DOI: 10.24996/ijs.2022.63.4.33

ISSN: 0067-2904

Caps by Groups Action on the $PG(3,8)$

Najm Abdulzahra Makhrib Al-seraji[1], Abeer Jabbar Al-Rikabi[2*], Emad Bakr Al-Zangana[1]

[1]Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
[2]Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq

Received: 6/6/2021 Accepted: 22/8/2021 Published: 30/4/2022

Abstract.

In this paper, the (k,r)-caps are created by action of groups on the three-dimensional projective space over the Galois field of order eight. The types of (k,r)-caps are also studied and determined either they form complete caps or not.

Keywords: Cap, Galois field, Projective space, Singer group.

أغطية بواسطة عمل الزمر على $PG(3,8)$

نجم عبدالزهرة مخرب السراجي[1], عبير جبار الركابي[2*], عماد بكر الزنكنة[3]

[1,3]الرياضيات, العلوم, المستنصرية, بغداد, العراق

[2]الرياضيات, التربية الاساسية, المستنصرية, بغداد, العراق

الخلاصة

في هذا البحث, تم انشاء الاغطية (k,r)- بفعل عمل الزمر على الفضاء الاسقاطي من البعد الثالث على حقل كالواز من الرتبة الثامنة. كذلك تم دراسة انواع الاغطية (k,r)- وتحديد فيما اذا كانت تشكل اغطية كاملة ام لا.

1. Introduction

The main aim of this paper is to formulate the cap using action of groups in the three-dimensional projective space over the Galois field of order eight. The (k,r)-caps for $r=2,3,6,9$ are studied and classified if they form complete cap or not, as well as the way that makes incomplete cap is complete is founded.

Let $PG(n,q)$ be the n-dimensional projective space (n-space) over the Galois field $GF(q)=F_q$ [1].

Definition 1.1[1][2]: A projectivity τ which permutes the $\theta(n,q)$ points of $PG(n,q)$ in a single cycle is called a cyclic projectivity (Singer cycle) and the group that generates is called a Singer group.

Definition 1.2[1][2]: A spread $\mathcal{F}$of $PG(n,q)$ by r-subspaces is a set of r-subspaces which partitions $PG(n,q)$; that is, every point of $PG(n,q)$ lies in exactly one r-subspace of $\mathcal{F}$. Hence any two r-subspaces of $\mathcal{F}$are disjoint.

*Email: abear9933.edbs@uomustansiriyah.edu.iq

A (k,l)-*set* in $PG(n,q)$ is a set of k l-subspaces. A k-*set* is a $(k,0)$-*set* that means a set of k points. The most general type of (k,l)-set that will be considered is a $(k,l;\ r,s;\ n,q)$-*set*; that is, a $(k\ ,l)$-set in $PG(n,q)$ with at most r l-subspaces in any s-subspaces.
The definition of $(k,l;\ r,s\ ;n,q)$-set is specialized as follows:

(1) a $(k\ ;r,s;\ n\ ,q)$-*set is* a $(k,0;r,s;n,q)$-set*;*
(2) a (k,r)-*cap* is a $(k;r,1;n,q)$-set with $n \geq 3$;
(3) a k-*cap* is a $(k,2)$-cap*;*
(4) a k-*arc* is a $(k;n,n-1;n,q)$-set*;*
(5) a (k,l)-*span or partial spread* is a $(k,l;1,2l;n,q)$-*set* with $l \geq 1$.

Definition 1.3[1][2]: A k-cap is a set of k points in $PG(n,q)$ with $n \geq 3$ such that no three are collinear. A k-cap is complete if it is not contained in a $(k+1)$-cap.
A m-*secan*t of a (k,r)-cap K in $PG(n,q)$ is a hyperplane (line) ℓ such that $|K \cap \ell| = m$. Let Q be a point of $PG(n,q)$ which is not on the (k,r)-cap K. Let $\sigma_i(Q)$ be the number of i-secants through Q. The number $\sigma_r(Q)$ of r-secant is called the *index* of Q with respect to K. Let c_i , and C_i be the number of points of index i ,and the points of index i, respectively.
For more details or background for (k,r)-cap in $PG(n,q)$, we refer to the references [1] and [3].
In [1], Hirschfeld gave rich theoretical details of finite projective spaces of three dimensions. In [4-6], many other researchers attempted to discuss some objects in three dimensional projective spaces for example, classify cap and arc, and they found the lower and maximum bound for the size of a complete cap and arc for fixed field. In addition, the study of the splitting of three-dimensional space into planes and lines is one of the important topics that authors studied and discussed [7,8].
Since the study in three-dimensional space is more difficult than in two-dimensional spaces, so that the researchers attempted to work on special cases for each of the previous problems that are mentioned. .For example, the idea of acting subgroups of on space was used to obtain caps as in [9]. The procedure of group action on the plane was used in the work of Al-Seraji in [10-13] and Al-Zangana in [14-18] used idea of group action on the line and plane. Topalova studied the three dimensional projective space over the Galois field of order five using the automorphisms of order thirteen [19].

