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Estimates of Coefficient for Certain Subclasses of k-Fold Symmetric Bi-Univalent Functions

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Abstract
In the present paper, the authors introduce and investigate two new subclasses,namely $\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\eta)$ and $\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\alpha)$ of the class κ-fold bi-univalent functions in the open unit disk . The initial coefficients for all of the functions that belong to them are determined, as well as the coefficients for functions that belong to a field determining these coefficients are also determined which require a complicated process. The bounds for the initial coefficients $|b_{\kappa+1}|$ and $|b_{2\kappa+1}|$ that contained among the remaining results in our analysis are obtained. In addition, some specific special improver results for the related classes are provided.

Keywords. Analytic functions , Bi-Univalent functions , κ-Fold Symmetric Function, Univalent functions

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تخمين المعاملات للدوال ثنائية التكافؤ لعدد من الصفوف الجزئية المتماثلة

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الخلاصة
في هذه الورقة, يقدم المؤلفون ويحققون في فئتين فرعيتين جديدتين $\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\eta)$و $\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\alpha)$, لعدد من الفئات الفرعية ثنائية التكافؤ المتماثلة في قرص الوحدة المفتوح. نحدد المعاملات الأولية لجميع الدوال التي تنتمي اليها , وكذلك معاملات الدوال التي تنتمي الى مجال تحديد هذه المعاملات يتطلب عملية معقدة . تم تحديد المعاملات الاولية $|b_{\kappa+1}|$ و $|b_{2\kappa+1}|$ تم تحديدها بين النتائج المتبقية في تحليلنا التي تم الحصول عليها . بالإضافة الى ذلك , يتم تحديد بعض نتائج الخاصة للفئات ذات الصلة.

1. Introduction

Let $\mathcal{A}$ refers to the class of analytic functions f in the open unit disk $U = \{z \in C \text{ and } |z| < 1\}$,

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that satisfy the normalized condition $f(0) = f'(0) - 1 = 0$. An analytic function $f \in \mathcal{A}$ has Taylor series expansion of the form:

$$f(z) = z + \sum_{j=2}^{\infty} b_j z^j \ , \quad z \in U \tag{1}$$

The class of all functions in $\mathcal{A}$ refers by $\mathcal{K}$ which are univalent in U. In [1], the Koebe one–Quarter theorem ascertains that the image of U under each $f \in \mathcal{K}$ has a disk of radius ¼ . Obviously, for each $f \in \mathcal{K}$ which contains an inverse function f^{-1} satisfies $f^{-1}(f(z)) = z$, $z \in U$ and

$$f(f^{-1}(w)) = w \quad ,\left(|w| < \rho_0(f); \rho_0(f) \geq \frac{1}{4}\right),$$

where $g(w) = f^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2 b_3 + b_4)w^4 + \cdots$. (2)

When both f and f^{-1} are univalent in U, then $f \in \mathcal{A}$ is known to be bi univalent functions. In U, we denote the class of bi-univalent functions by Σ ,which are normalized by (1).

Lewin [2] obtained a coefficient bound that is given by $|a_2| \leq 1.51$ for every $f \in \sum$ and he looked at the class Σ of bi-univalent functions. Following that, inspired by Lewin's (see [2]) work. In 1980 Brannan and Clunie [3] guessed that $|a_2| \leq 2$ for all $f \in \sum$. In reality, Sriveastava et al. [4] have invigorated the analysis of analytical and bi-univalent functions. Bulut [5] has been pursued such study , Adegani and et al. [6] , Guney et al. [7] .

In 2016, the class of bi univalent defined by quasi subordination was introduced by Magesh et al. [8] and the coefficient bounds were obtained. In [9] , Ma and Minda presented and investigated the united classes:

$$T(\Phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \Phi(z) : z \in U \right\},$$

$$S(\Phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z) : z \in U \right\},$$

where $\Phi(z)$ is an analytic and univalent function in the unit disk U with a positive real part that satisfies $\Phi(0) = 1$, and $\Phi'(0) > 0$. $\Phi(U)$ is a starlike region in relation to 1 and it is symmetric in relation to real axis. The functions in the classes $T(\Phi)$ and $S(\Phi)$ are starlike of Ma-Minda type and convex of Ma-Minda type, respectively. For more details see [9].

