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F-J-semi Regular Modules

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Abstract
Let R be a ring with identity and let A be a left R-module. If F is a proper submodule of A and $x \in A$, x is called F-J-semi regular element in A , If there exists a decoposition $A = B \oplus C$ such that B is projective submodule of Rx and $Rx \cap C \ll_J F$. The aim of this paper is to introduce properties of F-J-semi regular module. In particular, its characterizations are given. Furthermore, we introduce the concepts of F −Jacobson hollow semi regular module and CF-J-semiregular module. Finally, many results of F −Jacobson hollow semi regular module and CF-J-semiregular module are presented.

Keywords: F-J- semi regular modules, R-F-J- semi regular modules , F-Jacobson hollow semi regular modules , CF-J-semi regular modules

المقاسات شبه المنتظمة من النمط F−J

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الخلاصة
الهدف من هذا البحث هو تقديم خصائص للموديول الجزئي شبه المنتظم من النمط J وبشكل خاص للموديول الجزئي من النمط F . كذلك تم تقديم تعريف للمقاس شبه المنتظم المجوف من النمط F−J و تعريف المقاس شبه المنتظم من النمط CF−J . اخيرا تم تقديم الكثير من النتائج حول للمقاس شبه المنتظم المجوف من النمط F−J و المقاس شبه المنتظم من النمط CF−J.

1- Introduction

Let R be a ring with identity and let A be a unitary left R-module. A submodule B of A is called small submodule if whenever $B + C = A$,then $C = A$ for some submodule of A. The small submodule is denoted by $B \ll A$ [1]. The sum of all small submodules is called the jacobson radical of A which is denoted by $J(A)$[2].
In [3], authors introduced J-small submodule. A submodule B of A is called J-small if whenever $B + C = A$ with $J\left(\frac{A}{C}\right) = \frac{A}{C}$ implies that $A = C$. It is denoted by $B \ll_J A$, It is clear that every small submodule of A is J-small,however, the converse is not to be true see [2].
An element x in a module A is called regular if $\alpha(x)x = x$ for some $\in A^*$. A module is called regular if each of its elements is regular[4]. Zelmanowitz [4] proved that a module is

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regular if and only if every cyclic submodule is a projective summan. He also introduced a semi-regular module. A module A is called semi-regular if for every non-zero cyclic submodule Rx of A there exists a projective submodule $B \leq Rx$ such that $A = B \oplus C$, and $C \cap Rx \ll A$ [4]. A submodule D of an R-module A is called J-lie over a projective summand of A if there exists a decomposition $A = B + C$, where B is projective submodule of D and $C \cap D \ll_J A$ [5] .

In [5], authors introduced a J-semi regular module. An R-module is called J-semi regular module if every cyclic submodule of A is J-lying over a projective summand of A.

Let F be a submodule of , an element $x \in A$ is called F-semi regular element in A if there exists a decomposition $A = B \oplus C$ such that B is a projective submodule of Rx and $\cap C \leq F$. A module A is called an F-semi regular module if m is F-semi regular element for each $m \in A$ [6].

This paper is devoted to introduce F- J -semi regular module, R- F - J-semi regular , F-Jacobson hollow semi regular, and $CF - J$- semi regular modules by using the concept of J-small submodule .

2- F-Jacobson semi regular modules:-

In order to introduce $F - J -$semi regular module , we use the concept of J-small submodule that is appeared in [3].

Definition 2.1 [3] : Let A be any R–module a submodule B of A is called Jacobson small (for short J–small , denoted by $B \ll_J A$) if whenever $A = B + C$, $C \leq A$, and $J(\frac{A}{C}) = \frac{A}{C}$, then $A = C$.

Lemma 2.2 [3] **:** Let , c be two submodules of an R–module A , if $B \leq C \leq A$ and $J(\frac{A}{B}) = \frac{A}{B}$, then $J(\frac{A}{C}) = \frac{A}{C}$.

Proof:

Let $f: \frac{A}{B} \rightarrow \frac{A}{C}$ be an epimorphism function which is defined by f $(a + B) = a + C$. From [2] we have $f\ (J(\frac{A}{B})) \leq J(\frac{A}{C})$. Hence $(\frac{A}{B}) = J(\frac{A}{C}) \leq J(\frac{A}{C})$. Therefore $(\frac{A}{C}) = \frac{A}{C}$.

Corollary 2.3 [3]: Let A be any R–module and let , C be two submodules of A . If $(\frac{A}{B}) = \frac{A}{B}$, then $J(\frac{A}{B+C}) = \frac{A}{B+C}$.

Definition 2.4 [5]: A submodule B of an R-module A is called J-lie over a projective summand of A if there exists a decomposition $A = C \oplus \acute{C}$, where C is projective submodule of B and $\acute{C} \cap B \ll_J A$.

An R-module A is called J-semi regular module if every cyclic submodule of A is J-lying over a projective summand of A [5].

Definition 2.5: Let A be an R-module ,let F be a proper submodule of A , an element $x \in A$ is called F-J-semi regular element in A , If there exists a decoposition $A = B \oplus C$ such that B is projective submodule of Rx and $Rx \cap C \ll_J F$.

Example 2.6: Consider Q as Z-module , $F = 2Z$ and $N = Z$, since Q is indecomposable , then $\{0\}$ is only projective summand of Z and $Q \cap Z = Z$ is not containd in $2Z$.