2. Sets of subspaces in $PG(3,8)$

In [8], the space $PG(3,8)$ is introduced where the projectivity $T = M(A)$ that is given by the companion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tau^5 & \tau^3 & 1 & \tau^5 \end{bmatrix},$$

it is a cyclic projectivity ,where τ be the primitive element of F_8, it has been used to construct points, lines and planes. Also, the space is partitioned into subgeometries.
It is well known that every projective space has a cyclic projectivity $M(A)$. To get the matrix A, it is enough to have a companion matrix $A = C(F)$ with primitive characteristic polynomial F. Therefore, the points of $PG(3,8)$ computed by the following form:

$$P(i) = P(0) * A^i, \qquad i = 0, \ldots\ ,584.$$

The points of $PG(3,8)$ have a unique forms up to equivalence relation which are $[1,0,0,0]$, $[x,1,0,0]$, $[x\ ,y,1,0]$, $[x\ ,y\ ,z\ ,1]$ $\forall\ x\ ,y\ ,z\ \in\ F_8$ and the number of each form is $1\ ,q,q^2,q^3$, respectively. Hence, we have 585 points and planes, 4745 lines, 9 points on each line and 73 lines passing through each point.
Therefore, to construct all others 585 planes, the cyclic projectivity is used as follows: $\mathcal{P}(i) = X_w\ * A^i\ ,\ i = 0\ ,\ldots,584$, where X_w represents the plane with fourth coordinate is zero. The equation of the line X_w is $Y + \tau^3 W = 0$.

Form the analysis of the number of points, several movements can be deduced, and we will study each one as follows.
As $|PG(3,8)| = 585 = 3 * 3 * 5 * 13$. From Theorem (1.14) in [20], we conclude that there are ten subspaces of $PG(3,8)$. Let $p_1 = 3, p_2 = 3, p_3 = 5, p_4 = 13$.
From the action of A^i on the points of $PG(3,8)$, $i \in \{p_1, p_3, p_4\}$, the following results are deduced:
Throughout this paper, if A^i has j orbits, then the symbol A^i_j will denote the orbit j of A^i and $N^k_{A^i_j}$ =Number of lines which are intersect A^i_j of order k such that $0 \leq k \leq 9$.

Algorithm 2.1: The procedures that are used to prove the main theorem is as follows:
1- Find the orbits for each non-trivial integer factor of 585 p_i from the action of cyclic group $\langle A^{p_1} \rangle$ on $PG(3,8)$.
2- Find the intersection between lines and orbits to know the degree of the cap that they formed.
3- Determine if the caps are complete or incomplete by finding the points of index zero for each cap.
4- Add points to the incomplete cap from the set of points of index zero to make it complete.
Note that all the calculations in the paper have been done using the Gap programming, which can found at https://www.gap-system.org/.

