DOI:10.24996/ijs.2018.59.3B.14

ISSN: 0067-2904

Symmetrical Fibonacci and Lucas Wave Solutions for Some Nonlinear Equations in Higher Dimensions

M. Allami*, A. S. Rashid and A. K. Mutashar
Department of Mathematics, College of Education, Misan University, Iraq

Abstract
We consider some nonlinear partial differential equations in higher dimensions, the negative order of the Calogero-Bogoyavelnskii-Schiff (nCBS) equationin (2+1) dimensions, the combined of the Calogero-Bogoyavelnskii-Schiff equation and the negative order of the Calogero-Bogoyavelnskii-Schiff equation (CBS-nCBS) in (2+1) dimensions, and two models of the negative order Korteweg de Vries (nKdV) equations in (3+1) dimensions. We show that these equations can be reduced to the same class of ordinary differential equations via wave reduction variable. Solutions in terms of symmetrical Fibonacci and Lucas functions are presented by implementation of the modified Kudryashov method.

Keywords: nCBS equation, CBS-nCBS equation, nKdV equation, modified Kudryashov method, symmetrical Fibonacci and Lucas functions.

حلول فيبوناشي ولوكاس الموجية المتماثلة لبعض المعادلات غير الخطية في ابعاد عليا

محمد جبار حواس اللامي*، علي سامي رشيد ، احمد كريم مطشر
قسم الرياضيات، كلية التربية، جامعة ميسان، ميسان، العراق

الخلاصة
تناولنا بعض المعادلات التفاضلية الجزئية غير الخطية في ابعاد عليا، معادلة الرتبة السالبة كالوغيرو بوغويافيلينسكي شيف في 1+2 بعد ، المعادلة المجتمعة لمعادلتي كالوغيرو بوغويافيلينسكي شيف ومعادلة الرتبة السالبة كالوغيرو بوغويافيلينسكي شيف في 1+2 بعد، نموذجان من معادلة كورتواك دي فرايز الرتبة السالبة في 1+3 بعد . بينا ان هذه المعادلات يمكن تحويلها الى نفس صنف المعادلات التفاضلية الاعتيادية من خلال متغير خفض الرتبة. حلول بدلالة دوال فيبوناتشي ولوكاس المتماثلة تم الحصول عليها بتطبيق طريقة كودرياشوف المعدلة .

1 Introduction

The derivation of nonlinear partial differential equations by employing recursion operators has been attracted a considerable attention recently [1,2,3]. The recursion operator is known as an integro-differential operator that one of its uses is that to form differential equations. The author in [1] derived the nCBS equation in two spatial dimensions plus time. This is given by

$$u_{xxxt} + 4u_x u_{xt} + 2u_{xx} u_t + u_{xy} = 0, \tag{1}$$

and that by exploiting the inverse of the recursion operator for the KdV equation. Beside of that he used the simplified version of Hirota's method to get some types of solutions. In [2] the same

*Email: drmjh53@gmail.com

framework, the combined CBS-nCBS equation was built up by putting together the recursion operator of the CBS and its inverse operator. The combined CBS-nCBS equation [2], is then

$$u_{xt} + u_{xxxy} + u_{xxxt} + 4u_x(u_{xy} + u_{xt)}) + 2u_{xx}(u_y + u_t) = 0. \quad (2)$$

It was shown that it is a completely integrable equation in the sense that it passes Painelvè test [2]. Moreover the simplified version of Hirota's method was used to obtain soliton solutions. In [3], the Verosky approach with the recursion operator for the KdV equation were used to construct some new models in three fields. The author managed also to obtain multi-soliton solutions by applying the simplified version of Hirota's method. The nKdV model I equation in three fields plus time [3] reads

$$-u_{xxxt} + 4u_x u_{xt} + 2u_{xx}u_t - u_{xx} - u_{xy} - u_{xz} = 0, \quad (3)$$

and the nKdV model II equation [3] therefore

$$-u_{xxxt} + 4u_x u_{xt} + +2u_{xx}u_t - u_{xx} - u_{xy} + u_{xxxz} - 4u_x u_{xz} - 2u_{xx}u_z = 0. \quad (4)$$

A few analytic solutions for these equations were found. In the coming sections, we shall solve these equations by applying the modified Kudryashov method in order to gain exact solutions by means of symmetrical Fibonacci and Lucas functions. The organization of the paper as follows; in section two the modified Kudryashov method is described. The following three Sections are devoted to transform the nCBS equation, the nCBS-CBS equation, the nKdV model I equation and the nKdV model II equation into ordinary differential equation, the reduced equation, by using a wave reduction variable. In section six the reduced equation is solved by using the modified Kudryashov method which gives the base for constructing the solutions for the nCBS equation, the CBS-nCBS equation, the nKdV model I and model II equations in section seven. Finally, conclusion is given.

2 The modified Kudryashov method

The Kudryashov method is named after Kudryashov [4]. The Kudryashov method is a reliable method for getting solutions of nonlinear equations. It is applied and developed by many researchers see for instance [5-11]. We shall use Pandir's modification [5] for this method to solve the nonlinear equations under consideration. The method can be briefly described as follows

Consider a partial differential equation

$$G(u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, u_{xz}, u_{xt}, u_{xxx}, \dots.) = 0\,, \quad (5)$$

where u is a function of $x, y,\ z$ and t and $u_x, u_y, \dots$ refer to the partial derivatives with respect to the independent variables .

