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On Indeterminacy (Neutrosophic) of Hollow Modules

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Abstract

In this paper, we formulate and study a new property, namely indeterminacy (neutrosophic) of the hollow module. We mean indeterminacy hollow module is neutrosophic hollow module B (shortly Ne(B)) such that it is not possible to specify the conditions for satisfying it. Some concepts have been studied and introduced, for instance, the indeterminacy local module, indeterminacy divisible module, indeterminacy indecomposable module and indeterminacy hollow-lifting module. Also, we investigate that if Ne(B) is an indeterminacy divisible module with no indeterminacy zero divisors, then any indeterminacy submodule Ne(K) of Ne(B) is an indeterminacy hollow module. Further, we study the relationship between the indeterminacy of hollow and indecomposable modules. Finally, conclusions and discussions for our obtained results are given.

Keywords: Hollow module, Cyclic module, Indecomposable module, Local module, Indeterminacy hollow-lifting.

حول اللاتحديد (نيوتروسوفك) للمقاسات المجوفة

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الخلاصة

في هذا البحث قمنا بصياغة ودراسة خاصية جديدة تسمى اللاتحديد (نتر وسوفك) للمقاس المجوف. نعني بالمقاس المجوف غير المحدد هو المقاس المجوف B (اختصارا (Ne(B)) بحيث لا يمكن تحديد شروط تحققه. تمت دراسة بعض المفاهيم وتقديمها على سبيل المثال, المقاس المحلي اللامحدد , المقاس غير القابل للقسمة اللامحدد والمقاس الغير قابل للتحلل واللامحدد ومقاس الرفع– المجوف غير المحدد. أيضًا نتحرى ما إذا كانت Ne(B) مقاس قابل للقسمة لامحدد بدون قواسم صفرية, لذا فإن أي مقاس جزئي غير محدد (Ne K)) من Ne (B) هي مقاس مجوف غير محدد. كذ لك قمنا بدراسة العلاقات بين المقاس النمطي المحدد والمقاس النمطي اللامحدد الغير قابل للتحلل .اخيرا تم اعطاء استنتاجات ومناقشة للنتائج التي تم الحصول عليها

1. Introduction

All rings are commutative with unity and all modules are unital modules. In our daily life, there are many events in which no clear decision can be made. Therefore, we rely on the principle of indeterminacy as a way to study and take the appropriate decision neutrosophic (indeterminacy) as a new way of philosophy that studies the originality of the concepts.

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Many algebraic structures were studied such as rings, ideals, submodules, and modules [1- 6]. Some researchers have studied neutrosophic of some algebraic concepts such as neutrosophic set, neutrosophic submodule, neutrosophic module, neutrosophic group, and neutrosophic ring in general [7, 8, 9, 10, 11]. In this paper, we will choose one of the most important modules namely the hollow module and study its neutrosophic and its relationships with other concepts such as indecomposable module and hollow-lifting modules. Vildan et al. [12] studied neutrosophic submodules and neutrosophic modules. In [14], any r submodule N of M is called semismall, if either N=0 or for each nonzero submodule K of N, N/K is a small in M/K. More details about semismall submodules can be found in [15]. A neutrosophic zero divisors means; if we Ne(x) and Ne(y) are two nonzero neutrosophic elements, so Ne(x) Ne(y)=0. On the other hand, our notion of neutrosophic hollow module cannot be interpreted too broadly. Although neutrosophic local modules are neutrosophic hollow modules. However, the converse is true only when Ne(CB) is neutrosophic cyclic module. An A-module B is called neutrosophic hollow-lifting (Ne(HLB)) if every neutrosophic submodule Ne(K) of Ne(B) leads to Ne(B)/Ne(K) is a hollow and has a coessential neutrosophic submodule that is a neutrosophic direct summand of Ne(B) such that Ne(H) is called neutrosophic coessential submodule of Ne(K) in Ne(B) (briefly Ne(H) $\ll_{coess}$Ne(K) in Ne(B) if Ne(K) / Ne(H) $\ll$ Ne(B) / Ne(H).