Main Theorem 2.2:
I. The orbits $A^{p_1}_j; j = 1,2,3$ are complete $(195,6)$-caps.
II. 1. The orbits of $A^{p_3}_j; j = 1, \ldots, 5$ are incomplete $(117,9)$-caps.
2. The maximum complete cap can be formed from the orbits of $A^{p_3}_1$ is $(585,9)$-cap.
III. 1. The orbits of $A^{p_4}_j; j = 1, \ldots, 13$ are incomplete $(45,9)$-caps.
2. The maximum complete cap can be formed from the orbits of $A^{p_4}_j$ are $(585,9)$-caps.
IV. The orbits $A^{p_1p_2}_j; j = 1, \ldots, 9$. are complete 65-caps .
V. The orbits $A^{p_1p_3}_j, j = 1, \ldots, 15$. are complete $(39,3)$-caps.
VI. 1. The orbits of $A^{p_1p_4}_1;\ j = 1, \ldots, 39$ are incomplete $(15,3)$-caps.
2. The maximum complete cap can be formed from the orbits of $A^{p_1p_4}_1$ is $(60,3)$-caps.
3. The orbits of $A^{p_1p_4}_j$ are formed 39 copies of $PG(3,2)$, has 15 points and planes, 35 lines, 3 points on each line and 7 lines passing through each point.
VII. 1. The orbits of $A^{p_3p_4}_1$; $j = 1, \ldots, 65$ are incomplete $(9,9)$-caps.
2. The maximum complete cap can be formed from the orbits of $A^{p_3p_4}_1$ is $(585,9)$-caps.
3. The orbits $A^{p_3p_4}_j, j = 1, \ldots, 65$ are formed the spread of $PG(3,8)$.
VIII. 1. The orbits of $A^{p_1p_2p_3}_j, j = 1, \ldots, 45$ are incomplete 13-caps.
2. The maximum complete cap can be formed from the orbits of $A^{p_1p_2p_3}_j, j = 1, \ldots, 45$ is 28-caps.
IX. 1. The orbits of $A^{p_1p_2p_4}, j = 1, \ldots, 117$ are incomplete 5-caps.
2. The maximum complete cap can be formed from the orbits of $A^{p_1p_2p_4}, j = 1, \ldots, 117$ is 27-caps.
X. 1. The orbits of $A^{p_1p_3p_4}_j, j = 1, \ldots, 195$ are incomplete $(3,3)$-caps.
2. The maximum complete cap can be formed from the orbits of $A^{p_3p_4}_1$ is $(61,3)$-caps.

Proof:
I. The orbits of A^{p_1}: There are three orbits from the action of A^{p_1} on $PG(3,8)$, of size 195.

$$A^3_1 = \{0,3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48,51, \ldots, 579, 582\};$$
$$A^3_2 = \{1,4,7,10,13,16,19,22,25,28,31,34,37,40,43,46,49, \ldots, 580,583\};$$

$A_3^3 = \{2,5,8,11,14,17,20,23,26,29,32,35,38,41,44,47,50,\dots,581,584\}$.

The orbits $A_j^{p_1}, j = 1,2,3$ are (195; 6,1; 3,8)-sets of 195 points of degree 6 since $A_j^{p_1}, j = 1,2,3$ intersects each line in at most six points in $PG(3,8)$, as they are shown in the equation below.

$$N^k_{A_j^{p_1}} = \begin{cases} 390 & if\left|A_j^{p_1} \cap \ell_i\right| = 0 \\ 1170 & if\left|A_j^{p_1} \cap \ell_i\right| = 2 \\ 1625 & if\left|A_j^{p_1} \cap \ell_i\right| = 3 \\ 1170 & if\left|A_j^{p_1} \cap \ell_i\right| = 4 \\ 390 & if\left|A_j^{p_1} \cap \ell_i\right| = 6 \end{cases} \quad ;\ i = 1,\dots,4745.$$

Therefore, the orbits $A_j^{p_1}, j = 1,2,3$ are (195,6)-caps additionally, it is complete (195,6)-caps since there are no points of index zero for $A_j^{p_1}$; that is, $c_o = 0$.

II. **The orbits of A^{p_3}:** There are 5 orbits from the action of A^{p_3} on $PG(3,8)$ of size 117.

$A_1^5 = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70,\dots,580\}$;
$A_2^5 = \{1, 6, 11, 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71,\dots,581\}$;
$A_3^5 = \{2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67, 72,\dots,582\}$;
$A_4^5 = \{3, 8, 13, 18, 23, 28, 33, 38, 43, 48, 53, 58, 63, 68, 73,\dots,583\}$;
$A_5^5 = \{4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64, 69, 74,\dots,584\}$.

1. The orbits of $A_j^{p_3}, j = 1,\dots,5$ are (117; 9,1; 3,8)-sets of 117 points of degree 9 since $A_j^{p_3}, j = 1,\dots,5$ intersect each line in at most 9 points in $PG(3,8)$ as they are shown in the equation below.

$$N^k_{A_j^{p_3}} = \begin{cases} 520 & if\left|A_j^{p_3} \cap \ell_i\right| = 0 \\ 1638 & if\left|A_j^{p_3} \cap \ell_i\right| = 1 \\ 1404 & if\left|A_j^{p_3} \cap \ell_i\right| = 2 \\ 702 & if\left|A_j^{p_3} \cap \ell_i\right| = 3 \\ 468 & if\left|A_j^{p_3} \cap \ell_i\right| = 4 \\ 13 & if\left|A_j^{p_3} \cap \ell_i\right| = 9 \end{cases} \quad ;\ i = 1,\dots,4745.$$

Therefore, $A_j^{p_3}, j = 1,\dots,5$ are (117,9)-caps. And it is incomplete (117,9)-caps since $c_o = 468$.

