DOI: 10.24996/ijs.2022.63.4.29

ISSN: 0067-2904

Maximal and Minimal Regular β -Open Sets in Topological Spaces

Nafeha O. Rashid, Sami A. Hussein*
Department of mathematics, College of Basic Education, Salahaddin University-Erbil, Iraq

Received: 26/5/2021 Accepted: 11/7/2021 Published: 30/4/2022

ABSTRACT
In this paper, new concepts of maximal and minimal regular $\beta-$open sets are introduced and discussed. Some basic properties are obtained. The relation between maximal and minimal regular $\beta-$open sets and some other types of open sets such as regular open sets and β -open sets are investigated.

Keywords: $\beta-$open set, regular $\beta-open\ set$, $minimal\ regular\ \beta-open\ set$, maximal regular $\beta-$open set,

المجموعات العظمى والصغري مفتوحة المنتظمة من النمط β من الفضاءات التبولوجية

نافعة عمر رشيد، سامي علي حسين*
قسم الرياضيات، كلية التربية الاساسية ، جامعة صلاح الدين، اربيل ، العراق

الخلاصه
في هذا البحث عرفنا و درسنا نوع جديد من المجموعات العضمى والصغرى المفتوحة المنتظمة من النمط β, تم الحصول على بعض خواصه , درسنا علاقه بين المجموعات العضمى والصغرى المفتوحه المنتظمه من النمط β مع مجموعات المفتوحه اخرى مثل المجموعات المفتوحة المنتظمه و مجموعات المفتوحه من النمط β

I. INTRODUCTION
The concepts of maximal open sets and minimal open sets in topological spaces were introduced by Nakaok and Oda in [1-5,6, 7]. The study of α -open sets and their properties were initiated by Njastad in 1965[8]. Stone gave a new class of open sets called regular open sets in 1937,[9, 10]. In[1], the concepts of β -open sets have been introduced and studied. Latter, these β -open sets are recalled as semipreopen sets, which were introduced by D.Andrejevic in [2]. In [9] , As a simulation of these studies, minimal and maximal $\beta-$open sets have been introduced and studied. On the other hand as a simulation of minimal regular open set in [3] has been introduced and studied. Latter, Nasser [7] explains the study of notation of minimal and maximal regular open sets and he investigates several of their basic properties. The notation of minimal and maximal α -open set is introduced in[4]. In addition, a new class of open sets called regular $\beta-$open sets were given by authors in [11]. Our aim in presenting this article is to introduce the notation of a new class of maximal and

*Email: sami.hussein@su.edu.krd

minimal open sets which is called maximal regular β – open sets and minimal regular β –open sets. We also investigate several of their basic properties. Finally, we study their relationship with various types of open sets. Throughout this article ,the closure (Interior) of a subset U_1 of space X is symbolized by $\mathrm{Cl}(U_1)$ ($\mathrm{Int}(U_1)$).

II. Basic Backgrounds.

By $(X,\mathfrak{I})$ or simply X we mean a topological space for which no separation axioms are considered unless it is stated.

Definition 2.1. [6] The minimal open (closed) set in a space X is a proper nonempty open(closed) subset U of X which is included in U is Φ or U.

Definition 2.2. [5] The maximal open (closed) set in a space X is a proper nonempty open(closed) subset U of X that includes U is U or X.

Definition 2.3.[1] The β – open (semipre) set in a space X is a subset U of X such that $U \subseteq Cl(Int(Cl(U)))$. U is called β –closed set if U^c is β –open set. The collection of all β – open (β – closed) set will be denoted by (X)($\beta C(X)$).

Definition 2.4. [10] The regular open (regular closed) set in a space X is a subset V of X, such that $V = Int\big(Cl(V)\big)$ $(V = Cl\big(Int(V)\big)$. The collection of all regular open(regular closed) sets will be denoted RO(X) (RC(X)).

Definition 2.5. [8] The α – open set in a space X in a space X is a subset U of X such that $U \subseteq Int(Cl(Int(U)))$. U is called α – closed set if U^c is α –open set. The set of all α –open (α -closed) set will be denoted by $\alpha O(X)$($\alpha C(X)$).

Definition 2.6. [11] The regular β – open set in a space X is a subset U of X, such that $U = \beta\mathrm{Int}\big(\beta Cl(U)\big)$.

The collection of all regular β – open sets will be denoted $\mathrm{R\beta O}(X)$.

Definition 2.7. [9] The minimal β – open (β – closed) set in a space X is a proper nonempty β – open(β – closed) subset U of X which is included in U is Φ or U. The set of all minimal β – open(β – closed) sets will be denoted by $M_i\beta O(X)$ ($M_i\beta C(X)$).

Definition 2.8. [9] The maximal β – open (β – closed) set in a space X is a proper nonempty β –open(β –closed) subset U of X which is includes U is U or X. The set of all maximalβ – open(β – closed) sets will be denoted by $\mathrm{M_a\beta O(X)}$ ($\mathrm{M_a\beta C(X)}$).