Proposition 2.7: Let A be a regular module, then A is F-J-semi regular for any $\leq A$.

Proof : For any $a \in A$, Ra is projective and direct summand of A , then there exists B submodule of A such that $A = Ra \oplus B$ and $Ra \cap B = 0 \ll_J F$ for any F submodule in A , therefore A is F-J-semi regular module .

Proposition 2.8: Let A be F-J-semi regular module and K be a submodule of A such that $F \leq K$, then K is F-J-semi regular.

Proof : Let $Rx \leq K$, so $Rx \leq A$. Since A is F-J-semi regular, then there exists a decomposition $A = B \oplus C$ such that B is projective in Rx and $Rx \cap C \ll_J F$. $A \cap K = (B \oplus C) \cap K = B \oplus (C \cap K)$ by modular law, we have $B \leq Rx$ and $Rx \cap (C \cap K) = (Rx \cap C) \cap K \ll_J F \cap K = F$, therefore K is F-J-semi regular module.

Proposition 2.9 : Every semisimple projective R-module A is F-J-semi regular module for every proper submodule F of A .

Proof: Let $Rx \leq A$, then $A = Rx \oplus B$, where $B \leq A$. Rx is projective and $Rx \cap B = \{0\} \ll_J F$, therefore A is an F-J-semi regular module.

Proposition 2.10:- Let A be an R-module and let K be a submodule of A. If A is F-J-semi regular, then K is $(K \cap F)$-J-semi regular where F proper submodule of A.

Proof: Let $Rx \leq K$, then $Rx \leq A$. Since A is F-J-semi regular, then there exists a decomposition $A = B \oplus C \ll_J F$ .From the modular law, we have $A \cap K = (B \oplus C) \cap K = B \oplus (C \cap K)$ and $K = B \oplus (C \cap K)$ $(C \cap K) \cap Rx = (C \cap Rx) \cap K \ll_J F \cap K$, therefore K is $(K \cap F)$-J-semi regular.

Before we give next proposition we have to recall that a sub-module B of an R-module A is called fully invariant if $g(B) \leq B$ for every $g \in End(A)$, where $End(A)$ is the ring of endemorphisms of$A$**.** A module A is called duo module if every submodule of A is fully invariant [7].

Proposition 2.11:

Let A be R-module and F is fully invariant submodule of A , then for every $x \in A$ the following statements are equivalent:

1- There exists a decomposition $A = B \oplus C$ such that B is projective submodule of D and $\cap\, C \ll_J F$.

2- There exists a homomorphism $\alpha: A \to D$ such that $\alpha^2 = \alpha$, $\alpha(A)$ is projective and $(I - \alpha)(D) \ll_J F$.

3- D can be written as $D = B \oplus S$, where B is projective summand and $S \ll_J F$.

Proof : $1 \to 2$ Let D be a submodule of A by assumption $A = B \oplus C$ where B is a projective submodule of D and $D \cap C \ll_J F$. From the modular law, we have $D = B \oplus (C \cap D)$, let $\alpha: A \to B$ be the projective map , it is clear that $\alpha^2 = \alpha$ and $\alpha(A)$ is a projective. Now consider the map $(I - \alpha): A \to C$ such that $(I - \alpha)(D) \leq C$, let $x \in (I - \alpha)(D)$, then $x = d - \alpha(d)$ for some $d \in D$, however $\alpha(x) \in B \leq D$, therefore $x \in D$ and $x \in D \cap C \ll_J F$, this implies that $(I - \alpha)(D) \leq D \cap C \ll_J F$.

$2 \to 1$) Assume that there exists a homomorphism $\alpha: A \to D$ such that $\alpha^2 = \alpha$, $\alpha(A)$ is projective and $(I - \alpha)(D) \ll_J F$. We claim that $A = \alpha(A) \oplus (I - \alpha)(A)$, let $a \in A$, then $a = a + \alpha(a) - \alpha(a) = \alpha(a) + a - \alpha(a)$ and$A = \alpha(A) + (I - \alpha)(A)$. Now, let $x \in \alpha(A) \cap (I - \alpha)(A)$, then $x = \alpha(a_1)$ and $x = (I - \alpha)(a_2)$ for some $a_1 , a_2 \in A$, so that $\alpha(x) = \alpha(a_1) = \alpha(a_2) - \alpha(a_2) = 0$, hence $\alpha(a_1) = 0$. This implies that $x = 0$, and $\alpha(A)$ is projective, let $t \in D \cap (I - \alpha)(A)$ then $t \in D$ and $t \in (I - \alpha)(A)$ since $t \in (I - \alpha)(A)$ then $t = (I - \alpha)(a)$, where $a \in A$, now $t = a - \alpha(a)$, hence $a \in D$ so that $t \in (I - \alpha)(D)$, therefore $D \cap (I - \alpha)(A) \leq (I - \alpha)(D) \ll_J F$.

$3 \to 1$) Let D be submodule of A, then by our assumption $D = B \oplus S$ where B is projective summand of A and $S \ll_J F$ and $A = B \oplus C$, for some submodule C of A. From the modular law, we have$D = B \oplus (D \cap C)$. Let $P: A \to C$ be a projection map, we claim that $P(S) = P(D \cap C)$, and $P(D) = P(B) \oplus P(S) = P(S)$. On other hand $P(D) = P(B) \oplus P(D) \oplus P(D \cap C) = P(D \cap C)$, thus $D \cap C = P(S) \leq P(F)$. Since F is fully invariant submodule of A, therefore $P(F) \leq F$ implies that $D \cap C \ll_J F$.

