C} \mathcal{F}_2$, since $\mathcal{F}_2$ is FI-CS then by Corollary (3. 16), ker$\hbar$ is a DS of $\mathcal{F}_2$, say $\mathcal{F}_2$ = ker$\hbar \oplus \mathcal{F}_2{}''$, for some $\mathcal{F}_2{}'' \leq \mathcal{F}_2$. Since $\mathcal{S}oc(\mathcal{F}_2) = 0$ then $\mathcal{F}_2'' = 0$, and $\hbar = 0$.

Corollary (4.3): Let $\mathcal{F}_1$ be nonsingular semisimple *R–semimodule* with injective hull $\mathbb{E}(\mathcal{F}_1)$and $\mathcal{F}_2$ be an *R–semimodule* with $\mathcal{S}oc(\mathcal{F}_2) = 0$, if $\mathcal{F}_2$ is FI-CS and $0 \neq \hbar \in$ Hom($\mathcal{F}_2$, $\mathbb{E}$ $(\mathcal{F}_1)$) then ker$\hbar \ntrianglerighteq \mathcal{F}_2$.

Proof: Assume that $\hbar \in$ Hom($\mathcal{F}_2$, $\mathbb{E}(\mathcal{F}_1)$),then if ker$\hbar \trianglerighteq \mathcal{F}_2$ by Proposition (4.2) we have $\hbar$ = 0, this leads to a contradiction with hypotheses. Therefore ker$\hbar \ntrianglerighteq \mathcal{F}_2$. □

Proposition (4.4): If $\mathcal{F}$ is FI-CS and indecomposable then every fully invariant subsemimodule is essential in $\mathcal{F}$.

Proof: Assume that $0 \neq \mathcal{N} \trianglerighteq \mathcal{F}$, since $\mathcal{F}$ is FI-CS, then there exists a DS subsemimodule $\mathcal{D}$ of$\mathcal{F}$, such that $\mathcal{N} \leq^e \mathcal{D}$, since $\mathcal{F}$ indecomposable by hypotheses we have either $\mathcal{D} = 0$ or$\mathcal{D} = \mathcal{F}$, but $\mathcal{N}$is nonzero. Therefore $\mathcal{N} \leq^e \mathcal{F}$. □

Proposition (4.5): Let $\mathcal{F}$ be FI-CS, and $\mathcal{N} \trianglerighteq \mathcal{F}$. If $\mathcal{D} \trianglerighteq^{\mathrm{DS}} \mathcal{F}$, such that $(\mathcal{D}+\mathcal{N})/\mathcal{D}$ is nonsingular, then $\mathcal{N} \cap \mathcal{D}$ is a DS of $\mathcal{N}$.

Proof: Note that $(\mathcal{D} + \mathcal{N})/\mathcal{D} \cong \mathcal{N}/(\mathcal{N} \cap \mathcal{D})$, so by hypotheses $\mathcal{N}/(\mathcal{N} \cap \mathcal{D})$is nonsingular, hence by[15], $(\mathcal{N} \cap \mathcal{D})$ is closed in $\mathcal{N}$ but $\mathcal{N}$ is FI-CS by Proposition (3.6), so by Corollary (3.16), $(\mathcal{N} \cap \mathcal{D})$ is a DS of $\mathcal{N}$. □

Remark (4.6): If $\mathcal{F}_1$is singular and $\mathcal{F}_2$ is nonsingular, then $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0$.

*Proof***:** Assume that $h \in \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$, and $x \in \mathcal{F}_1$, since $\mathcal{F}_1$ is singular then there exists $\mathcal{I} \leq^e R$, such that $\mathcal{I}x = 0$, then we have $h(\mathcal{I}x) = \mathcal{I}h(x) = 0$, but $h(x) \in \mathcal{F}_2$, and $\mathcal{F}_2$ is nonsingular then $h(x) = 0$, so $h = 0$. □

Proposition (4.7): Let $\mathcal{F}$ be FI-CS, then $\mathcal{F} = Z(\mathcal{F}) \oplus \mathcal{P}$, for some nonsingular P of $\mathcal{F}$and P is $Z(\mathcal{F})$-injective.

Proof: It is easy to prove that hen $Z(\mathcal{F}) = 0$ or $\mathrm{Z}(\mathcal{F}) = \mathcal{F}$. Now suppose that $\mathrm{Z}(\mathcal{F}) \leq \mathcal{F}$, since $\mathrm{Z}(\mathcal{F})$ is a fully invariant by [15], and $\mathcal{F}$ is FI-CS, then there exists a DS subsemimodule $\mathcal{D}$ of $\mathcal{F}$, such that $\mathrm{Z}(\mathcal{F}) \leq^e \mathcal{D}$, and $\mathcal{F} = \mathcal{D} \oplus \mathcal{P}$ for some $\mathcal{P} \leq \mathcal{F}$, so $\mathrm{Z}(\mathcal{F}) = \mathrm{Z}(\mathcal{D}) \oplus \mathrm{Z}(\mathcal{P})$, however $\mathrm{Z}(\mathcal{D}) = \mathcal{D} \cap \mathrm{Z}(\mathcal{F}) = \mathrm{Z}(\mathcal{F})$. Therefore $\mathrm{Z}(\mathcal{P}) = 0$, hence $\mathcal{P}$ is nonsingular), by Remark (4.6), $\mathrm{Hom}(\mathcal{N}, \mathcal{P}) = 0$, so that $\mathcal{P}$ is $Z(\mathcal{F})$-injective . □

Proposition (4.8): For every FI-CS semimodule, $Z_2(\mathcal{F})$ is FI-CS direct summand of F.

Proof: Assume F is FI-CS, since $Z_2(\mathcal{F})$ is a fully invariant in F, then by Proposition (3.6), $Z_2(\mathcal{F})$ is FI-CS and there exists a direct summand F' of F such that$Z_2(\mathcal{F}) \leq^e F'$. But $Z_2(\mathcal{F})$ is closed, hence $Z_2(\mathcal{F})$ is a direct summand of F. □

5. Conclusion

In general every module is semimodule, however the converse is not true. Thus most of the results which are achieved in FI-CS module they are also achieved in FI-CS semimodule. In addition, the additive cancellative property has been added as a condition in both Lemma (3.10) and Proposition (3.11) in order to obtain the results of this study. Likewise, some Propositions have been proven, for example, Lemma (3.11), Proposition (3. 12), Proposition (4.2), and Corollary (4.3). We have assumed that the existence of the injective hull, considering that this feature is always present in the module, which is not necessarily available in the semimodule. As a final result, it is clear that the purpose of this study was achieved.

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