For all $\in \mathcal{K}$, a function r which is given by $r(z) = \sqrt[\kappa]{f(z^\kappa)}$ $\kappa \in N = \{1,2,\ldots\}$ is univalent in U and maps into a region of κ-fold symmetry. If the normalized condition is satisfied and the function can be written as the form:

$$f(z) = z + \sum_{j=1}^{\infty} b_{\kappa j+1} z^{\kappa j+1} \qquad \kappa \in N = \{1,2,\ldots.\}, z \in U . \tag{3}$$

it is namely κ-fold symmetric [10,11].

The class of κ-fold symmetrical univalent function that are normalized by the previous series extension , it is denoted by $\mathcal{K}_\kappa$ in equation(3). If $\kappa = 1$ the function in class $\mathcal{K}$ is symmetric one-fold. Like the concept of symmetrical κ-fold. A conventional definition can be taken of bi-univalent symmetric κ-fold functions.

Srivastava et al. described κ-fold symmetric biunivalent functions antecedents to in [12]. The κ-fold symmetric bi-univalent function of a function f of the class $\sum$ for any positive integer κ is generated. The function f is normalized form of is defined in (3) and f^{-1} is defined in the following:

$g(w)=f^{-1}(w)=w-b_{\kappa+1}w^{\kappa+1}+[(\kappa+1)b_{\kappa+1}^2-b_{2\kappa+1}]w^{2\kappa+1}-\Big[\frac{1}{2}(\kappa+1)(3\kappa+2)b_{\kappa+1}^3-(3\kappa+2)b_{\kappa+1}b_{2\kappa+1}+b_{3\kappa+1}\Big]w^{3\kappa+1}+\cdots.$

(4)

where $g=f^{-1}$.

The κ-fold symmetric biunivalent functions class is denoted by $\sum_\kappa$. For κ=1, the formula (4) corresponds to the formule of class (2). Some examples of κ-fold symmetic bi-univalent functions are given as follows:

$\left(\frac{z^\kappa}{1-z^\kappa}\right)^{\frac{1}{\kappa}}$ and $\left[\frac{1}{2}\log\left(\frac{1+z^\kappa}{1-z^\kappa}\right)^{\frac{1}{\kappa}}\right]$,

according to the inverse functions, we have

$\left(\frac{w^\kappa}{1-w^\kappa}\right)^{\frac{1}{\kappa}}$ and $\left(\frac{e^{w^\kappa}-1}{e^{w^\kappa}}\right)^{\frac{1}{\kappa}}$.

Many experiments have been conducted to investigate estimates κ-fold bi-univalent functions with different subclasses, for detail [13-19]. The aim of this paper is to determine Taylor-Maclaurin coefficients $|b_{\kappa+1}|$ and $|b_{2\kappa+1}|$ for functions belong to the a new subclasses . We are also discussed some important findings that presented here.

Lemma1.[20],[21]. If $t\in P$, then $|t_i|\le 2$ for all i , where P is the collection of each analytic function t in U for which $Re\{t(z)\}>0, z\in U$ such that $t(z)=1+t_1z+t_2z^2+t_3z^3+\ldots, (z\in U)$.