Proposition 2.12: Let A_1 and A_2 be R-modules such that $A = A_1 \oplus A_2$ is duo module, if A_1 and A_2 are F_1-J-semi regular and F_2-J-semi regular, respectively. Then A is $(F_1 \oplus F_2)$-J-semi regular module**.**

Proof : Let D be a cyclic submodule of A, so that $(D \cap A_i)$ for i =1,2 are cyclic submodule of A. Since A is duo module, then $D = (D \cap A_1) \oplus (D \cap A_2)$. Now, since A_i are F_i-J-semi regular, then there exists projective direct summand submodules of $D_i = (A_i \cap D)$ such that $A_i = B_i \oplus C_i$ and $D_i \cap C_i \ll_J F_i$ $for\ i = 1,2)$, and $A = A_1 \oplus A_2 = (B_1 \oplus C_1) \oplus (B_2 \oplus C_2)$. Since $B_1$and B_2 are projective then we have $B_1 \oplus B_2$ is also projective.
Now, from [3] we get $D \cap (C_1 \oplus C_2) = (D \cap A_1 \oplus D \cap A_2) \cap (C_1 \oplus C_2) = (D_1 \cap C_1) \oplus (D_2 \cap C_2) \ll_J F_1 \oplus F_2$.

Proposition 2.13: Let $A = \bigoplus_{i \in I} A_i$ be a direct sum of the submodules $\{A_i\}_{i \in I}$ of A, if A is F-J-semi regular, then each A_i is F_i-J-semi regular where $F_i = F \cap A_i$.

Proof : Let $x_i \in A_i$, since $x_i \in A$ and A is F-J-semi regular, it is implies that there exists $B_i \leq Rx_i$, where B_i is projective and it is a direct summand of A , since $A = B_i \oplus C_i$ such that $C_i \leq A\ and\ Rx_i \cap C_i \ll_J F$ for all $i \in I$. Now $A_i \cap A = A_i \cap (B_i \oplus C_i) = B_i \oplus (A_i \cap C_i)$, since B_i is a direct summand of A_i, where B_i is projective and $(C_i \cap A_i) \cap Rx_i \leq Rx_i \cap C_i \ll_J F$. Hence $(C_i \cap A_i) \cap Rx_i \leq A_i \cap F \leq F$, but $= \bigoplus_{i \in I} A_i$. Therefore, $F = \bigoplus_{i \in I} (A_i \cap F)$. Since $A_i \cap F$ is direct summand of F and $(A_i \cap C_i) \cap Rx_i \ll_J A_i \cap F$. Therefore, A_i is F_i-J-semi regular module for all i such that $i \in I$.

3- $\boldsymbol{R - F - J}$ −semi regular modules

This section is devoted to introduce $\boldsymbol{R - F - J}$ -semi regular module , which is a generalization of the $F - J$ −semi regular module .

Definition 3.1:-

Let A be any R-module and let F be a proper submodule of . A module A is called ***R*-*F*-*J*-semi regular** module if for each $x \in A$ such that $Rad_J(A) \leq F$, then there exists a projective summand submodule B of Rx such that $A = B⊕C$, $\leq A$, and $C \cap Rx \ll_J F$.

Examples 3.2:-

1. Z_6 as Z_6-module is R-F-J-semi regular module for every proper submodule F of Z_6.
2. Z_4 as Z-module is not R-$\langle\bar{2}\rangle$-J-semi regular module .
3. Every $F - J$ −semi regular module is $R - F - J$ −semi regular module. However, the converse is not true by (2)

Proposition 3.3:-

Let $A = A_1 \oplus A_2$ be a direct sum of a projective submodules A_i of for $i = 1,2$. If A is R-F-J-semi regular , then A_i is R-F_i-J-semi regular module for all $i = 1,2$, where $F = F_1 \oplus F_2$

Proof : Let $Rx_1 \leq A_1$ such that $Rad_J(A_1) \leq Rx_1$ so that $Rad_J(A) \leq RX_1 + Rad_J(A)$, since A is R-F-J-semi regular , then there exists a projective submodule B $B \leq Rx$, where B is a direct summand of A , and $A = B \oplus C$ for some $C \leq A$ and $C \cap (Rx_1 + Rad_J(A)) \ll_J F$. Hence $A_1 = (B \oplus C) \cap A_1 = (B \cap A_1) \oplus (C \cap A_1)$, since A_1 is projective , then $B \cap A_1$ is projective , now $(C \cap A_1) \cap (Rx_1 + Rad_J(A)) \leq C \cap Rx_1 + Rad_J(A) \ll_J F = F_1 \oplus F_2$, and $B \cap Rx_1 \leq F_1$,because of F_1 is direct summand of F , then $C \cap Rx_1 \ll_J F_1$, by[3] . By the same way one can get for A_2 .