Taking the wave reduction variable

$$\xi = \alpha x + \beta y + \gamma z + \delta t,$$

and

$$u(x, y, z, t) = U(\xi), \qquad \xi = \xi(x, y, z, t),$$

where α, β, γ and δ are arbitrary constants.

Equation (5) is transformed into an ordinary differential equation

$$F(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, \dots\dots..) = 0, \quad (6)$$

where $U_{(r)} = \frac{d^r U}{d\xi^r}$.

We search for solutions in the form

$$U(\xi) = \sum_{k=-n}^{n} c_k Q(\xi)^k, \quad (7)$$

where $c_k,\ k = 0, \pm 1, \pm 2, \pm 3, \dots\dots, \pm n$ are arbitrary constants and $Q\,(\xi)$ satisfies the auxiliary differential equation

$$Q_\xi(\xi) = \ln a\big(Q^2(\xi) - Q(\xi)\big), \quad (8)$$

where a is a constant . The solution for auxiliary equation (8) is

$$Q(\xi) = \frac{1}{1 \pm a^\xi}\ , a \neq 1.$$

The number of the terms, positive number n , in the formula (7) can be deduced by balancing highest order nonlinear terms in the equation (6) . Inserting determined equations (7) and (8) into equation (6) and next collecting the coefficients of the $Q(\xi)^k$ $(k = 0, \pm 1, \pm 2, \pm 3, \dots)$ and then equating each coefficient to zero to gain a system of algebraic equations can be solved to get the solutions of the equations under study.

3 The negative order of the Calogero – Bogoyavelnskii-Schiff equation

The nCBS equation (1) is written by

$$u_{xxxt} + 4u_x u_{xt} + 2u_{xx}u_t + u_{xy} = 0, \tag{9}$$

where u is a function of x, y and t . It is a fourth order nonlinear partial differential equation. Clearly, the terms u_{xxxt} and u_{xy} represent the linear terms while the terms $u_x u_{xt}$ and $u_{xx}u_t$ are the nonlinear terms. The nCBS equation (9) can be transformed into ordinary differential equation and that by reducing the number of independent variables to only one independent variable ξ by using wave reduction variable

$$u(x, y, z, t) = U(\xi), \quad \xi = \alpha x + \beta y + \delta t, \tag{10}$$

where α, β and δ are arbitrary constants.

Substituting equation (10) into equation (9), after using the chain rule, gives

$$\alpha^3 \delta U_{\xi\xi\xi\xi} + 6\alpha^2 \delta U_\xi U_{\xi\xi} + \alpha\beta U_{\xi\xi} = 0,$$

integrating with respect to ξ

$$\alpha^3 \delta U_{\xi\xi\xi} + 3\alpha^2 \delta {U_\xi}^2 + \alpha\beta U_\xi + c^* = 0,$$

where c^* is a constant of integration.

Multiplying $U_{\xi\xi}$ and integrating it again lead to

$$\frac{\alpha^3\delta}{2} {U_{\xi\xi}}^2 + \alpha^2 \delta {U_\xi}^3 + \frac{\alpha\beta}{2} {U_\xi}^2 + c^* U_\xi + c^{**} = 0,$$

where c^{**} is a constant of integration . Rewriting the last equation

$${U_{\xi\xi}}^2 + \frac{2}{\alpha} {U_\xi}^3 + \frac{\beta}{\alpha^2\delta} {U_\xi}^2 + \frac{2c^*}{\alpha^3\delta} U_\xi + \frac{2c^{**}}{\alpha^3\delta} = 0.$$

Now by using the dependent variable transformation

$$U = \int_\xi \Phi d\xi \ , \tag{11}$$

that gives

$$\Phi_\xi^2 + A_1\Phi^3 + B_1\Phi^2 + C_1\Phi + D_1 = 0, \tag{12}$$

where

$$A_1 = \frac{2}{\alpha}, \quad B_1 = \frac{\beta}{\alpha^2\delta}, \quad C_1 = \frac{2c^*}{\alpha^3\delta} \quad and \quad D_1 = \frac{2c^{**}}{\alpha^3\delta}, \tag{13}$$

and all the constants are arbitrary .

4 The combined of the Calogero-Bogoyavelnskii-Schiff equation and the negative order of the Calogero-Bogoyavelnskii-Schiff equation

We again use a wave reduction variable to transform the combined CBS- nCBS equation (2), reads

$$u_{xt} + u_{xxxy} + u_{xxxt} + 4u_x(u_{xy} + u_{xt}) + 2u_{xx}(u_y + u_t) = 0, \tag{14}$$

into ordinary differential equation and that by reducing the number of independent variables to only one independent variable ξ by exploiting

$$u(x, y, z, t) = U(\xi), \qquad \xi = \alpha x + \beta y + \delta t, \tag{15}$$

where α, β and δ are arbitrary constants.