In this paper, we formulate an idea of a neutrosophic hollow module. Then, we study this idea and try to improve the strong algebraic idea of a neutrosophic hollow module. Also, we introduce important results about neutrosophic hollow modules that relate to the neutrosophic indecomposable module, neutrosophic lifting module, and neutrosophic hollow-lifting modules.

2. Preliminaries

Definition (2.1). [12] Let Ω be an initial universe. A fuzzy set is S: $\Omega \to [0,1]$. So we have a degree of membership of p in FΩ, p$\in \Omega$.

Example (2.2). Let $\Omega = \{2, 4, 7, 8, 9\}$, so the fuzzy set $\Phi=\{(2,0), (4, 0.1), (7, 0.2), (8, 0.8), (9, 1)\}$. We can say the number 2 is not maximum, hence 2 has the membership is 0. However, we say that numbers from 4 to 9 have degrees 0.1, 0.2 and 0.8. Finally, 9 is a maximum and has a complete membership degree.

Remark (2.3): From example (2.2), we can define neutrosophic by:

X(I)=y+2I: y, 2∈R or ℂ, and y is a specified part on X(I) and ZI defined with $Z_1I+Z_2I=(Z_1+Z_2)I$.

Now we can introduce a definition of neutrosophic set by the following:

Definition(2.4). [10]. Let X ≠ φ be a set. Any set which is generated by X and I is called neutrosophic and denoted by X(I).

Definition (2.5). [10]. Let (B, +, .) be a left A-module. We consider Ne(B) as a neutrosophic and denoted by $_RE$ and I. Then, Ne(B) = (E(I), +, .) is called a weak neutrosophic A-module.

Definition (2.6). [11]. Let Ne(B) be a neutrosophic module over neutrosophic ring A. We say Ne(K) is a neutrosophic submodule if the following conditions hold:

1) Ne (K) ≠ A.
2) Ne(K) ⊆ Ne(B).
3) Ne(K) is an A-module and has a proper subset.

Definition(2.7). Let K be a neutrosophic submodule of neutrosophic module B. We say that K is a neutrosophic small submodule of Ne(B) {simply Ne(K)<<Ne(B)} if there exists Ne(H) submodule of Ne(B) (i.e. Ne(H)↪Ne(B)) ∋ Ne(H)+Ne(K)=Ne(B), so Ne (H)=Ne(B).

Remark (2.8). Any neutrosophic A-module B has only one neutrosophic maximal submodule is called a neutrosophic local module. Also, neutrosophic maximal submodule is equal to the neutrosophic of radical B (i.e. Ne(K) = Ne(Rad (B)) and hence Ne(Rad (B)) << Ne(B).

3. Neutrosophic Hollow Modules

In this section, we introduce some results about a neutrosophic hollow module that relates to other concepts such as the neutrosophic local module and neutrosophic indecomposable module.

Definition (3.1). Any module B is called neutrosophic hollow if every proper neutrosophic submodule Ne(K) is a neutrosophic small in Ne(B), (i.e. Ne(K) << Ne(B)).

Remarks and Examples: (3. 2).

1) Z_4 is a neutrosophic hollow module.
2) Z as a Z-module is not a neutrosophic module.
3) Every neutrosophic local module is a neutrosophic hollow module. In general the converse is not true.

Theorem (3.3). Let Ne(B) be a neutrosophic module over a neutrosophic ring Ne(A). If B is a neutrosophic hollow cyclic module Ne(HC), then it is a neutrosophic local module.

Proof:

Suppose that B is neutrosophic hollow cyclic module (Ne(HC)). Then B is a neutrosophic finitely generated module that means Ne(B)=∑Ne(A)Ne(x_1)+Ne(A)Ne(x_2)+...+Ne(A)Ne(x_n). Hence, B has neutrosophic maximal submodule K. Let Ne(H) be a neutrosophic submodule of neutrosophic module B. If

Ne(H) $\nsubseteq$ Ne(K), then Ne(B) = Ne(K)+Ne(H).