2. We can create a complete cap from the (117,9)-cap by including external points such that the maximum complete $(k,9)$-cap that is constructed from A_1^5 is $\xi_1 = A_1^5 \cup C_o$ is complete (585,9)-cap since every nine points are located on line. And $A_j^{p_3}, j = 2,\dots,5$ are incomplete (117,9)-caps since $c_o \neq 0$. The maximum complete $(k,9)$-cap constructed from A_j^5 is $\xi_j = A_j^5 \cup Z_j$ where $Z_j \subseteq C_0^j$ =set of all external points of $A_j^5, j = 2,\dots,5$.

III. The orbits of A^{p_4}: There are 13 orbits from the action of A^{p_4} on $PG(3,8)$, of size 45.

$A_1^{13} = \{0, 13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182, 195, 208, 221, 234, 247,\dots,572\}$;
$A_2^{13} = \{1, 14, 27, 40, 53, 66, 79, 92, 105, 118, 131, 144, 157, 170, 183, 196, 209, 222, 235, 248,\dots,573\}$;
$A_3^{13} = \{2, 15, 28, 41, 54, 67, 80, 93, 106, 119, 132, 145, 158, 171, 184, 197, 210, 223, 236, 249,\dots,574\}$;

⋮

A_{13}^{13}
$=\{12,25,38,51,64,77,90,103,116,129,142,155,168,181,194,207,220,233,246,\ldots,584\}$.

1. The orbits $A_j^{p_4}, j=1,\ldots,13$ are intersection of 5 lines in at most 9 points, as they are shown as follows:

$$N_{A_j^{p_4}}^{k}=\begin{cases}2220 & if\left|A_j^{p_4}\cap \ell_i\right|=0\\ 1890 & if\left|A_j^{p_4}\cap \ell_i\right|=1\\ 540 & if\left|A_j^{p_4}\cap \ell_i\right|=2\\ 90 & if\left|A_j^{p_4}\cap \ell_i\right|=3\\ 5 & if\left|A_j^{p_4}\cap \ell_i\right|=9\end{cases} \quad ,i=1,\ldots,4745.$$

Thus the orbits of $A_j^{p_4}, j=1,\ldots,13$ are (45; 9,1; 3,8)- sets; that is, (45,9)-caps. Since $c_o=540$, then it is incomplete caps

2. Since degree of caps is 9, then by Remark 4 the maximum complete $(k,9)$-cap that is constructed from A_1^{13} is $\xi_1=A_1^{13}\cup C_o$ is (585,9)-cap, and $A_j^{p_4}, j=2,\ldots,13$ are incomplete (45,9)-caps since $c_o\neq 0\ \forall j$. The maximum complete (585,9)-cap constructed from A_j^{13} is $\xi_j=A_j^{13}\cup C_0^j$ where C_0^j =set of all external points of A_j^{13}, j=2,…,13.

IV. The orbits of $A^{p_1p_2}$**:** There are 9 orbits from the action of $A^{p_1p_2}$ on $PG(3,8)$, of size 65.

A_1^{9}
$=$
$\{0,9,18,27,36,45,54,63,72,81,90,99,108,117,126,135,144,153,162,171,180,189,\ldots,576\}$;
A_2^{9}
$=$
$\{1,10,19,28,37,46,55,64,73,82,91,100,109,118,127,136,145,154,163,172,181,190,\ldots,577\}$;
$\vdots$
A_9^{9}
$=$
$\{8,17,26,35,44,53,62,71,80,89,98,107,116,125,134,143,152,161,170,179,188,197,\ldots,584\}$.

The orbits $A_j^{p_1p_2}, j=1,\ldots,9.$ are intersection of 2080 lines in at most 2 points in $PG(3,8)$, as they are shown as follows:

$$N_{A_j^{p_1p_2}}^{k}=\begin{cases}2080 & if\ \left|A_j^{p_1p_2}\cap \ell_i\right|=0\\ 585 & if\left|A_j^{p_1p_2}\cap \ell_i\right|=1\\ 2080 & if\ \left|A_j^{p_1p_2}\cap \ell_i\right|=2\end{cases} \quad ;i=1,\ldots,4745.$$

Therefore, the orbits $A_j^{p_1p_2}, j=1,\ldots,9$ are $(65;2;\ 3,8)$- sets; that is, 65-caps since no three points of $A_j^{p_1p_2}$ are collinear, and the orbits $A_j^{p_1p_2}; j=1,\ldots,9$ are complete 65-caps, since there are no points of index zero for $A_j^{p_1p_2}$; that is $c_o=0$.