2. The subclass $\mathcal{N}_{\sum_\kappa}(\gamma,\lambda,\delta;\eta)$

Definition2. A function $f(z)\in\sum_\kappa$ that defined by (3) is called in the class $\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\eta)$, if the following conditions are satisfied:

$$\left|arg\left(\frac{\gamma z^2f''(z)}{(1-\gamma)z+zf'(z)}+\beta\frac{z^2f''(z)}{f(z)}+\delta z^2f'''(z)\right)\right|<\frac{\eta\pi}{2}\quad,\ z\in U \tag{5}$$

and

$$\left|arg\left(\frac{\gamma w^2g''(w)}{(1-\gamma)w+wg'(w)}+\beta\frac{w^2g''(w)}{g(w)}+\delta w^2g'''(w)\right)\right|<\frac{\eta\pi}{2}\,,\quad w\in U, \tag{6}$$

where $(0\le\gamma\le1, \beta\ge0,\ 0\le\delta<1$ and $0<\eta\le1)$, $g=f^{-1}$ is given by (4) .

Theorem3. Let $f(z)$ be a function that defined by (3) in a class $\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\eta)$.Then

$$|b_{\kappa+1}|\le min\left\{\frac{2|\eta|}{\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]},\frac{2|\eta|}{\sqrt{|2\kappa(\kappa+1)[(2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)[\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]]|}}\right\} \tag{7}$$

and

$$|b_{2\kappa+1}|\le min\left\{\frac{2\eta^2}{(\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1))^2}+\frac{\eta}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]},\frac{2\eta^2}{2\kappa(\kappa+1)(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)][\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(1+\kappa)+\beta]}+\frac{\eta}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]}\right\}. \tag{8}$$

Proof.

Let $f\in\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\eta)$. Then

$$\frac{\gamma z^2f''(z)}{(1-\gamma)z+zf'(z)}+\beta\frac{z^2f''(z)}{f(z)}+\delta z^2f'''(z)=[h(z)]^\eta \tag{9}$$

and

$$\frac{\gamma w^2 g''(w)}{(1-\gamma)w+wg'(w)}+\beta\frac{w^2 g''(w)}{g(w)}+\delta w^2 g'''(w)=[u(z)]^{\eta}\ , \qquad (10)$$

where $g=f^{-1},\ h(z)$ and $u(w)$ is a function in P and have the form

$$h(z)=1+h_{\kappa}z^{\kappa}+h_{2\kappa}z^{2\kappa}+h_{3\kappa}z^{3\kappa}+\cdots \qquad (11)$$

and

$$u(w)=1+u_{\kappa}w^{\kappa}+u_{2\kappa}w^{2\kappa}+u_{3\kappa}w^{3\kappa}+\cdots\quad . \qquad (12)$$

Now, from (9) and (10), we get the following

$$\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]b_{\kappa+1}=\eta h_{\kappa}\quad , \qquad (13)$$

$$2\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]b_{2\kappa+1}-\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]b_{\kappa+1}^2 = \eta h_{2\kappa} + \frac{1}{2}\eta(\eta-1)h_{\kappa}^2\ , \qquad (14)$$

$$-\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]b_{\kappa+1}=\eta u_{\kappa}\quad , \qquad (15)$$

and

$$-2\kappa(1+2\kappa)[\gamma+\beta+\delta(2\kappa-1)]b_{2\kappa+1} + \left[2\kappa(\kappa+1)(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]-\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]\right]b_{\kappa+1}^2 = \eta u_{2\kappa} + \frac{1}{2}\eta(\eta-1)u_{\kappa}^2\ . \qquad (16)$$

From (11) and (13), we get

$$h_{\kappa}=-u_{\kappa} \qquad (17)$$

and

$$2\big((\kappa+1)[\gamma+\beta+\delta(\kappa-1)]\big)^2 b_{\kappa+1}^2=\eta^2(h_{\kappa}^2+u_{\kappa}^2)\quad . \qquad (18)$$

Now, add (14) to (16) , we get

$$2\kappa(\kappa+1)[(1+2\kappa)(\gamma+\beta+\delta(2\kappa-1)[\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1) + \beta]]b_{\kappa+1}^2 = \eta(h_{\kappa}+u_{\kappa}) + \frac{1}{2}\eta(\eta-1)(h_{\kappa}^2+u_{\kappa}^2)\quad . \qquad (19)$$

By applying Lemma1 for the coefficients $h_{\kappa}, u_{\kappa}, h_{\kappa}^2$ and u_{κ}^2 .It follows from (17) and (18),the following

$$|b_{\kappa+1}|\le\frac{2|\eta|}{\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]} \qquad (20)$$

and

$$|b_{\kappa+1}|\le\frac{2|\eta|}{\sqrt{|2\kappa(\kappa+1)[(2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)[\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]]|}}, \qquad (21)$$

which yield coefficient $b_{\kappa+1}$ in (7).