But we have B is a neutrosophic hollow module. Hence, Ne(K)=Ne(B) and this contradiction. Therefore, any neutrosophic submodule of Ne(B) is contained in Ne(K). So, Ne(B) has the largest neutrosophic submodule. Thus, Ne(B) is a neutrosophic local module.

Definition (3.4). Any module Ne(B) is called neutrosophic divisible if Ne(B) = Ne(b)Ne(B) such that Ne(b) is an element of Ne(B).

Theorem (3.5). Let Ne(B) be a neutrosophic module of neutrosophic ring Ne(A). If B is a neutrosophic divisible with no neutrosophic zero divisors, Ne(K) is a neutrosophic hollow module, and for all Ne (I), Ne(K) = Ne (I) Ne(B), then Ne(B) is a neutrosophic local module.

Proof: Assume that 0≠Ne(H) = Ne(I) Ne(B and Ne(H) is a neutrosophic submodule of Ne(B). Then, Ne(H) =Ne(I) Ne(B)=Ne(B). Hence B is a neutrosophic simple module (neutrosophic cyclic module). But B is a neutrosophic hollow module. Therefore, B is a neutrosophic local module.

Example (3.6). Let Ne(B) be a neutrosophic A-module. If 0 ≠ Ne(B), Ne(r)∈Ne(A) and neutrosophic homomorphism Ne(f):Ne(B)→Ne(B) produced by Ne(r) is onto or neutrosophic nilpotent, then Ne(B) is called neutrosophic secondary.

Remark (3.7). Any neutrosophic secondary multiplication module over Ne(A) is a neutrosophic finitely generated module.

Now from Definition(3.5) and Remark(3.6), we can introduce the following result.

Corollary (3.8). Let B be a neutrosophic module If Ne(K)<<Ne(B) and Ne(B) is a neutrosophic secondary multiplication module, then E is neutrosophic local module.

Definition (3.9). Let B be a A-module. Then, B is called neutrosophic semi-hollow module if a neutrosophic submodule Ne(K) of Ne(B) is a neutrosophic semi-small submodule (if Ne(K)=0, or ∀0 ≠ Ne(H) ≤ Ne(K) and Ne($\frac{K}{H}$)<<Ne($\frac{B}{H}$)).

Recall that an A-module B is called neutrosophic injective if f: Ne(H)→Ne(K) and g: Ne(H)→Ne(B), there exists h: Ne(K) → Ne(B) such that g=f∘h.

Also, let Ne(B) be a neutrosophic A-module. Then, E is called the neutrosophic multiplication module in the case for every Ne(K) of Ne(B), there exists $Ne(J)$ an neutrosophic ideal of Ne(A) such that $Ne(K)$=$Ne(J)$ Ne(B).

Lemma (3.10). Let Ne(B) be a neutrosophic injective A-module. Then Ne(B) is a neutrosophic divisible module**.**

Proof: Suppose that Ne(B) be a neutrosophic injective module. Let Ne(b) ∈ Ne(B) and Ne(r) ∈Ne(A) such that Ne(r) is not zero divisor in Ne(A). Take f:Ne(Ar)→Ne(B)∋ f(Ne(sr))=Ne(sr) ∋ Ne(s) ∈ Ne(A). Since Ne(r) is not zero divisor and if Ne(sr)=0, then Ne(s)=0 and hence f(Ne(sr))==0. Therefore, there exists a homomorphism g:Ne(A)→Ne(B). Thus Ne(b) =f(Ne(r))=g(Ne(r))- Ne(r)g(1) (because g agree with f on Ne(Ar)). Thus, Ne(B) is neutrosophic divisible module**.**

Theorem (3.11) . Let Ne(B) be a neutrosophic module if the following conditions hold:
(1) Ne(B) is a neutrosophic injective module over neutrosophic integral domain Ne(A).
(2) Ne(B) is a neutrosophic semi-hollow module.
(3) For every Ne(I), Ne(K) = Ne(I) Ne(B).
Then, Ne(B) is a neutrosophic local module.