V. The orbits of $A^{p_1p_3}$**:** There are 15 orbits from the action of $A^{p_1p_3}$ on $PG(3,8)$, of size 39.

A_1^{15}
$=\{0,15,30,45,60,75,90,105,120,135,150,165,180,195,210,225,240,255,270,285,\ldots,570\}$;
A_2^{15}
$=$
$\{1,16,31,46,61,76,91,106,121,136,151,166,181,196,211,226,241,256,271,286,\ldots,571\}$;
$\vdots$

A_{15}^{15}
$=\{14,29,44,59,74,89,104,119,134,149,164,179,194,209,224,239,254,269,284,299,\dots,584\}$.
The orbits $A_j^{p_1p_3}, j=1,\dots,15$. are intersection of 169 lines in at most 3 points in $PG(3,8)$ such that

$$N^k_{A_j^{p_1p_3}} = \begin{cases} 2470 & if\left|A_j^{p_1p_3}\cap \ell_i\right| = 0 \\ 1872 & if\left|A_j^{p_1p_2}\cap \ell_i\right| = 1 \\ 234 & if\left|A_j^{p_1p_2}\cap \ell_i\right| = 2 \\ 169 & if\left|A_j^{p_1p_2}\cap \ell_i\right| = 3 \end{cases} ;i=1,\dots,4745.$$

Thus the orbits $A_j^{p_1p_3}, j=1,\dots,15$ are (39; 3,1; 3,8)-sets; that is, (39,3)-caps, and it is complete since $c_o=0$

VI. The orbits of $A^{p_1p_4}$: There are 39 orbits from the action of $A^{p_1p_4}$ on $PG(3,8)$, of size 15.

$A_1^{39}=\{0,39,78,117,156,195,234,273,312,351,390,429,468,507,546\}$;
$A_2^{39}=\{1,40,79,118,157,196,235,274,313,352,391,430,469,508,547\}$;
$A_3^{39}=\{2,41,80,119,158,197,236,275,314,353,392,431,470,509,548\}$;
$A_4^{39}=\{3,42,81,120,159,198,237,276,315,354,393,432,471,510,549\}$;
⋮
$A_{39}^{39}=\{38,77,116,155,194,233,272,311,350,389,428,467,506,545,584\}$.

1. The orbits $A_j^{p_1p_4}, j=1,\dots,39$ are intersection of 35 lines in at most 3 points, as they are shown as follows:

$$N^k_{A_j^{p_1p_4}} = \begin{cases} 3720 & if\left|A_j^{p_1p_4}\cap \ell_i\right| = 0 \\ 990 & if\left|A_j^{p_1p_4}\cap \ell_i\right| = 1 \\ 35 & if\left|A_j^{p_1p_4}\cap \ell_i\right| = 3 \end{cases} ,i=1,\dots,4745.$$

Therefore, the orbits $A_j^{p_1p_4}, j=1,\dots,39$ are (15; 3,1; 3,8)- sets; that is, (15,3)-caps, which are incomplete, since $c_o=360$.

2. We can create a complete cap from the (15,3)-cap by including external points such that $Z=\{1,2,3,4,5,6,8,9,10,11,12,15,16,20,21,27,30,44,71,72,75,79,80,$
$81,93,102,111,114,118,119,$
$127,132,139,161,181,196,219,237,432,433,444,471,568,570,571\}\subseteq C_o$.

The maximum complete $(k,3)$-cap that is constructed from A_1^{39} is $\xi_1=A_1^{39}\cup Z$ is complete $(60,3)$-cap since $c_o=0$.

And $A_j^{p_1p_4}, j=2,\dots,39$ are incomplete (15,3)-caps since $c_o\neq 0\ \forall j$. Thus the maximum complete $(k,3)$-cap constructed from $A_j^{39}$is $\xi_j=A_j^{39}\cup Z_j$ where $Z_j\subseteq C_0^j$ =set of all external points of $A_j^{39}, j=2,\dots,39$ of index zero.

3. By Theorem 12 in [17].

VII. The orbits of $A^{p_3p_4}$: There are 65 orbits from the action of $A^{p_3p_4}$ on $PG(3,8)$, of size 9.

$A_1^{65}=\{0,65,130,195,260,325,390,455,520\}$;
$A_2^{65}=\{1,66,131,196,261,326,391,456,521\}$;
⋮
$A_{65}^{65}=\{64,129,194,259,324,389,454,519,584\}$.