Definition 2.9. [7] The maximal regular open (regular closed) set in a space X is a proper nonempty regular–open(regular −closed) subset U of X which is includes U is U or X. The set of all maximal regular open(regular closed) sets will be denoted by $\mathrm{M_aRO(X)}$ ($\mathrm{M_aRC(X)}$).

Definition 2.10. [7] The minimal regular open (regular closed) set in a space X is a proper nonempty regular open(regular closed) subset U of X which is included in U is Φ or U.

The set of all minimal regular open(regular closed) sets will be denoted by $\mathrm{M_iRO(X)}$ ($\mathrm{M_iRC(X)}$).

Definition 2.11. [9] X is said to be $T_{\beta\min}$ if $\Phi \neq G \in \beta O(X)$ implies $G \in M_{\mathrm{i}}\beta O(X)$ for each G.

Definition 2.12. [9] X is called $T_{\beta\max}$ space if $\Phi \neq G \in \beta O(X)$ implies $G \in M_{\mathrm{a}}\beta O(X)$ for each G. **Theorem 2.13.** [11] $U \in R\beta O(X)$ if and only if $U \in \beta O(X) \cap \beta C(X)$.

Theorem 2.14. [9] The subset U of X, is a minimal β –closed if and only if U^c is maximal β –open set.

Theorem 2.15. [9] A space X is $T_{\beta\min}$ if and only if it is $T_{\beta\max}$.

Theorem 2.16. [9] For each pair $A, B \in M_{\mathrm{i}}\beta O(X)$ of $T_{\beta\min}$ such that $A \neq B$, we have $A \cap B = \Phi$.

Theorem 2.17. [9] For each pair $A, B \in M_i\beta O(X)$ of $T_{\beta\min}$, such that $A \neq B$, we have $A \cup B = X$.
Theorem 2.18. [11]$U \in R\beta O(X)$ if and only if $Int(U) \in RO(X)$ and $Cl(U) \in RC(X)$.
Theorem 2.19.[11] If $A_1 \in RO(X)$ and $A_2 \in R\beta O(X)$, then $A_1 \cap A_2 \in R\beta O(X)$.
Theorem 2.20. [11] $\alpha O(X) \cap R\beta O(X) = RO(X)$.
Theorem 2.21. [11] $\forall A_1 \subseteq X$, we have $\beta Int(\beta Cl(A_1))$, $\beta Cl(\beta Int(A_1))$, $\beta Int(Cl(A_1))$, $\beta Cl(Int(A_1))$ and $Int(Cl(A_1))$ are regular β –open sets.
Theorem 2.22. [11]The following conditions are equivalent for each subset W of X:
(i) $W \in R\beta O(X)$, (ii) $W \in \beta O(X) \cap \beta C(X)$, (iii) $X \backslash W \in R\beta O(X)$, (iv)$W = \beta Cl(\beta Int(W))$,
(v) $\beta Fr(W) = \varphi$, (vi) $Int(Cl(Int(W))) \subset W \subset Cl(Int(Cl(W)))$, open s (vii) $W \in \beta O(X)$ and $X \backslash W \in \beta O(X)$.
Theorem 2.23.[3] i- If the intersection of a minimal regular open set and a regular open set is not equal to Φ, then the minimal regular open set is a subset of the regular open set.
ii- If the intersection of each pair of a minimal regular open set is not equal to Φ, then it is equal to each other.
Theorem 2.24. [4] Let U_1, U_2 and U_3 be maximal α –open sets such that $U_1 \neq U_2$. If $U_1 \cap U_2 \subset U_3$, then either $U_1 = U_3$ or $U_2 = U_3$.
Theorem2.25. [4] Let U_1, U_2 and U_3 be maximal α –open sets which are various from each ether. Then, $(U_1 \cap U_2) \not\subset (U_1 \cap U_3)$.
Theorem 2.26.[7] If A_1 is proper closed set includes a maximal regular open set A_2, then $Int(A_1) = A_2$.

III. MAXIMAL AND MINIMAL REGULAR β – OPEN SETS

In this section, the concepts of minimal and maximal regular $\beta - o$pen sets are introduced. Then, we discuss several properties of these and their relationships with several types of nearly open sets.