Subtracting (16) from (14), we obtain

$$4\kappa(2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)b_{2\kappa+1} = 2\kappa(\kappa+1)[2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)]b_{\kappa+1}^2+\eta(h_{\kappa}-u_{\kappa}) + \frac{1}{2}\eta(\eta-1)(h_{\kappa}^2-u_{\kappa}^2)\quad . \qquad (22)$$

By subtracting from (18) and (19) , putting (22) and we use Lemma1 to find that

$$|b_{2\kappa+1}|\le\frac{2\eta^2}{(\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1))^2} + \frac{\eta}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]} \qquad (23)$$

and

$$|b_{2\kappa+1}| \leq \frac{2\eta^2}{2\kappa(\kappa+1)(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)][\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]} + \frac{\eta}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]} \quad . \tag{24}$$

(23) and (24) yield the estimate in (8) .
The proof is completed.
In case $\gamma = 0$ and $\beta = 0$, we can obtain the following results.
Corollary4. Let $f(z)$ that defined by (3) be in the class$\mathcal{N}_{\sum_\kappa}(\delta;\eta)$. Then

$$|b_{\kappa+1}| \leq min\left\{\frac{2|\eta|}{\kappa(\kappa+1)[\delta(\kappa-1)]}, \frac{2|\eta|}{\sqrt{|2\kappa(\kappa+1)[(2\kappa+1)(\delta(2\kappa-1)[\delta(2\kappa-1)]|}}\right\}$$

and

$$|b_{2\kappa+1}| \leq min\left\{\frac{2\eta^2}{(\kappa(\kappa+1)[\delta(\kappa-1))^2} + \frac{\eta}{\kappa(2\kappa+1)[\delta(2\kappa-1)]}, \frac{2\eta^2}{2\kappa(\kappa+1)(2\kappa+1)[\delta(2\kappa-1)][\delta(2\kappa-1)} + \frac{\eta}{\kappa(2\kappa+1)[\delta(2\kappa-1)]}\right\}.$$

In case $= 1$, $\beta = 0$ and $\delta = 0$, we get the next Corollary
Corollary5. Let $f(z)$ that defined by(3) be in the class $\mathcal{N}_{\sum_\kappa}(\eta)$.Then

$$|b_{\kappa+1}| \leq min\left\{\frac{2|\eta|}{\kappa(1+\kappa)} \ , \quad \frac{2|\eta|}{\sqrt{|2\kappa(1+\kappa)^2[(2\kappa+1)[1-2\kappa(\kappa+1)]|}}\right\}$$

and

$$|b_{2\kappa+1}| \leq min\left\{\frac{2\eta^2}{\left(\kappa(\kappa+1)\right)^2} + \frac{\eta}{\kappa(2\kappa+1)}, \frac{2\eta^2}{2\kappa(\kappa+1)^2(2\kappa+1)[1-2\kappa(\kappa+1)} + \frac{\eta}{\kappa(2\kappa+1)}\right\}.$$

In case $\gamma = 1$ and $\delta = 0$, we obtain the following Corollary
Corollary6 Let (z) , which is defined by(3) be in the class$\mathcal{N}_{\sum_\kappa}(\beta,\eta)$.Then

$$|b_{\kappa+1}| \leq min\left\{\frac{2|\eta|}{\kappa(\kappa+1)[1+\beta]}, \frac{2|\eta|}{\sqrt{|2\kappa(\kappa+1)[(2\kappa+1)(1+\beta)[(1+\beta)-2\kappa(\kappa+1)[(\kappa+1)+\beta]]|}}\right\}$$

and

$$|b_{2\kappa+1}| \leq min\left\{\frac{2\eta^2}{(\kappa(\kappa+1)[1+\beta)^2} + \frac{\eta}{\kappa(2\kappa+1)[1+\beta]}, \frac{2\eta^2}{2\kappa(\kappa+1)(2\kappa+1)[1+\beta][1+\beta-2\kappa(\kappa+1)[(\kappa+1)+\beta]} + \frac{\eta}{\kappa(2\kappa+1)[1+\beta]}\right\}.$$