1. The orbits $A_j^{p_3p_4}, j=1,\dots,65$ are intersection of 1 line in at most 9 points as shown below.

$$N^{k}_{A_j^{p_3p_4}} = \begin{cases} 4096 & if \ |A_j^{p_3p_4} \cap \ell_i| = 0 \\ 648 & if \ |A_j^{p_3p_4} \cap \ell_i| = 1 \ ; \ i = 1, \dots, 4745.. \\ 1 & if \ |A_j^{p_3p_4} \cap \ell_i| = 9 \end{cases}$$

Thus the orbits $A_j^{p_3p_4}, j = 1, \dots, 65$ are (9; 9,1; 3,8)- sets; that is, (9 ,9)-caps and it is incomplete (9,9)-caps since $c_o = 576$.

2. Since degree of caps is 9, then by Remark 4 the maximum complete $(k, 9)$-caps that are constructed from A_j^{65} are complete (585,9)-caps have the formula $\xi_j = A_j^{65} \cup C_o^j$; $j = 1, \dots, 65$.

3. By Theorem 10 in [17].

VIII. The orbits of $A^{p_1p_2p_3}$: There are 45 orbits from the action of $A^{p_1p_2p_3}$ on $PG(3,8)$, of size 13.

$A_1^{45} = \{0, 45, 90, 135, 180, 225, 270, 315, 360, 405, 450, 495, 540\}$;

$A_2^{45} = \{1, 46, 91, 136, 181, 226, 271, 316, 361, 406, 451, 496, 541\}$;

⋮

$A_{45}^{45} = \{44, 89, 134, 179, 224, 269, 314, 359, 404, 449, 494, 539, 584\}$.

1. The orbits $A_j^{p_1p_2p_3}, j = 1, \dots, 45$ are intersection of 78 line in at most 2 points as they are shown in an equation:

$$N^{k}_{A_j^{p_1p_2p_3}} = \begin{cases} 3874 & if \ |A_j^{p_1p_2p_3} \cap \ell_i| = 0 \\ 793 & if \ |A_j^{p_1p_2p_3} \cap \ell_i| = 1 \ ; \ i = 1, \dots, 4745. \\ 78 & if \ |A_j^{p_1p_2p_3} \cap \ell_i| = 2 \end{cases}$$

For that reason the orbits $A_j^{p_1p_2p_3}, j = 1, \dots, 45$ are (13; 2,1; 3,8)- sets; that is, 13-caps, and the orbits of $A_j^{p_1p_2p_3}$ are incomplete 13-caps since $c_o = 182$.

2. We can create a complete cap from the k-cap by including external points such that $Z = \{3, 6, 9, 21, 27, 39, 57, 78, 99, 141, 186, 276, 393, 411, 456\} \subseteq C_o$.

The maximum complete 28-cap that is constructed from A_1^{45} is $\xi_1 = A_1^{45} \cup Z$. And $A_j^{p_1p_2p_3}$, $j = 2, \dots, 45$ are incomplete 13-caps since $c_o \neq 0$. The maximum complete k-cap that is constructed form A_j^{45} is $\xi_j = A_j^{45} \cup Z_j$ where $Z_j \subseteq C_0^j$ =set of all external points of $A_j^{45}, j = 1, \dots, 45$ of index zero.

IX. The orbits of $A^{p_1p_2p_4}$: There are 117 orbits from the action of $A^{p_1p_2p_4}$ on $PG(3,8)$, of size 5.

$A_1^{117} = \{0, 117, 234, 351, 468\}$;

$A_2^{117} = \{1, 118, 235, 352, 469\}$;

⋮

$A_{177}^{117} = \{116, 233, 350, 467, 584\}$.

1. The orbits $A^{p_1p_2p_4}, j = 1, \dots, 117$ are intersection of 10 lines in at most 2 points as they are shown as follows:

$$N^{k}_{A_j^{p_1p_2p_4}} = \begin{cases} 4390 & if \ |A^{p_1p_2p_4} \cap \ell_i| = 0 \\ 345 & if |A^{p_1p_2p_4} \cap \ell_i| = 1 \ ; \ i = 1, \dots, 4745.. \\ 10 & if |A^{p_1p_2p_4} \cap \ell_i| = 2 \end{cases}$$

Therefore, the orbits $A^{p_1p_2p_4}, j = 1, \dots, 117$ are (5; 2; 3,8)- sets; that is, 5-caps and it is incomplete 5-caps since $c_o = 510$.