Proposition 3.4:-

Let A be a duo module such that $A = A_1 \oplus A_2$. If A_i is R-F-J-semi regular module $(\forall i = 1,2)$, then A is R-F-J-semi regular module where $F = F_1 \oplus F_2$

Proof : Let Rx be a cyclic submodule of A such that $Rad_J(A) \leq Rx$, then $Rad_J(A) \cap A_i \leq Rx \cap A_i, i = 1,2$, but $Rad_J(A_i) \leq Rad_J(A) \cap A_i \leq Rx \cap A_i$ and A_i is R-F-J-semi regular module for all $i = 1,2$, then there exists a projective submodule $B_i \leq Rx \cap A_i$ and $A_i = B_i \oplus C_i$ for some $C_i \leq A_i$, $(i = 1,2)$ with $C_i \cap Rx_i \ll_J F_i$. Now $= A_1 \oplus A_2 = (B_1 \oplus B_2) \oplus (C_1 \oplus C_2)$, $B_1 \oplus B_2$ is projective and $(C_1 \oplus C_2) \cap Rx = (C_1 \oplus C_2) \cap (Rx \cap A_1) \oplus (Rx \cap A_1) \oplus (Rx \cap A_2)) = (C_1 \cap Rx \cap A_1) \oplus (C_2 \cap Rx \cap A_2) \leq (C_1 \cap Rx_1) \oplus (C_2 \cap Rx_2) \ll_J F_1 \oplus F_2 = F$ [3] .

proposition 3.5:-

Let A be an R-module and D be a submodule of A such that $Rad_J(A) \leq D$ and F is a proper fully invariant submodule of , the following are equivalent :

1. There exists a decomposition $A = B \oplus C$ such that B is projective submodule of D and $\cap C \ll_J F$.
2. There exists a homomorphism $\alpha: A \to D$ such that $\alpha^2 = \alpha$, $\alpha(A)$ is projective and $(I - \alpha)(D) \ll_J F$.
3. D can be written as $D = B \oplus S$, where B is projective summand and $S \ll_J F$.

Proof :$1 \to 2$) Let D be a submodule of A , From our assumption, we have $A = B \oplus C$ where B is a projective submodule of D and $D \cap C \ll_J F$ by the modular law , $D = B \oplus (C \cap D)$. Let $\alpha: A \to B$ be the projective map , it is clear $\alpha^2 = \alpha$ and $\alpha(A)$ is a projective. Now consider the map $(I - \alpha): A \to C$ and $(I - \alpha)(D) \leq C$ if $\in (I - \alpha)(D)$, then $x = d - \alpha(d)$ for some $d \in D$ but $\alpha(x) \in B \leq D$. Therefore $x \in D$ and $x \in D \cap C \ll_J F$. This implies that $(I - \alpha)(D) \leq D \cap C \ll_J F$**.**

$2 \to 1$) Assume that there exists a homomorphism $\alpha: A \to D$ such that $\alpha^2 = \alpha$, $\alpha(A)$ is projective and $(I - \alpha)(D) \ll_J F$. Claim that $A = \alpha(A) \oplus (I - \alpha)(A)$ if $a \in A$, then $= a + \alpha(a) - \alpha(a) = \alpha(a) + a - \alpha(a)$, thus $A = \alpha(A) + (I - \alpha)(A)$. Now let $x \in \alpha(A) \cap (I - \alpha)(A)$ and $= \alpha(a_1)$, $x = (I - \alpha)(a_2)$ for some $a_1 , a_2 \in A$, so that $\alpha(x) = \alpha(a_1) = \alpha(a_2) - \alpha(a_2) = 0$, hence $\alpha(a_1) = 0$ and hence $x = 0$ $\alpha(A)$ is projective , if $t \in D \cap (I - \alpha)(A)$, then $t \in D$ and $t \in (I - \alpha)(A)$, since $t \in (I - \alpha)(A)$ it implies that $t = (I - \alpha)(a)$

Where $\in A$, now $t = a - \alpha(a)$ and hence $a \in D$ so that $t \in (I - \alpha)(D)$.Therefore $D \cap (I - \alpha)(A) \leq (I - \alpha)(D) \ll_J F$.

$3 \to 1$) Let D be submodule of A so that by our assumption $D = B \oplus S$ where B is projective summand of A and $S \ll_J F$ and then $A = B \oplus C$ for some submodule C of A , by the modular law $D = B \oplus (D \cap C)$ let $P: A \to C$ the projection map , claim that $P(S) = P(D \cap C)$, $P(D) = P(B) \oplus P(S) = P(S)$. On other hand, $P(D) = P(B) \oplus P(D) \oplus P(D \cap C) = P(D \cap C)$ thus $D \cap C = P(S) \leq P(F)$, because of F is fully invariant submodule of , therefore $P(F) \leq F$ and $D \cap C \ll_J F$.

4- FJ-hollow semi regular and CF-J-semi regular modules

We introduce the concepts of F –Jacobson hollow semi regular module and CF-J-semiregular module was introduced. Some of their properties are also investigated .

Definition 4 . 1 :- Let A be R-module and F a proper submodule of A. A proper submodule A is called F-jacobson hollow semi regular (for short FJ –hollow semi regular) if for any cyclic submodule B of A with $\frac{A}{B}$ is J –hollow , then there exists C a projective submodule of B such that $A = C \oplus \acute{C}$ where $\acute{C} \leq A$ and $\acute{C} \cap B \ll_J F$.