In the case of $\kappa = 1$, Theorem3 can be reduced to the next results.
Corollary7. Let a function $f(z)$ which is defined by (3) be in the class$\mathcal{N}_{\sum}(\gamma,\beta,\delta;\eta)$.Then

$$|b_2| \leq min\left\{\frac{|\eta|}{[\gamma+\beta]}, \quad \frac{2|\eta|}{\sqrt{|4[3(\gamma+\beta+\delta[\gamma+\beta+\delta-4)[2\gamma^2+\beta]]|}}\right\}$$

and

$$|b_3| \leq min\left\{\frac{\eta^2}{2([\gamma+\beta+])^2} + \frac{\eta}{3[\gamma+\beta+\delta]}, \frac{\eta^2}{6[\gamma+\beta+\delta][\gamma+\beta+\delta-4[2\gamma^2+\beta]} + \frac{\eta}{3[\gamma+\beta+\delta]}\right\}$$

3. The subclass $\mathcal{N}_{\sum_\kappa}(\gamma,\lambda,\delta;\alpha)$

Definition8. A function $f(z) \in \sum_\kappa$ defined by (3) is called in the class $\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\alpha)$, if the following requirements are met:

$$Re\left(\frac{\gamma z^2 f''(z)}{(1-\gamma)z + zf'(z)} + \beta\frac{z^2 f''(z)}{f(z)} + \delta z^2 f'''(z)\right) > \alpha \ , \ z \in U \tag{25}$$

and

$$Re\left(\frac{\gamma w^2 g''(w)}{(1-\gamma)w + wg'(w)} + \beta\frac{w^2 g''(w)}{g(w)} + \delta w^2 g'''(w)\right) > \alpha \ , \qquad w \in U, \tag{26}$$

where ($0 \leq \gamma \leq 1$, $\beta \geq 0$, $0 \leq \delta < 1$ and $0 \leq \alpha < 1$) , $g = f^{-1}$ is given by (4) .

Theorem9. Let $f(z)$ be a function defined by (3) in the class$\mathcal{N}_{\sum_\kappa}(\gamma,\beta,\delta;\alpha)$.Then

$$|b_{\kappa+1}| \leq \min\left\{\frac{2(1-\alpha)}{\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]}, \sqrt{\frac{2(1-\alpha)}{\kappa(\kappa+1)[(2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)[\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]]}}\right\} \quad (27)$$

and

$$|b_{2\kappa+1}| \leq \min\left\{\frac{2(1-\alpha)^2}{(\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)])^2} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]}, \frac{2(1-\alpha)^2}{2\kappa(\kappa+1)(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)][\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]}\right\}. \quad (28)$$

Proof.

From (27) and (28) there is $h, u \in P$, such that

$$\frac{\gamma z^2 f''(z)}{(1-\gamma)z + zf'(z)} + \beta\frac{z^2 f''(z)}{f(z)} + \delta z^2 f'''(z) = \alpha + (1-\alpha)h(z) \quad (29)$$

and

$$\frac{\gamma w^2 g''(w)}{(1-\gamma)w + wg'(w)} + \beta\frac{w^2 g''(w)}{g(w)} + \delta w^2 g'''(w) = \alpha + (1-\alpha)h(z) \quad , \quad (30)$$

where $g = f^{-1}$, $h(z)$ and $u(w)$ is a function in P and has the form

$$h(z) = 1 + h_\kappa z^\kappa + h_{2\kappa} z^{2\kappa} + h_{3\kappa} z^{3\kappa} + \cdots \quad (31)$$

and

$$u(w) = 1 + u_\kappa w^\kappa + u_{2\kappa} w^{2\kappa} + u_{3\kappa} w^{3\kappa} + \cdots \quad . \quad (32)$$