Substituting equation (15) into equation (14), after using the chain rule, gives

$$\alpha^3(\beta + \delta)U_{\xi\xi\xi\xi} + 6\alpha^2(\beta + \delta)U_\xi U_{\xi\xi} + \alpha\delta U_{\xi\xi} = 0,$$

integrating once

$$\alpha^2(\beta + \delta)U_{\xi\xi\xi} + 3\alpha(\beta + \delta){U_\xi}^2 + \delta U_\xi + c^* = 0,$$

where c^* is a constant of integration.

Multiplying $U_{\xi\xi}$ and integrating again leads to

$$\alpha^2(\beta + \delta){U_{\xi\xi}}^2 + 2\alpha(\beta + \delta){U_\xi}^3 + \delta {U_\xi}^2 + 2c^* U_\xi + 2c^{**} = 0,$$

where c^{**} is a constant of integration.

Using the dependent variable transformation, equation (11), one can obtain

$$\Phi_\xi^2 + A_2\Phi^3 + B_2\Phi^2 + C_2\Phi + D_2 = 0, \tag{16}$$

where

$$A_2 = \frac{2}{\alpha}, \quad B_2 = \frac{\delta}{\alpha^2(\beta+\delta)}, \quad C_2 = \frac{2c^*}{\alpha^2(\beta+\delta)} \quad and \quad D_2 = \frac{2c^{**}}{\alpha^2(\beta+\delta)}, \tag{17}$$

and all the constants are arbitrary .

5. The negative order Korteweg de Vries model I and model II equations

We start off with the nKdV model I equation (3), reads

$$-u_{xxxt} + 4u_x u_{xt} + 2u_{xx}u_t - u_{xx} - u_{xy} - u_{xz} = 0. \tag{18}$$

Putting

$$u(x,y,z,t) = U(\xi), \ \xi = \alpha x + \beta y + +\gamma y + \delta t, \tag{19}$$

where α, β, γ and δ are arbitrary constants.

Substituting equation (19) into equation (18), after employing the chain rule, to obtain

$$-\alpha^3\delta U_{\xi\xi\xi\xi} + 6\alpha^2\delta U_\xi U_{\xi\xi} - \alpha(\alpha+\beta+\gamma)U_{\xi\xi} = 0,$$

integrating once with respect to ξ

$$-\alpha^3\delta U_{\xi\xi\xi} + 3\alpha^2\delta {U_\xi}^2 - \alpha(\alpha+\beta+\gamma)U_\xi + c^* = 0,$$

where c^* is a constant of integration.

Multiplying $U_{\xi\xi}$ and integrating again gives

$$-\alpha^3\delta U_{\xi\xi}^2 + 2\alpha^2\delta U_\xi^2 - \alpha(\alpha+\beta+\gamma)U_\xi^2 + 2c^*U_\xi + 2c^{**} = 0,$$

where c^{**} is a constant of integration.

Using the dependent variable transformation, equation (11), one can obtain

$$\Phi_\xi^2 + A_3\Phi^3 + B_3\Phi^2 + C_3\Phi + D_3 = 0, \tag{20}$$

and

$$A_3 = -\frac{2}{\alpha}, \quad B_3 = \frac{\alpha+\beta+\gamma}{\alpha^2\delta}, \quad C_3 = -\frac{2c^*}{\alpha^3\delta} \quad and \quad D_3 = -\frac{2c^{**}}{\alpha^3\delta} \tag{21}$$

where all the constants are arbitrary .

We arrive at the nKdV model II equation (4), that is

$$-u_{xxxt} + 4u_x u_{xt} + 2u_{xx}u_t - u_{xx} - u_{xy} + u_{xxxz} - 4u_x u_{xz} - 2u_{xx}u_z = 0, \tag{22}$$

applying

$$u(x,y,z,t) = U(\xi), \ \xi = \alpha x + \beta y + \gamma z + \delta t, \tag{23}$$

where α, β, γ and δ are arbitrary constants.

Substituting equation (23) into equation (22), after using the chain rule, yields

$$\alpha^3(\gamma-\delta)U_{\xi\xi\xi\xi} + 6\alpha^2(\delta-\gamma)U_\xi U_{\xi\xi} - \alpha(\alpha+\beta)U_{\xi\xi} = 0,$$

integrating once

$$\alpha^3(\gamma-\delta)U_{\xi\xi\xi} + 3\alpha^2(\gamma-\delta){U_\xi}^2 - \alpha(\alpha+\beta)U_\xi + c^* = 0,$$

where c^* is a constant of integration.

Multiplying $U_{\xi\xi}$ and integrating again gives

$$\alpha^3(\gamma-\delta)U_{\xi\xi}^2 + 2\alpha^2(\delta-\gamma)U_\xi^3 - \alpha(\alpha+\beta)U_\xi^2 + 2c^*U_\xi + 2c^{**} = 0,$$

where c^{**} is a constant of integration .

Using the dependent variable transformation, equation (11), leads to

$$\Phi_\xi^2 + A_4\Phi^3 + B_4\Phi^2 + C_4\Phi + D_4 = 0, \tag{24}$$

where

$$A_4 = -\frac{2}{\alpha}, \ B_4 = -\frac{\alpha+\beta}{\alpha^2(\gamma-\delta)}, \ C_4 = \frac{2c^*}{\alpha^3(\gamma-\delta)} \quad and \quad D_4 = \frac{2c^{**}}{\alpha^3(\gamma-\delta)}, \tag{25}$$

and all the constants are arbitrary.