2. We can create a complete cap from the k -cap by including external points such that $Z = \{1,2,3,4, 6,7,8,14,15,25,28,32,36,41,47,54,65,92,119,200,262,275\} \subseteq C_o$. The maximum complete k-cap that is constructed from A_1^{117} is $\xi_1 = A_1^{117} \cup Z$ is complete 27-cap. And $A_j^{p_1p_2p_4}, j = 2, \dots, 117$ are incomplete 5-caps since $c_o \neq 0$. The maximum complete k-cap that is constructed from $A_j^{117}$is $\xi_j = A_j^{117} \cup Z_j$ where $Z_j \subseteq C_0^j$ =set of all external points of A_j^{117}, j = 1, … , 117 of index zero.

X. The orbits of $A_j^{p_1p_3p_4}$: There are 195 orbits from the action of $A_j^{p_1p_3p_4}$ on $PG(3,8)$, of size 3.

$$A_1^{195} = \{0, 195, 390\};$$
$$A_2^{195} = \{1, 196, 391\};$$
$$\vdots$$

$A_{195}^{195} = \{194, 389, 584\}$.

1. The orbits $A_j^{p_1p_3p_4}, j = 1, \dots, 195$ are intersection of 1 line in at most 3 points as shown as follows:

$$N^k_{A_j^{p_1p_3p_4}} = \begin{cases} 4528 & if\ |A_j^{p_1p_3p_4} \cap \ell_i| = 0 \\ 216 & if\ |A_j^{p_1p_3p_4} \cap \ell_i| = 1 \\ 1 & if\ |A_j^{p_1p_3p_4} \cap \ell_i| = 3 \end{cases} ;\ i = 1, \dots, 4745.$$

Thus the orbits $A_j^{p_1p_3p_4}, j = 1, \dots, 195$ are (3; 3,1; 3,8)- sets; that is, (3 ,3)-caps and incomplete since $c_o = 576$.

2. We can create a complete cap from the $(k, 3)$-cap by including external points such that $Z = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18,19,20,21,22,23,24,25,26,27,28,29,30,32,33, 35,36,37,39,40,41,42,43,47,48,54,63,66,70,74,77,78,82,93,95,99,120,146,159,222,249,459\} \subseteq C_o$. The maximum complete $(k, 3)$-cap that is constructed from A_1^{195} is $\xi_1 = A_1^{195} \cup Z$ is complete (61,3)-cap. And $A_j^{p_1p_3p_4}, j = 2, \dots, 195$ are incomplete (3,3)-caps since $c_o \neq 0$. The maximum complete $(k, 3)$-cap that is constructed from $A_j^{195}$is $\xi_j = A_j^{195} \cup Z_j$ where $Z_j \subseteq C_0^j$ =set of all external points of A_j^{195}, j = 2, … , 195 of index zero.

Corollary 2.3: Any incomplete cap Γ in the Main Theorem of degree nine, that constructs from it, is a complete (585,9)-cap.

Proof: Since Γ has degree nine, so that any line will contact Γ in at most nine points which is the order of the line. Hence, all the points out of Γ will be of index zero. Thus, for points in $PG(3,8)\backslash\Gamma$, say $x_1 \dots x_k$, the intersection of $\Gamma \cup \{x_1 \dots x_k\}$ with any line will no longer be more than nine points. Therefore, the maximum complete cap formed from Γ is just the pace $PG(3,8)$, which is (585,9)-cap**.**

3. Conclusion

In this paper, we are found four types distinct (k, r)- caps in $PG(3,8)$ with respect to r where $r = 2, 3, 6, 9$, they are given as follows:

1. When $r = 2$ we created by action of groups 171 complete caps and 162 incomplete caps, classified according to their size as follows:

- 9 complete 65-caps
- 45 complete 28-caps.
- 117 complete 27-caps.
- 45 incomplete 13-caps.
- 117 incomplete 5-caps.

2. When $r = 3$ we created by action of groups 249 complete caps and 234 incomplete caps, classified according to their size as follows:

- 15 complete (39,3)-caps.

- 39 complete $(60,3)$-caps.
- 195 complete $(61,3)$-caps.
- 39 incomplete $(15,3)$-caps.
- 195 incomplete $(3,3)$-caps.

3. When $r = 6$ we created by action of groups 3 complete caps, classified according to their size as follows:

- 3 complete $(195,6)$-caps.