Proof: Since Ne(B) is a neutrosophic injective module, then Ne(B) is a neutrosophic divisible module. We have Ne(B) is a neutrosophic semi-hollow module. Then, B is a neutrosophic hollow module. For all Ne(I), Ne(K)=Ne(I)Ne(B), we obtain Ne(B) is a neutrosophic multiplication module , Ne(K) ≤ Ne(B). Thus, Ne(B) is a neutrosophic local module.

Theorem (3.12). Let Ne(A) be a neutrosophic ring. If Ne(LA) is a neutrosophic local ring, then every neutrosophic cyclic A-module is a neutrosophic hollow module.

Proof: Suppose that Ne(LA) be a neutrosophic local ring. Let B be a neutrosophic cyclic module (Ne(CB)). Clearly, $\frac{Ne(A)}{Ne(ann(B))}$ is a neutrosophic local ring. Thus, Ne(B) is a neutrosophic hollow module**.**

Definition (3.13). Let B be a neutrosophic A-module. Then, Ne(B) is called neutrosophic semi multiplication module if for every neutrosophic prime ideal (Ne(P_I)) of Ne(A) implies Ne(B_{PI}) is a neutrosophic finitely generated and $\dim_{pl}$ Ne(B) = 0

Theorem (3.14). Let Ne(B) be a neutrosophic module over neutrosophic ring A (Ne(A)). Then every Ne(SM)Ne(LA) is a neutrosophic hollow module.

proof: Let Ne(SM) be a neutrosophic semi multiplication module over neutrosophic semi-local ring A Ne(SLA). So for each Ne(PI) of Ne(A) we get Ne(B_p) is a cyclic. Hence Ne(B) is a cyclic. But every Ne(SM) over Ne(SLA) is cyclic. So A has Ne(SM) is a hollow module.

Definition (3.15). Let Ne(B) be a neutrosophic A-module. The Ne(B) is called a neutrosophic serial module if every two neutrosophic submodules of Ne(B) are comparable with respect to inclusion.

Theorem (3.16). Let Ne(B) be a neutrosophic serial module over Ne(A). Then every Ne(K) of Ne(B) is a neutrosophic submodule hollow module.

proof: Let Ne(K) be a neutrosophic submodule of Ne(B). Assume that Ne(Max(K)) ≠ 0 and Ne(K) =Ne(K_1)+Ne(K_2) where Ne(K_1) is a meutrosophic maximal submodule of Ne(K) and Ne(K_2) is a neutrosophic submodule of Ne(K). Since Ne(K_1) and Ne(K_2) are neutrosophic submodules of Ne(B) and Ne (B) is a neutrosophic serial module, then Ne(K_1) ≤ Ne(K_2) or Ne(K_2) ≤ Ne(K_1) and Ne(K_1) ≤ Ne(K_2) this gives Ne(K) = Ne(K_1) + Ne(K_2). If Ne(K_2) ≤ Ne(K_1)
because Ne(K_1) is a neutrosophic maximal submodule of Ne(K).

Corollary (3.17). Let Ne(B) be a neutrosophic module if the following conditions hold:
(1) Ne(B) is a neutrosophic multiplication module.
(2) Every Ne(K_1) and Ne(K_2) are neutrosophic comparable. Then Ne(K)↪ Ne(B) such that Ne(B) is a neutrosophic hollow module.

proof: Let Ne(K_1) and Ne(K_2) be neutrosophic submodules of Ne(B). Then Ne(K_1) = Ne(I) Ne(B) and Ne(K_2) = Ne(J) Ne(B) where Ne(I) and Ne(J) are Neutrosophic ideals of Ne(A), Ne(I) ⊆ Ne(J). Then Ne(K_1) ⊆ Ne(K_2) and Ne(J) ⊆Ne(I). So Ne(K_2) ⊆ Ne(K_1). Hence Ne(K) is a neutrosophic submodule of neutrosophic hollow module Ne(B).

Lemma (3.18). [13]. Let Ne(VA) be a neutrosophic V-ring. A neutrosophic A-module (Ne(B)) is a neutrosophic lifting module if and only if it is a neutrosophic hollow module.

Theorem (3.19). Let Ne(A) be a neutrosophic Noetherian ring. If Neutrosophic A-module Ne(B) satisfies the following:

(1) Every neutrosophic simple A-module is a neutrosophic injective A-module.