6. Solutions of the nonlinear reduced equation

We move in this section to solve the nonlinear equation

$$\Phi_\xi^2 + A\Phi^3 + B\Phi^2 + C\Phi + D = 0. \tag{26}$$

One may notice that this equation is similar to equations (12), (16), (20) and (24). In order to solve equation (26), we follow the same approach that was employed by Pandir [5], by doing balance between Φ_ξ^2 and Φ^3, one can deduce that n =2 in equation (7)

$$\Phi(\xi) = c_{-2}Q^{-2} + c_{-1}Q^{-1} + c_0 + c_1Q^1 + c_2Q^2, \tag{27}$$

where c_{-2}, c_{-1}, c_0, c_1 and c_2 are constants to be determined and $Q(\xi)$ satisfies the auxiliary equation

$$Q_\xi(\xi) =, \tag{28}$$

where a is a constant . The solution for the auxiliary equation (28) is

$$Q(\xi) = \frac{1}{1 \pm a^{\xi}}, a \neq 1.$$

Now differentiating equation (27) once with respect to ξ

$$\Phi_\xi = -2c_{-2}Q_\xi Q^{-3} - c_{-1}Q_\xi Q^{-2} + c_1 Q_\xi + 2c_2 Q_\xi\, Q. \tag{29}$$

Substituting equation (27) and equation (29) into equation (26) yields

$$\left(-2c_{-2}Q_\xi Q^{-3} - c_{-1}Q_\xi Q^{-2} + c_1 Q_\xi + 2c_2 Q_\xi Q\right)^2 + A(c_{-2}Q^{-2} + c_{-1}Q^{-1} + c_0 + c_1 Q^1 + c_2 Q^2)^3 + B(c_{-2}Q^{-2} + c_{-1}Q^{-1} + c_0 + c_1 Q^1 + c_2 Q^2)^2 + C(c_{-2}Q^{-2} + c_{-1}Q^{-1} + c_0 + c_1 Q^1 + c_2 Q^2) + D = 0.$$

Inserting equation (28) into the last equation and simplifying to obtain

$$\begin{aligned} & Ac_{-2}^3 + DQ^6 + BQ^8 c_1^2 + 6Ac_{-2}c_{-1}Q^4 c_1 + 6Ac_{-2}c_{-1}Q^5 c_2 + 6Ac_{-2}c_1 Q^7 c_2 + 6Ac_{-2}c_{-1}Q^3 c_0 \\ & + 6Ac_{-2}c_0 Q^5 c_1 + 6Ac_{-2}c_0 Q^6 c_2 + 6Ac_{-1}c_2 Q^8 c_1 + 6Ac_{-1}c_0 Q^6 c_1 + 6Ac_{-1}c_0 c_2 Q^7 \\ & + 6Ac_1 c_0 c_2 Q^9 + Ac_0^3 Q^6 + Ac_1^3 Q^9 + Ac_2^3 Q^{12} + Bc_{-2}^2 Q^2 + Bc_{-1}^2 Q^4 + Bc_2^2 Q^{10} \\ & + Bc_0^2 Q^6 + Cc_{-2}Q^4 + Cc_{-1}Q^5 + Cc_0 Q^6 + Cc_1 Q^7 + Cc_2 Q^8 + Ac_{-1}^3 Q^3 \\ & + 3Ac_{-2}c_2^2 Q^8 + 3Ac_{-2}c_0^2 Q^4 + 3Ac_1 c_{-1}^2 Q^5 + 3Ac_2 c_{-1}^2 Q^6 + 3Ac_0 c_{-1}^2 Q^4 \\ & + 3Ac_{-1}c_1^2 Q^7 + 3Ac_{-1}c_2^2 Q^9 + 3Ac_{-1}c_0^2 Q^5 + 3Ac_1 c_0^2 Q^7 + 3Ac_2 c_0^2 Q^8 + 3Ac_0 c_1^2 Q^8 \\ & + 3Ac_0 c_2^2 Q^{10} + 3Ac_2 c_1^2 Q^{10} + 3Ac_1 c_2^2 Q^{11} + 2BQ^3 c_{-2}c_{-1} + 2BQ^5 c_{-2}c_1 \\ & + 2BQ^6 c_{-2}c_2 + 2BQ^6 c_1 c_{-1} + 2BQ^7 c_2 c_{-1} + 2BQ^9 c_2 c_1 + 2BQ^4 c_{-2}c_0 \\ & + 2BQ^5 c_0 c_{-1} + 2BQ^7 c_0 c_1 + 2BQ^8 c_0 c_2 + 3Ac_{-1}c_{-2}^2 Q + 3Ac_1 c_{-2}^2 Q^3 + 3Ac_2 c_{-2}^2 Q^4 \\ & + 3Ac_0 c_{-2}^2 Q^2 + 3Ac_{-2}c_{-1}^2 Q^2 + 3Ac_{-2}c_1^2 Q^6 + Q^4 (\ln a)^2 c_{-1}^2 + Q^8 (\ln a)^2 c_1^2 \\ & + 4Q^{10} (\ln a)^2 c_2^2 + Q^6 (\ln a)^2 c_{-1}^2 - 2Q^5 (\ln a)^2 c_{-1}^2 + Q^{10} (\ln a)^2 c_1^2 \\ & - 2Q^9 (\ln a)^2 c_1^2 \\ & + 4Q^{12} (\ln a)^2 c_2^2 - 8Q^{11} (\ln a)^2 c_2^2 + 4Q^2 (\ln a)^2 c_{-2}^2 + 4Q^4 (\ln a)^2 c_{-2}^2 \\ & - 8Q^3 (\ln a)^2 c_{-2}^2 + 4Q^3 (\ln a)^2 c_{-2}c_{-1} - 4Q^5 (\ln a)^2 c_{-2}c_1 - 8Q^6 (\ln a)^2 c_{-2}c_2 \\ & - 2Q^6 (\ln a)^2 c_{-1}c_1 - 4Q^7 (\ln a)^2 c_{-1}c_2 + 4Q^9 (\ln a)^2 c_2 c_1 + 4Q^5 (\ln a)^2 c_{-2}c_{-1} \\ & - 8Q^4 (\ln a)^2 c_{-2}c_{-1} - 4Q^7 (\ln a)^2 c_{-2}c_1 + 8Q^6 (\ln a)^2 c_{-2}c_1 - 8Q^8 (\ln a)^2 c_{-2}c_2 \\ & + 16Q^7 (\ln a)^2 c_{-2}c_2 - 2Q^8 (\ln a)^2 c_{-1}c_1 + 4Q^7 (\ln a)^2 c_{-1}c_1 - 4Q^9 (\ln a)^2 c_{-1}c_2 \\ & + 8Q^8 (\ln a)^2 c_{-1}c_2 + 4Q^{11} (\ln a)^2 c_2 c_1 - 8Q^{10} (\ln a)^2 c_2 c_1 = 0. \end{aligned}$$