Proposition 4.2:-

Let $A = A_1 \oplus A_2$ be a duo module , if A_i FJ-hollow semi regular $(i = 1,2)$, then A is FJ-hollow semi regular where $F = F_1 \oplus F_2$ provided that $B \cap A_i \neq A_i$ $(i = 1,2)$.

Proof : let B be a cyclic submodule of A such that $\frac{A}{B}$ J-hllow , so that $B = B \cap A_1 \oplus B \cap A_2$ $\frac{A}{B} \cong \frac{A_1 \oplus A_2}{(B\cap A_1)\oplus(B\cap A_2)} \cong \frac{A_1}{B\cap A_1} \oplus \frac{A_2}{B\cap A_2}$,because of $\frac{A}{B}$ is J- hollow , then by [8] , $\frac{A_1}{B\cap A_1}$ and $\frac{A_2}{B\cap A_2}$ are J-hollow . Thus, there exists $C_i \leq A_i$ where C_i is projective summand of A_i i.e $\exists \acute{C_\iota} \leq A_i$ such that $A_i = c_i \oplus \acute{C_\iota}$ and $\acute{C_\iota} \cap (B \cap A_i) \ll_i F_i$

Now $A = A_1 \oplus A_2$=$(C_1 \oplus \acute{C_1}) \oplus (C_2 \oplus \acute{C_2}) = (C_1 \oplus C_2) \oplus \acute{C_1} \oplus \acute{C_2})$, $C_1 \oplus C_2$ is projective and $B \cap \acute{C_1} \oplus \acute{C_2}) = (B \cap A_1) \oplus (B \cap A_2) \cap (\acute{C_1} \oplus \acute{C_2}) = (B \cap A_1 \cap \acute{C_1}) \oplus (B \cap A_2 \cap \acute{C_2}) \ll_J F_1 \oplus F_2$ by [9], then $B \cap \left(\acute{C_1} \oplus \acute{C_2}\right) \ll_J F$. Therefore A is FJ −hollow semi regular module .

Proposition 4 . 3 :- Let $A = A_1 \oplus A_2$, if A is FJ-hollow semi regular module, then A_1 and A_2 are F_iJ-hollow semi regular where $F = F_1 \oplus F_2$ provided that $\frac{A}{B_i}$ is J-hollow for each $B_i \leq A_i$ $(i = 1,2)$

Proof : let $B_1 \leq A_1$ such that B_1 is cyclic and $\frac{A_1}{B_1}$ is J-hollow , $B_1 \leq A_1 \leq A$, so that B_1 is a cyclic submodule of A , but $\frac{A}{B_i}$ is J-hollow , then there exists $C_1 \leq B_1$, C_1 is projective summand of A and $A = C_1 \oplus \acute{C_1}$ $\acute{C_1} \leq A_1$, and $B_1 \cap \acute{C_1} \ll_J F$, since $B_1 \cap \acute{C_1} \leq F_1 \leq F$ and F_1 is a direct summand of F , then $B_1 \cap \acute{C_1} \ll_J F_1$ [3]. Now $A_1 \cap A = A_1 \cap (C_1 \oplus \acute{C_1}) = C_1 \oplus (A_1 \cap \acute{C_1})$

$B_1 = C_1 \oplus S_1$ and $S_1 \ll_J A$, and $A_1 \cap A = A_1 \cap (C_1 \oplus \acute{C_1}) = C_1 \oplus (A_1 \oplus \acute{C_1})$ thus C_1 is projective summand of A_1 in B , $B \cap (A_1 \cap \acute{C_1}) = A_1 \cap B \cap \acute{C} \leq B \cap \acute{C_1} \ll_J F_1$, then $B \cap (A_1 \cap \acute{C_1}) \ll_J F_1 \to A_1$ is F_1J −hollow semi regular .

Corollary4. 4 :- If $A = A_1 \oplus A_2 \oplus ... \oplus A_n$ be a duo module , then A is FJ −hollow semi regular if and only if A_i is F_iJ −hollow semi regular where $F = F_1 \oplus F_2 \oplus ... \oplus F_n$ provided $A_i \cap B \neq A_i$,$\forall B \leq A$ $\forall i = 1,2, \dots, n$.

Proposition 4.5:- Let A_1 and A_2 be J −hollow modules provided that $A_i \cap B \neq A_i$ $(i = 1,2)$, where B is a cyclic submodule of A .

if $A = A_1 \oplus A_2$, then the following are equivalent

1. A is FJ −hollow semi regular .
2. A is FJ −semi regular .

Proof : 1 → 2) Let B be a cyclic submodule of A . since $A_1$and A_2 are J −hollow , then A is J −hollow and $\frac{A}{B}$ J −hollow by [9] but A is FJ −hollow semi regular , then there exists C a submodule projective of B such that $A = C \oplus \acute{C}$, $\acute{C} \leq A$ and $\acute{C} \cap A \ll_J F$, then A is FJ −semi regular .

2 → 1) it is easy to prove that .

Recall that a submodule B of an R-module A is called cofinite if $\frac{A}{B}$ is finitely generated [10].

Definition 4.6:- Let A be an R-module and F be a proper submodule of A , then A is called cofinitely F-Jacobson semi regular module (for short CFJ −semi regular module) if for any coffinite submodule B of A , there exists C projective submodule of B such that $A = C \oplus \acute{C}$ and $B \cap \acute{C} \ll_J F$ where $\acute{C} \leq A$.