4. When $r = 9$ we created by action of groups only one complete cap and 73 incomplete caps, classified according to their size as follows:

- Complete $(585,9)$-cap is the perfect space.
- 5 incomplete $(117,9)$-caps.
- 13 incomplete $(45,9)$-caps.
- 65 incomplete $(9,9)$-caps.

Acknowledgments

The author thanks the University of Mustansiriyah for their support; also, thanks the Department of Mathematics at College of Science and Department of Mathematics at College of Basic Education for providing a good environment to do this research.

References

[1] J.W.P. Hirschfeld, *Finite Projective spaces of three dimensions.* New York: Ox-ford Mathematical Monographs, The Clarendon Press, Oxford University Press. **1985**.

[2] J.W.P. Hirschfeld, *Projective geometries over finite fields*. 2nd edn., New York: Ox-ford Mathematical Monographs, The Clarendon Press, Oxford University Press **1998**..

[3] J.W.P. Hirschfeld, and J.A. Thas, Open problems in finite projective spaces. *Finite Fields Appl.*, 32: 44-81**, 2015**.

[4] A. SH., Al-Mukhtar, Complete arcs and surfaces in three dimensional projective space over Galois field. Ph.D. Thesis, University of Technology, Baghdad, Iraq. **2008**.

[5] A.A., Abdulla, and N.Y., Kasm Yahya, Application of algebraic geometry in three dimensional projective space PG $(3,7)$. *J. Phys.: Conf. Ser.* 1591, **2020**.

[6] F.F., Kareem, Complete (k, r)-cap in $PG(3, p)$ over Galois field $GF(4)$. *IBN AL-Haitham J. for pure and appl.sci.*, 24(2), 2011.

[7] M.G., Oxenham, *On n-covers of $PG(3, q)$ and related structures*. PhD. Thesis, University of Adelaide, Australia, **1991**.

[8] N.A.M., Al-Seraji, A.J., Al-Al-Rikabi, E.B., Al-Zangana,. Represent the space $PG(3,8)$ by subspaces and subgeometries. Sixth National Scientific/Third International Conference at College of Education for Pure Sciences / Kerbala University, Iraq, *AIP Conference Proceedings*. To appear, **2021**.

[9] E.B., Al-Zangana, and N.Y., Kasm Yahya, Subgroups and orbits by companion matrix in three dimensional projective space. *Baghdad Sci. J.* 19(4):0805, **2021**.

[10] Al-Subahawi, S.M.A. and Al-Seraji, N.A.M. **2020**. Partitions on the projective plane over Galois field of order 11^m, m $= 1$,2 ,3.*Iraqi Journal of Science*, SI.1(28): 204-214.

[11] Al-Seraji, N.A.M., Al-Nussairy, E.A. and Jafar, Z.S. **2020**. The group action on the finite projective planes of orders 53, 61, 64. *Journal of Discrete Mathematical Sciences and Cryptography*, 23(8):1573-1582.

[12] Al-Seraji, N.A.M., Bakheet, A. and Jafar, Z.S. **2020**. Study of orbits on the finite projective plane. *Journal of Interdisciplinary Mathematics*, 23(6):1187-1195.

[13] Al-Seraji, N.A.M. and Al-Ogali, R.A.B. **2017.** The group action on a projective plane over finite field of order sixteen. *Iraqi Journal of Science*, 58(3).

[14] Al-Zangana, E.B. and Joudah, S. A. **2018.** Action of groups on the projective plane over the field $GF(41)$. *J. Phys.: Conf. Ser.* 1003, 012059.

[15] Al-Zangana, E.B**. 2021**. Projective MDS codes over $GF(27)$. *Baghdad Sci. J.* 18(2(Suppl.):1125.

[16] Al-Zangana, E.B. **2021**. Splitting of $PG(1,27)$ by sets and orbits, and arcs on the conic. *Iraqi Journal of Science (IJS),* 62(6).
[17] Al-Zangana, E.B. and Shehab, E.A. **2021**. Certain types of linear codes over the finite field of order twenty-five. *Iraqi Journal of Science (IJS),* 62(11).
[18] Ibrahim, M.M., Al-Zangana, E.B. **2021**. Classification of subsets in finite projective line over Galois field of order twenty-seven. *J. Phys.: Conf. Ser.* 1818(1), 012087.
[19] Topalova, S.Z. **2013**. Parallelisms of $PG(3,5)$ with automorphisms of order 13. Seventh International Workshop on Optimal Codes and Related Topics September, Albena, Bulgaria, 176-180.
[20] Lidl, R. and Niederreiter, H. **1997**. Finite fields. 2nd edn., Cambridge University Press.