Collecting the coefficients of Q for $k = 0,1,2,3,..,12$ to get the following system of algebraic equations

$$Ac_{-2}^3 = 0,$$

$$3Ac_{-2}^2 c_{-1} = 0\ ,$$

$$Ac_2^3 + 4(\ln a)^2 c_2^2 = 0\ ,$$

$$3Ac_1 c_2^2 - 8(\ln a)^2 c_2^2 + 4\,(\ln a)^2 c_1 c_2 = 0,$$

$$6Ac_{-2}c_{-1}c_0 + 4\,(\ln a)^2\, c_{-2}c_{-1} + Ac_{-1}^3 + 2\,B\,c_{-2}c_{-1} + 3Ac_{-2}^2 c_1 - 8\,(\ln a)^2\, c_{-2}^2 = 0,$$

$$\begin{aligned} & Bc_0^2 + 6Ac_{-1}c_0 c_1 + +3Ac_{-1}^2 c_2 - 2\,(\ln a)^2\, c_{-1}c_1 - 8\,(\ln a)^2\, c_{-2}c_2 + 3Ac_1^2 c_{-2} + Cc_0 \\ & +6Ac_2 c_0 c_{-2} + 2Bc_{-1}c_1 + Ac_0^3 + 2Bc_{-2}c_2 + (\ln a)^2\, c_{-1}^2 + 8\,(\ln a)^2\, c_{-2}c_1 + D = 0, \end{aligned}$$

$$\begin{aligned} & Bc_{-1}^2 + Cc_{-2} + 3Ac_{-2}^2 c_2 + 3Ac_{-2}c_0^2 + 6Ac_{-2}c_{-1}c_1 - 8\,(\ln a)^2\, c_{-2}c_{-1} + (\ln a)^2\, c_{-1}^2 \\ & +2\,B\,c_{-2}c_0 + 3Ac_{-1}^2 c_0 + 4\,(\ln a)^2\, c_{-2}^2 = 0, \end{aligned}$$

$$\begin{aligned} & 6Ac_{-2}c_{-1}c_2 - 4\,(\ln a)^2\, c_{-2}c_1 - 2\,(\ln a)^2\, c_{-1}^2 + 3Ac_{-1}^2 c_1 + Cc_{-1} + 4\,(\ln a)^2\, c_{-2}c_{-1} \\ & +2Bc_{-2}c_1 + 6Ac_{-2}c_0 c_1 + 2Bc_{-1}c_0 + +3Ac_{-1}c_0^2 = 0, \end{aligned}$$

$$3Ac_{-2}c_{-1}^2 + Bc_{-2}^2 + 4\,(\ln a)^2\, c_{-2}^2 + 3Ac_{-2}^2 c_0 = 0,$$

$$\begin{aligned} & (\ln a)^2\, c_1^2 + 6Ac_{-1}c_1 c_2 - 2\,(\ln a)^2\, c_{-1}c_1 + Cc_2 + 3Ac_0 c_1^2 + 2Bc_0 c_2 + 8\,(\ln a)^2\, c_{-1}c_2 \\ & -8\,(\ln a)^2\, c_{-2}c_2\, 3Ac_{-2}c_2^2 + 3Ac_0^2 c_2 + Bc_1^2 = 0, \end{aligned}$$