Examples 4.7:-

1. Z as z-module , let $F = 3Z$ $2Z < Z$, such that $\frac{Z}{2Z} \cong Z_2$ is finitely generated , there exists $0 < 2Z$ such that 0 is projective and $Z = 0 \oplus Z$ $Z \cap 2Z = 2Z$ is not J −small in $3Z$ since $2Z$ is not containd $3Z$, then Z as Z-module is not $C3Z$ −semi regular .

2. Z_6 as Z_6-module , $F = \langle\bar{3}\rangle$ $\langle\bar{2}\rangle < Z_6$, $\frac{Z_6}{\langle\bar{2}\rangle} \cong \langle\bar{3}\rangle$ is finitely generated , then there exists $\langle\bar{2}\rangle \leq Z_6$ is projective $\langle\bar{2}\rangle \oplus \langle\bar{3}\rangle = Z_6$ $\langle\bar{2}\rangle \cap \langle\bar{3}\rangle = 0 \ll_J \langle\bar{3}\rangle$. Therefore, Z_6 is $C\langle\bar{3}\rangle$ −semi regular .

Proposition 4.8:- Let A be CFJ −semi regular R-module , let $B \leq F$ submodule of A such that $\frac{B+C}{B}$ is projective for any C projective submodule of A , then $\frac{A}{B}$ is $C\ {}^{F}/_{B} - J$ −semi regular module .

Proof: Let $\frac{D}{B}$ cofinite submodule of $\frac{A}{B}$. $\frac{A/_B}{D/_B} \cong \frac{A}{D}$ is finitely generated , since A is CFJ −semi regular , then there exists C is projective submodule of D such that $A = C \oplus \acute{C}$, $\acute{C} \cap D \ll_J F$. $\frac{A}{B} = \frac{C \oplus \acute{C}}{B} = \frac{C+B}{B} \oplus \frac{\acute{C}+B}{B}$, $\frac{C+B}{B}$ is projective in $\frac{D}{B}$, now $\frac{\acute{C}+B}{B} \cap \frac{D}{B} = \frac{(\acute{C}+B)\cap D}{B} \ll_J \frac{F}{B}$, then $\frac{A}{B}$ is $C\, {}^{F}/_{B} J$ −semi regular module .

Proposition 4.9:- Let $= A_1 \oplus A_2$, then A_1 and A_2 are $CF_i - J$ −semi regular if and only if A is $CF - J$ −semi regular where $F = F_1 \oplus F_2$.

Proof : let L be a cofinite submodule of A , $\frac{A}{L} = \frac{A_1+L}{L} \oplus \frac{A_2+L}{L}$, so that $\frac{A/_L}{A_2+L/_L} \cong \frac{A_1+L}{L} \cong \frac{A_1}{A_1 \cap L}$ is finitely generated ,then $A_1 \cap L$ is cofinite in A_1 , since A_1 is CF_1J −semi regular then there exists C_1 a projective submodule of $A_1 \cap L$ such that $A_1 = C_1 \oplus \acute{C}_1$ and $\acute{C}_1 \cap L \ll_J F_1$ and simillarly there exists C_2 a projective submodule of $A_2 \cap L$ such that $A_2 = C_2 \oplus \acute{C}_2$ and $\acute{C}_2 \cap L \ll_J F_2$

$A = A_1 \oplus A_2 = (C_1 \oplus C_2) \oplus (\acute{C}_1 \oplus \acute{C}_2)$, $C_1 \oplus C_2$ is projective and $(\acute{C}_1 \oplus \acute{C}_2) \cap L = (\acute{C}_1 \cap L) \oplus (\acute{C}_2 \cap L) \ll_J F_1 \oplus F_2$, then $(\acute{C}_1 \oplus \acute{C}_2) \cap L \ll_J F$, then A is $CF - J$ −semi regular .

Conversly, let L cofinite submodule in A_1 , $\frac{A}{L} = \frac{A_1 \oplus A_2}{L} = \frac{A_1}{L} \oplus \frac{A_2+L}{L}$, $\frac{A/_L}{A_2+L/_L} \cong \frac{A_1}{L}$, so that $\frac{A}{L}$ finitely generated , thus there exists C a projective submodule of L such that $A = C \oplus \acute{C}$ and $L \cap \acute{C} \ll_J A$, $A = C \oplus \acute{C}$ and $A_1 \cap A = A_1 \cap (C \oplus \acute{C}) = C \oplus (A_1 \cap \acute{C})$, $(A_1 \cap \acute{C}) \cap L = (A_1 \cap (\acute{C} \cap L)) \leq \acute{C} \cap L \ll_J F$ and F_1 is direct summand of F . Hence, $A_1 \cap \acute{C} \cap L \ll_J F_1$ by [3] , therefore A_1 is $CF_1 - J$ −semi regular module .

Proposition 4.10 :- If A is a projective R-module and CFJ −semi regular module where F is proper submodule of A , then $\frac{A}{B}$ has projective J-cover for every cofinite submodule B of A .