Now, from (29) and (30), we have

$$\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]b_{\kappa+1} = (1-\alpha)h_\kappa \quad , \quad (33)$$

$$2\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]b_{2\kappa+1} - \kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]a_{\kappa+1}^2 = (1-\alpha)h_{2\kappa} \quad , \quad (34)$$

$$-\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]b_{\kappa+1} = (1-\alpha)u_\kappa \quad , \quad (35)$$

and

$$-2\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]b_{2\kappa+1} + \left[2\kappa(\kappa+1)(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)] - \kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]\right]b_{\kappa+1}^2 = (1-\alpha)u_{2\kappa} \quad . \quad (36)$$

From (31) and (33), we obtain

$$h_\kappa = -u_\kappa \quad (37)$$

and

$$2\big((\kappa+1)[\gamma+\beta+\delta(\kappa-1)]\big)^2 b_{\kappa+1}^2 = (1-\alpha)^2(h_\kappa^2 + u_\kappa^2) \quad (38)$$

Adding (34) and (36) , we get

$$2\kappa(\kappa+1)[(2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)[\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]]b_{\kappa+1}^2 = (1-\alpha)(h_{2\kappa} + u_{2\kappa}) \quad (39)$$

From Lemma1, for the coefficients $h_{2\kappa}, u_{2\kappa}, h_\kappa^2$ and u_κ^2, its follows from (37) and (38), we get

$$|b_{\kappa+1}| \leq \frac{2(1-\alpha)}{\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1)]} \quad (40)$$

and

$$|b_{\kappa+1}| \leq \sqrt{\frac{2(1-\alpha)}{\kappa(\kappa+1)[(2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)[\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]]}} , \qquad (41)$$

which yields coefficient $b_{\kappa+1}$ in (27).
If we subtract (16) from (14), then we obtain

$$4\kappa(2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)b_{2\kappa+1} = 2\kappa(\kappa+1)[2\kappa+1)(\gamma+\beta+\delta(2\kappa-1)]b_{\kappa+1}^2 + (1-\alpha)(h_{2\kappa}-u_{2\kappa}) . \qquad (42)$$

Upon subtracting from (38) and (39) and we put the result in (42). Hence from Lemma1, we find that

$$|b_{2\kappa+1}| \leq \frac{2(1-\alpha)^2}{(\kappa(\kappa+1)[\gamma+\beta+\delta(\kappa-1))^2} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]} \qquad (43)$$

and

$$|b_{2\kappa+1}| \leq \frac{2(1-\alpha)^2}{2\kappa(\kappa+1)(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)][\gamma+\beta+\delta(2\kappa-1)-2\kappa(\kappa+1)[\gamma^2(\kappa+1)+\beta]} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[\gamma+\beta+\delta(2\kappa-1)]} . \qquad (44)$$

The equations (43) and (44) yield the estimate in (28) .
The theorem's proof is now completed.
In case $\gamma = 0$ and $\beta = 0$, we get the following Corollary.
Corollary10. Let $f(z)$ be a function that efined by (3) in class$\mathcal{N}_{\sum_\kappa}(\delta;\alpha)$.Then

$$|b_{\kappa+1}| \leq min\left\{\frac{2(1-\alpha)}{\kappa(\kappa+1)[\delta(\kappa-1)]}, \quad \sqrt{\frac{2(1-\alpha)}{\kappa(\kappa+1)[(2\kappa+1)(\delta(2\kappa-1)[\delta(2\kappa-1)]}}\right\}$$

and

$$|b_{2\kappa+1}| \leq min\left\{\frac{2(1-\alpha)^2}{(\kappa(\kappa+1)[\delta(\kappa-1))^2} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[\delta(2\kappa-1)]}, \frac{2(1-\alpha)^2}{2\kappa(\kappa+1)(2\kappa+1)[\delta(2\kappa-1)][\delta(2\kappa-1)} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[\delta(2\kappa-1)]}\right\}.$$