$$3Ac_0 c_2^2 + 4\,(\ln a)^2\, c_2^2 - 8\,(\ln a)^2\, c_1 c_2 + Bc_2^2 + (\ln a)^2\, c_1^2 + 3Ac_1^2 c_2 = 0,$$

$$\begin{aligned} & 16\,(\ln a)^2\, c_{-2}c_2 + 3Ac_{-1}c_1^2 + 3Ac_0^2 c_1 - 4\,(\ln a)^2\, c_{-1}c_2 + 6Ac_{-2}c_1 c_2 + 2Bc_0 c_1 \\ & +2Bc_{-1}c_2 + Cc_1 + 6Ac_{-1}c_0 c_2 + 4\,(\ln a)^2\, c_{-1}c_1 - 4\,(\ln \mathrm{a})^2\, c_{-2}c_1 = 0, \end{aligned}$$

$$6Ac_0 c_1 c_2 + Ac_1^3 + 3Ac_{-1}c_2^2 + 2Bc_1 c_2 - 2\,(\ln a)^2\, c_1^2 + 4\,(\ln a)^2\, c_1 c_2 - 4\,(\ln a)^2\, c_{-1}c_2 = 0.$$

Solving the system of algebraic equation to gain the following solution for $c_0c_1c_2 \neq 0$

$$A=-\frac{4\,(\ln a)^2}{c_2},\qquad B=\frac{(\ln a)^2(-c_2+12c_0)}{c_2},\qquad C=-\frac{2\,(\ln a)^2\,c_0(-c_2+6c_0)}{c_2}$$

$$D=\frac{(\ln a)^2\,c_0^2(-c_2+4c_0)}{c_2},c_0=c_0,c_1=-c_2,\,c_2=c_2,\,c_{-1}=0,\quad c_{-2}=0, \tag{30}$$

where all the constants are arbitrary. From equation (27), solutions for the reduced equation (26) are established. In the next section, we shall use these results to get the solutions for the nCBS equation, the CBS-nCBS equation and the nKdV model I and model II equations rely on analogues of the coefficients.

7. Solutions for the nCBS equation, the CBS - nCBS equation, the nKdV model I and model II equations

This section is dedicated to construct solutions for the nCBS equation, the CBS-nCBS equation, the nKdV model I equation and the nKdV model II equation by taking into account the results in pervious section.

In order to get solutions for the nCBS equation (1), comparing the coefficients of the equation (13) and equation (30) to deduce that

$$c_2=-2\,(\ln a)^2\,\alpha\ ,\qquad c^*=\alpha^2\delta c_0((\ln a)^2\,\alpha+3c_0)\ ,$$

$$\alpha=\alpha\,,\ \ c^{**}=-\frac{1}{2}c_0^2\alpha^2\delta(2c_0+(\ln a)^2\ \alpha),\qquad c_1=-c_2\ ,$$

$$c_0=\ c_0\,,\qquad \beta=-\alpha\delta((\ln a)^2\ \alpha+6\,c_0)\,,\qquad \delta=\delta,$$

using equation (27), we get the solutions for equation (12)

$$\Phi(\xi)=c_0+\ 2\alpha\big(\ln(a)\big)^2\frac{1}{1\mp a^{\xi}}-2\alpha\big(\ln(a)\big)^2\frac{1}{(1\mp a^{\xi})^2},$$

employing equation (11) and the appendix, the solutions for the nCBS equation (1) are

$$u(x,y,t)=c_0\xi-\alpha\ln a\frac{cLs\left(-\frac{1}{2}\xi\right)+sLs\left(-\frac{1}{2}\xi\right)}{cLs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t)=c_0\xi-\alpha\ln a\frac{sLs\left(\frac{1}{2}\xi\right)-cLs\left(\frac{1}{2}\xi\right)}{sLs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t)=c_0\xi-\alpha\ln a\left(1-tLs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t)=c_0\xi-\alpha\ln a\left(1-ctLs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t)=c_0\xi-\alpha\ln a\frac{c\,Fs\left(-\frac{1}{2}\xi\right)+sFs\left(-\frac{1}{2}\xi\right)}{cFs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t)=c_0\xi-\alpha\ln a\frac{sFs\left(\frac{1}{2}\xi\right)-cFs\left(\frac{1}{2}\xi\right)}{sFs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t)=c_0\xi-\alpha\ln a\left(1-tFs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t)=c_0\xi-\alpha\ln a\left(1-ctFs\left(\frac{1}{2}\xi\right)\right),$$

where $\xi=\alpha x-\ \alpha\delta(\ln(a)^2\,\alpha+6c_0)y+\delta t$ and all the constants are arbitrary.