(2) Ne(B) is a neutrosophic semisimple module.

(3) Ne(B) is a neutrosophic indecomposable module.

Then, Ne(B) is a neutrosophic hollow module.

Proof: Since every neutrosophic simple module (Ne(B)) over neutrosophic ring A is a neutrosophic injective module, so A is a neutrosophic V-ring. From condition (2}, we have Ne(B) is a neutrosophic semisimple module. Hence, B is a neutrosophic lifting module by Lemma 3.18. Then, for all $Ne(K) \hookrightarrow Ne(B)$, there exists $Ne(B) = Ne(K)+ Ne(A)$ such that $Ne(H) \hookrightarrow Ne(K)$.

Also, we have

$Ne(K) \cap Ne(H) \ll Ne(B)$.

But Ne(A) is a neutrosophic indecomposable module. Hence Ne(B) is a neutrosophic hollow module.

We know that a neutrosophic submodule Ne(K) of neutrosophic module Ne(B) is a neutrosophic lies over a summand if it is a supplement neutrosophic submodule of B**.**

Proposition (3.20). Let $0 \neq Ne(B)$ and the following are true:

1) $Ne(B) \neq Rad(Ne(B))$.

2) $Ne(K) \hookrightarrow Ne(B)$ is a neutrosophic lies over a summand of Ne(B).

3) Ne(B) is a neutrosophic indecomposable module. Thus Ne(B) is a neutrosophic hollow module.

Proof: Suppose that $N(B) \neq 0$. Let $Ne(K) \hookrightarrow N(B)$ with neutrosophic lies over a summand of N(B). So

$N(B)= N(H) \oplus N(L) \ni N(L) \hookrightarrow N(K)$ and $Ne(H) \cap Ne(K) \ll Ne(B)$.

Since $N(L) \neq N(B)$, $N(H)=N(B)$ and $N(K) = N(H) \cap Ne(K) \subseteq Rad(Ne(B))$, then Rad(Ne(B)) is a unique neutrosophic local module. Thus, N(B) is a neutrosophic hollow module by Remark 3.2, (3).

Corollary (3.21). Let $Ne(B)=(Ne(B_1) \cap Ne(B_2))$ be a neutrosophic indecomposable module. If Ne(B) is a neutrosophic hollow-lifting module, then it is a neutrosophic hollow module.

Proof: Assume that $(Ne(B_1) \cap Ne(B_2))$ has a hollow factor module. There exists $N(K) \hookrightarrow (Ne(B_1) \cap Ne(B_2))$ such that $(Ne(B_1) \cap Ne(B_2))/N(K)$ is a neutrosophic hollow module. Since $Ne(B_1) \cap Ne(B_2)$ is a neutrosophic hollow-lifting module, then there exists a Ne(H) is a neutrosophic direct summand of $Ne(B_1) \cap Ne(B_2)$ such that $(N(K)/N(H)) \ll (Ne(B_2)/Ne(H))$ and since $Ne(B_1) \cap Ne(B_2)$ is a neutrosophic indecomposable module, then $Ne(B_1) \cap Ne(B_2) \neq 0$ and it is not a neutrosophic direct sum of two nonzero neutrosophic submodules ($Ne(K)=0$ and $Ne(K) \ll Ne(B_1) \cap Ne(B_2)$)). So $Ne(B_1) \cap Ne(B_2)= Ne(B)$ is a neutrosophic hollow module.

3. Conclusions

The Neutrosophic module is one of many important notions in module theory. In this paper, we have defined neutrosophic hollow modules. Several properties have been presented in this article. In Theorem 3.11, we have studied the relationship between the neutrosophic semi-hollow module and the neutrosophic local module. Also, we have proved that if B is a neutrosophic indecomposable module Ne(B) and is a neutrosophic hollow-lifting module, then it is a neutrosophic hollow module. Finally, we have introduced a neutrosophic injective module over neutrosophic integral domain Ne(A) and neutrosophic semi-hollow module to get neutrosophic local module.

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