Proof : Let A be a projective and CFJ—semi regular module and B a cofinite submodule of A, so that there exists a decomposition $A = C \oplus \acute{C}$ such that C is a projective submodule of B and $B \cap \acute{C} \ll_J F$ since $F \leq A$, then $B \cap \acute{C} \ll_J A$, consider $\pi: \acute{C} \to \frac{\acute{C}}{(\acute{C} \cap B)}$ epimorphism and Ker (π) = $(\acute{C} \cap B)$. By the second isomorphism theorem $\frac{A}{B} = \frac{B+\acute{C}}{B} \cong \frac{\acute{C}}{B \cap \acute{C}}$, since $(B \cap \acute{C}) \leq \acute{C} \leq A$ and $(B \cap \acute{C}) \ll_J A$ and $\acute{C}$ is a direct summand of A , then $(B \cap \acute{C}) \ll_J \acute{C}$. Therefore $\frac{A}{B}$ has a projective J-cover.

Proposition 4.11:- Let A be an indecomposable finitely generated R-module and F is a proper submodule of A , if A is $CF - J$ −semi regular , then A is J-semi hollow .

Proof : Let $B < A$ be a proper cofinite submodule in A and since A is $CF - J$ −semi regular , then there exists a decomposition $A = C \oplus \acute{C}$ such that $C \leq B$ and C is projective and $B \cap \acute{C} \ll_J F$, but A is indecomposable , then either $C = 0$ or $C = A$ if $C = A$, then $B = A$,

we get a contradiction , thus $C = 0$ and $\acute{C} = A$ imply that $B \cap \acute{C} = B \ll_J F \leq A$ and $B \ll_J A$, therefore A is J-semi hollow.
Recall that $Rad_J(A)$ is the sum of all J-small submodules of A **[3] .** It is clear that $(A) \leq Rad_J(A)$. However, the converse in general is not true see **[3] .**
Definition 4 . 12 :- Let A be an R-module. If F proper in A , then A is called F-Rad_J-semi regular module and if for each cyclic Rx in A such that $Rad_J(A) \leq Rx$, then there exists a decomposition $A = C \oplus \acute{C}$, where C is a projective submodule of Rx and $\acute{C} \cap Rx \ll_J F$.

Example 4.13:-
1- Consider Z_6 as Z_6-module, and $F = \langle\bar{2}\rangle$ proper in Z_6 $Rad_J(Z_6) = Z_6$, then $Rad_J(Z_6) \leq \langle\bar{1}\rangle$ and Z_6 is projective summand $Z_6 = Z_6 \oplus \{0\}$ and $Z_6 \cap \{0\} = \{0\} \ll_J F$ then Z_6 is F-Rad_J-semi regular module .
2- Consider Z as Z-module , and F any proper in Z , $Rad_J(Z) = Z$, then Z has projective summand $Z = Z \oplus \{0\}$, and $Z \cap \{0\} = \{0\}$ $\{0\} \ll_J F$, then Z is F-Rad_J-semi regular module .
3- Consider Z_4 as Z-module and $F = \langle\bar{2}\rangle$, $Rad_J(Z_4) = \langle\bar{2}\rangle$ since $Z_4 = Z_4 \oplus \{0\}$ such that only $\{0\}$ is projective submodule in Z_4 such that $Rad_J(Z_4) \leq Z_4$ and $Z_4 \cap Z_4 = Z_4$ but Z_4 is not J-small in F , then Z_4 as Z-module is not F-Rad_J-semi regular module .
Proposition4.14:- If A is a non-cyclic J-semi hollow R-module and F is proper direct summand of A , then A is F-Rad_J-semi regular module .
Proof : Let Rx be a submodule of A such that $Rad_J(A) \leq Rx$, $A = A \oplus \{0\}$ and $Rx \cap A = Rx$ since $Rx \neq A$, then $Rx \ll_J A$ and $Rx \ll_J F$, therefore A is F-Rad_J-semi regular module .
Proposition 4.15:- Let A_1 , A_2 be R-modules and $A = A_1 \oplus A_2$ be a duo module, then A_1 and A_2 are F-Rad_J-semi regular module if and only if A is F-Rad_J-semi regular module, where $= F_1 \oplus F_2$, F_i proper in F_i for $i = 1,2$**.**
Proof : Let $Rx \leq A$ such that $Rad_J(A) \leq Rx$, since A is duo module , then $Rx = (Rx \cap A_1) \oplus (Rx \cap A_2)$ and $Rad_J(A_i) \leq Rad_J(A) \cap A_i \leq Rx \cap A_i$ [3].
Since A_i is F-Rad_J-semi regular module $(i = 1,2)$, then there exists a projective summand of $A_i , C_i \leq A_i \cap Rx , (i = 1,2)$ such that $A_i = C_i \oplus \acute{C}_i$ for $\acute{C}_i \leq A_i , (i = 1,2)$ and $\acute{C}_i \cap (A_i \cap Rx_i) \ll_J F_i$, put $C = C_1 \oplus C_2$, where C is a projective summand of A and $A = (C_1 \oplus C_2) \oplus (\acute{C}_1 \oplus \acute{C}_2)$, $(\acute{C}_1 \oplus \acute{C}_2) \cap Rx = (\acute{C}_1 \oplus \acute{C}_2) \cap (Rx \cap A_1) \oplus (Rx \cap A_2) =$
$(\acute{C}_1 \cap Rx \cap A_1) \oplus (\acute{C}_2 \cap Rx \cap A_2) \ll_J F_1 \oplus F_2 = F.$
Conversely , let $Rx_i \leq A_i$, such that $Rad_J(A_i) \leq Rx_i, (i = 1,2)$, then from [3], we have $Rad_J(A_1) \oplus Rad_J(A_2) = Rad_J(A) \leq Rx_1 \oplus Rx_2$, since A is F-Rad_J-semi regular module , then there exists a projective summand C of A such that $C \leq Rx_1 \oplus Rx_2$, $A = C \oplus \acute{C}$, $\acute{C} \leq A$, and $\acute{C} \cap (Rx_1 \oplus Rx_2) \ll_J F$. Now $A_i = A_i \cap \left(C \oplus \acute{C}\right) = A_i \cap C \oplus \left(A_i \cap \acute{C}\right)$. Since $x_i \leq A_i$, then $A_i \cap C \leq A_i$ and $Rx_i \cap C \leq A_i \, for \, i = 1,2$ $\left(A_i \cap \acute{C}\right) \cap Rx_i \leq \left(\acute{C} \cap Rx_i\right) \ll_J F, \acute{C} \cap Rx_i \leq F_i$ since F_i is a direct summand of F $for \, i = 1, 2,$ from [8],we get$\left(\acute{C} \cap Rx_i \cap A_i\right) \ll_J F_i$.
Proposition 4.16:- If A be an R-module , $Rad_J(A) \leq B$ is a submodule of A , and F is proper fully invariant submodule of A , then the following statements are equivalent:
1- There exists a decomposition $A = C \oplus \acute{C}$, such that$C \leq B$ and C is a projective summand of A and $B \cap \acute{C} \ll_J F$
2- There exists a homomorphism $\alpha: A \to B$ such that $\alpha^2 = \alpha, \alpha(A)$ is projective and $(I - \alpha)(A) \ll_J F$