The next set of results for the CBS- nCBS equation (2). Calling equation (17) and equation (30) to gain

$$c_0=-\frac{1}{6}\frac{(\delta+(\ln a)^2\,\alpha^2\beta+(\ln a)^2\,\alpha^2\delta)}{(\beta+\delta)\alpha},\qquad c_1=-c_2\,,\qquad c_2=-2\,(\ln a)^2\,\alpha\ ,\quad \alpha=\alpha\ ,$$

$$\beta=\beta\,,\qquad \delta=\delta$$

$$c^{*}=-\frac{1}{12}\frac{(\ln(a)^2\alpha^2\beta+(\ln a)^2\alpha^2\delta-\delta)(\delta+\ln(a)^2\alpha^2\beta+(\ln a)^2\alpha^2\delta)}{(\beta+\delta)\alpha},$$

$$c^{**}=-\frac{1}{216}\frac{(2(\ln a)^2\alpha^2\beta+2(\ln a)^2\alpha^2\delta-\delta)(\delta+(\ln a)^2\alpha^2\beta+(\ln a)^2\alpha^2\delta)^2}{(\beta+\delta)^2\alpha^2}.$$

From equation (27), we get the following solutions for equation (16),

$$\Phi(\xi)=-\frac{1}{6}\frac{(\delta+(\ln a)^2\alpha^2\beta+(\ln a)^2\alpha^2\delta)}{(\beta+\delta)\alpha}+2\alpha\big(\ln(a)\big)^2\frac{1}{1\mp a^{\xi}}-2\alpha\big(\ln(a)\big)^2\frac{1}{(1\mp a^{\xi})^2}.$$

By using equation (11) and the appendix, the solutions for the CBS- nCBS equation (2) are

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\frac{cLs\left(-\frac{1}{2}\xi\right)+sLs\left(-\frac{1}{2}\xi\right)}{cLs\left(\frac{1}{2}\xi\right)}$$

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\frac{sLs\left(\frac{1}{2}\xi\right)-cLs\left(\frac{1}{2}\xi\right)}{sLs\left(\frac{1}{2}\xi\right)}$$

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\left(1-tLs\left(\frac{1}{2}\xi\right)\right)$$

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\left(1-ctLs\left(\frac{1}{2}\xi\right)\right)$$

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\frac{cFs\left(-\frac{1}{2}\xi\right)+sFs\left(-\frac{1}{2}\xi\right)}{cFs\left(\frac{1}{2}\xi\right)}$$

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\frac{sFs\left(\frac{1}{2}\xi\right)-cFs\left(\frac{1}{2}\xi\right)}{sFs\left(\frac{1}{2}\xi\right)}$$

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\left(1-tFs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t)=-\left(\frac{\delta}{6(\beta+\delta)\alpha}+\frac{1}{6}\frac{\alpha(\ln a)^2\beta}{\beta+\delta}+\frac{\alpha}{6}\frac{(\ln a)^2\delta}{\beta+\delta}-2(\ln a)^2\alpha\right)\xi-\alpha\ln a\left(1-ctFs\left(\frac{1}{2}\xi\right)\right),$$

where $\xi=\alpha x+\beta y+\delta t$ and all the constants are arbitrary.

To get solutions for nKdV Model I equation (3). Using equation (21) and equation (30) to have

$$c_2=2\ln(a)^2\alpha,\quad \beta=-\alpha-\gamma-\ln(a)^2\alpha^2\delta+6\alpha\delta c_0,\ c_1=-c_2,$$
$$\alpha=\alpha,\quad \gamma=\gamma,\ \delta=\delta,\ c_0=c_0,$$
$$c^{*}=-\alpha^2\delta c_0(\ln(a)^2\alpha-3c_0),\ c^{**}=\frac{1}{2}c_0^2\alpha^2\delta(\ln(a)^2\alpha-2c_0),$$

Calling equation (27), the solutions for equation (20) are given by

$$\Phi(\xi) = c_0 - 2\alpha\big(\ln(a)\big)^2 \frac{1}{1 \mp a^{\xi}} + 2\alpha\big(\ln(a)\big)^2 \frac{1}{(1 \mp a^{\xi})^2}.$$

By using equation (11) and the appendix, the solutions for nKdV Model I equation (3) are

$$u(x,y,t) = c_0\xi + \alpha \ln a \frac{cLs\left(-\frac{1}{2}\xi\right) + sLs\left(-\frac{1}{2}\xi\right)}{cLs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \frac{sLs\left(\frac{1}{2}\xi\right) - cLs\left(\frac{1}{2}\xi\right)}{sLs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \left(1 - tLs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \left(1 - ctLs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \frac{cFs\left(-\frac{1}{2}\xi\right) + sFs\left(-\frac{1}{2}\xi\right)}{cFs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \frac{sFs\left(\frac{1}{2}\xi\right) - cFs\left(\frac{1}{2}\xi\right)}{sFs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \left(1 - tFs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \left(1 - ctFs\left(\frac{1}{2}\xi\right)\right),$$

where $\xi = \alpha x - (\alpha + \gamma + (\ln a)^2\alpha^2\delta - 6\alpha\delta c0)y + \gamma z + \delta t$ and all the constants are arbitrary.