In case $= 1$, $\beta = 0$,and $\delta = 0$, we the next result can be got.
Corollary11 Let $f(z)$ be a function that defined by (3) in class$\mathcal{N}_{\sum_\kappa}(\alpha)$. Then

$$|b_{\kappa+1}| \leq min\left\{\frac{2(1-\alpha)}{\kappa(\kappa+1)}, \quad \sqrt{\frac{2(1-\alpha)}{\kappa(\kappa+1)^2[(2\kappa+1)[1-2\kappa(\kappa+1)]}}\right\}$$

and

$$|b_{2\kappa+1}| \leq min\left\{\frac{2(1-\alpha)^2}{\left(\kappa(\kappa+1)\right)^2} + \frac{(1-\alpha)}{\kappa(2\kappa+1)}, \frac{2(1-\alpha)^2}{2\kappa(\kappa+1)^2(2\kappa+1)[1-2\kappa(\kappa+1)} + \frac{(1-\alpha)}{\kappa(2\kappa+1)}\right\}.$$

In case $\gamma = 1$ and $\delta = 0$, the following corollary will be obtained.

Corollary12. Let $f(z)$ be a function that defined by (3) in class$\mathcal{N}_{\sum_\kappa}(\beta,\alpha)$. Then

$$|b_{\kappa+1}| \leq min\left\{\frac{2(1-\alpha)}{\kappa(\kappa+1)[1+\beta]}, \quad \sqrt{\frac{2(1-\alpha)}{\kappa(\kappa+1)[(2\kappa+1)(1+\beta)[(1+\beta)-2\kappa(\kappa+1)[(\kappa+1)+\beta]]}}\right\}$$

and

$|b_{2\kappa+1}| \leq min\left\{\frac{2(1-\alpha)^2}{(\kappa(\kappa+1)[1+\beta)^2} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[1+\beta]}, \frac{2(1-\alpha)^2}{2\kappa(\kappa+1)(2\kappa+1)[1+\beta][1+\beta-2\kappa(\kappa+1)[(\kappa+1)+\beta]} + \frac{(1-\alpha)}{\kappa(2\kappa+1)[1+\beta]}\right\}.$

In the case of $\kappa = 1$,one can reduce theorem 9 to the next result.

Corollary13. Let $f(z)$ be a function that defined by (3) in the class $\mathcal{N}_{\Sigma}(\gamma,\beta,\delta;\alpha)$.Then

$$|b_2| \leq min\left\{\frac{(1-\alpha)}{[\gamma+\beta]}, \quad \sqrt{\frac{(1-\alpha)}{[3(\gamma+\beta+\delta)[\gamma+\beta+\delta-4[2\gamma^2+\beta]]}}\right\}$$

and

$$|b_3| \leq min\left\{\frac{(1-\alpha)^2}{2([\gamma+\beta)^2} + \frac{(1-\alpha)}{3[\gamma+\beta+\delta]}, \frac{(1-\alpha)^2}{6[\gamma+\beta+\delta][\gamma+\beta+\delta-4[2\gamma^2+\beta]} + \frac{(1-\alpha)}{3[\gamma+\beta+\delta]}\right\}.$$

Conclusion : As a result of research, can say that, when applying the new two recent subclasses $\mathcal{N}_{\Sigma_\kappa}(\gamma,\beta,\delta;\eta)$ and $\mathcal{N}_{\Sigma_\kappa}(\gamma,\beta,\delta;\alpha)$, κ-fold symmetric to the geometric functions are included, $|b_{\kappa+1}| and\ |b_{2\kappa+1}|$ were determined for each class κ-fold symmetric bi-univalent, in complex analysis it is useful. Several improved results are provided in $U$for these two new subclasses.

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