3- Let B can be written as $B = C \oplus S$, where C is projective summand of A and $S \ll_J F$

Proof : $1 \to 2$) Let $B \leq A$ such that $Rad_J(A) \leq B$ by our assumption, we get $A = C \oplus \acute{C}$, where C is projective submodule of B and $\acute{C} \leq A$ with $B \cap \acute{C} \ll_J F$. Therefore $B = B \cap (C \oplus \acute{C}) = C \oplus (B \cap \acute{C})$.

Let $\alpha: A \to B$ the projection homomorphism .It is clear that $\alpha^2 = \alpha$ and $\alpha(A)$ is projective. Now consider the map $(I-\alpha): A \to \acute{C}, (I-\alpha)(\acute{C}) \leq \acute{C}$, let $x \in (I-\alpha)(B)$ then there exists $b \in B$ such that $x = (I-\alpha)(b) = b - \alpha(b)$, but $\alpha(x) \in C \leq B$, thus $x \in B$ and $x \in B \cap \acute{C} \ll_J F$ and $(I-\alpha)(B) \leq B \cap \acute{C} \ll_J F$.

$2 \to 3$) Suppose that there exists a homomorphism $\alpha: A \to B$ such that $\alpha^2 = \alpha$, $\alpha(A)$ is projective and $(I-\alpha)(A)A \ll_J F$. Claim that $A = \alpha(A) \oplus (I-\alpha)(A)$, let $a \in A$, then $a = a + \alpha(a) - \alpha(a) = \alpha(a) + a - \alpha(a) = \alpha(a) + (I-\alpha)(a)$, thus $A = \alpha(A) + (I-\alpha)(A)$, if $x \epsilon \alpha(A) \cap (I-\alpha)(A)$, then $x = \alpha(a_1)$ and $x = (I-\alpha)(a_2)$ for some $a_1, a_2 \in A$, it implies that $\alpha(x) = \alpha(a_1) = \alpha(a_2) - \alpha^2(a_2) = \alpha(a_2) - \alpha(a_2) = 0$, therefore $x = 0$ and $A = \alpha(A) \oplus (I-\alpha)(A)$, $\alpha(A)$ is projective. Now, let $y \in B \cap (I-\alpha)(A)$ so that $\in B$, $y \in (I-\alpha)(A)$, and $\in (I-\alpha)(B)$, hence $B \cap (I-\alpha)(A) \leq (I-\alpha)(A) \ll_J F$. If one takes $\alpha(A) = C, (I-\alpha)(A) = \acute{C}$, then we get the statement 1.

$3 \to 1$) Let $B \leq A$ such that $ad_J(A) \leq B$ so that $B = C \oplus S$, where C is projective and $S \ll_J F$ hence $A = C \oplus \acute{C}$ for $\acute{C} \leq A$ such that $\acute{C} \cap B = (C \oplus S) \cap \acute{C} = S \cap \acute{C}$, but $S \cap \acute{C} \leq S \ll_J F$, then $B \cap \acute{C} \ll_J F$.

Proposition 4.17:- Every semi simple projective R-module A is F-Rad_J-semi regular module.

Proof : Let Rx be submodule of A such that Rad_J-$(A) \leq Rx$ and F is proper submodule of A, since A is semi simple , then Rx is a direct summand of A , $A = Rx \oplus C$ for some submodule C of M and since A is projective , then Rx is projective . Now, $Rx \cap \acute{C} = \{0\} \ll_J F$, therefore A is F-Rad_J-semi regular module.

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