For gaining solutions for the nKdV Model II equation (4). Using equation (25) and equation (30) to get

$$c_2 = 2(\ln a)^2\alpha, \quad \beta = -\alpha + (\ln a)^2\alpha^2(\gamma - \delta) - 6\alpha c_0(\gamma - \delta),$$

$$c_0 = c_0, \quad \alpha = \alpha, \quad \gamma = \gamma, \quad c_1 = -c_2, \quad \delta = \delta$$

$$c^* = c_0\alpha^2((\ln a)^2\alpha\gamma - \delta(\ln a)^2\alpha - 3c_0\gamma + 3c_0\delta),$$

$$c^{**} = -\frac{1}{2}c_0^2\alpha^2(-2c_0\gamma + 2c_0\delta + (\ln a)^2\alpha\,\gamma - \delta(\ln a)^2\alpha).$$

Applying equation (27), we get the solutions for equation (24) in the form

$$\Phi(\xi) = c_0 - 2\alpha(\ln a)^2 \frac{1}{1 \mp a^{\xi}} + 2\alpha(\ln a)^2 \frac{1}{(1 \mp a^{\xi})^2},$$

and by using equation (11) and the appendix, the solutions for the nKdV Model II equation (4) are

$$u(x,y,t) = c_0\xi + \alpha \ln a \frac{cLs\left(-\frac{1}{2}\xi\right) + sLs\left(-\frac{1}{2}\xi\right)}{cLs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \frac{sLs\left(\frac{1}{2}\xi\right) - cLs\left(\frac{1}{2}\xi\right)}{sLs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \left(1 - tLs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \left(1 - ctLs\left(\frac{1}{2}\xi\right)\right),$$

$$u(x,y,t) = c_0\xi + \alpha \ln a \frac{\left(cFs\left(-\frac{1}{2}\xi\right) + sFs\left(-\frac{1}{2}\xi\right)\right)}{cFs\left(\frac{1}{2}\xi\right)},$$

$$u(x,y,t)=c_0\xi+\alpha\ln a\frac{sFs\left(\frac{1}{2}\xi\right)-cFs\left(\frac{1}{2}\xi\right)}{sFs\left(\frac{1}{2}\xi\right)},$$
$$u(x,y,t)=c_0\xi+\alpha\ln a\left(1-tFs\left(\frac{1}{2}\xi\right)\right),$$
$$u(x,y,t)=c_0\xi+\alpha\ln a\left(1-ctFs\left(\frac{1}{2}\xi\right)\right),$$

where $\xi=\alpha x-(\alpha-(\ln a)^2\alpha^2(\gamma-\delta)+6\alpha c_0(\gamma-\delta))\,y+\gamma z+\delta t$ and all the constants are arbitrary.

Conclusion

In this work, we have dealt with four nonlinear equations, the nCBS equation, the nCBS-CBS equation, the nKdV model I equation and the nKdV model II equation in higher dimensions. We stress that these equation can be reduced to the same nonlinear ordinary differential equation by employing wave reduction variable. Types of solutions in terms of symmetrical Fibonacci and Lucas functions are established. The modified version of the Kudryashov method is used to achieve our goal, the method is reliable to get exact solutions to nonlinear equations. The structure of the solutions of the nKdV model I equation is seem to be similar to that of the nKdV model II equation .

Appendix

Symmetrical representation of Lucas and Fibonacci functions and some their properties [12].

Symmetrical Lucas and Fibonacci sine and cosine functions

$$sLs(\xi)=a^{\xi}-a^{-\xi}\quad ,cLs=a^{\xi}+a^{-\xi}$$
$$sFs(\xi)=\frac{a^{\xi}-a^{-\xi}}{\sqrt{5}}\quad ,cFs=\frac{a^{\xi}+a^{-\xi}}{\sqrt{5}}.$$

Symmetrical Lucas and Fibonacci tangent and cotangent functions

$$tLs(\xi)=\frac{a^{\xi}-a^{-\xi}}{a^{\xi}+a^{-\xi}}\quad ,ctLs=\frac{a^{\xi}+a^{-\xi}}{a^{\xi}-a^{-\xi}}$$
$$tFs(\xi)=\frac{a^{\xi}-a^{-\xi}}{a^{\xi}+a^{-\xi}}\quad ,ctFs=\frac{a^{\xi}+a^{-\xi}}{a^{\xi}-a^{-\xi}}.$$

The correlations of the symmetrical Lucas and Fibonacci functions

$$sFs(\xi)=\frac{sLs(\xi)}{\sqrt{5}},\quad cFs(\xi)=\frac{cLs(\xi)}{\sqrt{5}}.$$

Some identities for the symmetrical Lucas and Fibonacci function

$$\left(cLs(\xi)\right)^2-\left(sLs(\xi)\right)^2=4,\ \left(cFs(\xi)\right)^2-\left(sFs(\xi)\right)^2=\frac{4}{5}$$
$$sFs(\xi)=-sFs(-\xi)\,,\quad sLs(\xi)=-sLs(\xi),$$
$$\left(sLs(\xi)\right)^2+2=cLs(2\xi),\ \left(cLs(\xi)\right)^2-2=cLs(2\xi),$$
$$cFs(\xi)=cFs(-\xi),\ cLs(\xi)=cLs(-\xi),$$
$$\frac{4}{\sqrt{5}}cFs(2\xi)=sFs(\xi).cFs(\xi),\ sLs(2\xi)=sLs(\xi).cLs(\xi).$$

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