Definition 3.1.

i) The minimal regular $\beta -$ open set in X is a proper nonempty regular $\beta -$ open subset U of X which is included in U is Φ or U.

ii) The maximal $\beta -$ open set in X is a proper nonempty regular $\beta -$ open subset U of X which is included in U is U or X

The collection of all minimal regular β –open (maximal regular β –open) sets in a space X is symbolized by $M_iR\beta O(X)$ ($MaR\beta O(X)$).
Remark 3.2. From the fact that $R\beta O(X)=R\beta C(X)$, by theorem 2.13. Then in any space X minimal regular β –open(maximal regular β –open) sets and minimal regular β –closed (maximal regular β –closed) sets are equivalent. It means that $M_iR\beta O(X) = M_iR\beta C(X)$ and $M_aR\beta O(X) = M_aR\beta C(X)$
Example 3.3. Consider $X = \{u_1, u_2, u_3, u_4\}$ and let $\mathfrak{J} = \{\Phi, \{u_1\}, \{u_2\}, \{u_1, u_2\}, X\}$ then $R\beta O(X)=\{\Phi, \{u_1\}, \{u_2\}, \{u_1, u_3\}, \{u_1, u_4\}, \{u_2, u_3\}, \{u_2, u_4\}, \{u_1, u_3, u_4\}, \{u_2, u_3, u_4\}, X\}$.So, the set $\{u_1\}$ is minimal regular β –open and the set $\{u_2, u_3, u_4\}$ is maximal regular β –open.
Theorem 3.4. $A_1 \in M_iR\beta O(X)$ if and only if ${A_1}^c \in M_aR\beta O(X)$
Proof. Let $A_1 \in M_iR\beta O(X)$, then by theorem 2.13, ${A_1}^c \in R\beta O(X)$. Let $A_2 \in R\beta O(X)$. such that ${A_1}^c \subseteq A_2$, then ${A_2}^c \subseteq A_1$. Since, ${A_2}^c$ is regular $\beta -$ open set included in a minimal regular $\beta -$ open set A_1, then either ${A_2}^c = \Phi$ or ${A_2}^c = A_1$ that is either $A_2 = X$ or $A_2 = {A_1}^c$.Therefore, ${A_1}^c$ is maximal regular β –open set.
Corollary 3.5. Let X be a space contain at least two elements and $a_1 \in X$. Then, if $\{a_1\}$ is a regular $\beta -$ open set then it is minimal regular $\beta -$ open set and $X \backslash \{a_1\}$ is maximal regular $\beta -$ open set.

Proof. Let $\{a_1\}$ be a regular β – open set. Since there is no nonempty regular β – open set including properly in the set $\{a_1\}$, then $\{a_1\}$ is minimal regular β – open set and by theorem 3.4 $X\backslash\{a_1\}$ is maximal regular β – open set.

Lemma 3.6. If $A_1 \in M_i R\beta O(X)$ and $A_2 \in R\beta O(X)$, then $A_1 \cap A_2 \in R\beta O(X)$ **.**

Proof. Let $A_1 \in M_i R\beta O(X)$ and $A_2 \in R\beta O(X)$, . We have now, $A_1 \cap A_2 \subseteq A_1$ .If $A_1 \cap A_2 = \Phi$ then, the prove is completed. If $A_1 \cap A_2 \neq \Phi$ and A_1 is a minimal regular β – open set, then $A_1 \cap A_2 = A_1$.This means that, $A_1 \cap A_2$ is regular β – open set.

Theorem 3.7. i- If the intersection of a minimal regular β – open set and a regular β – open set is not equal to Φ, then the minimal regular β – open set is a subset of the regular β – open set.

ii- If the intersection of each pair of a minimal regular β – open set is not equal to Φ, then equal to each other.

Proof. The proof is similar to theorem 2.23, then it is omitted.

Remark 3.8. If $a_1 \in A_1 \in M_i R\beta O(X)$, then $A_1 \subseteq A_2$, for each $a_1 \in A_2 \in R\beta O(X)$

Proof. Let $a_1 \in A_1$. If A_2 is regular β – open set including a_1. Then, $A_1 \cap A_2 \neq \Phi$.So by Theorem 3.7 $A_1 \subseteq A_2$.

Theorem 3.9. i- If A_1 is a $\beta -$ closed set including properly in a minimal regular – open set A_2. Then, $\beta\text{Int}(A_1) = \Phi$.

ii- If A_1 is a $\beta -$ closed set contain properly a maximal regular β –open set A_2. Then, $\beta\text{Int}(A_1) = X$.

iii- If A_1 is a $\beta -$ open set contain properly a maximal regular $\beta -$ open set A_2. Then, $\beta Cl\,(A_1) = X$.

Proof. i- Let $A_2 \in M_i R\beta O(X)$ and $A_1 \in \beta C(X)$ such that $A_1 \subset A_2$.So, $\beta\text{Int}(A_1)$ is regular β – open set and A_2 is a minimal regular β – open set. Then, $\beta\text{Int}(A_1) = \Phi$.

ii- Let $A_2 \in M_a R\beta O(X)$ and $A_1 \in \beta C(X)$such that $A_2 \subset A_1$.So by Theorem 2.21 $\beta Int(A_1) \in R\beta O(X)$ and A_2 is a maximal regular β –open set. Then, $\beta\text{Int}(A_1) = X$.

iii- Let $A_2 \in M_a R\beta O(X)$ and $A_1 \in \beta O(X)$, such that $A_2 \subset A_1$.So, by Theorem 2.21 $\beta Cl\,(A_1) \in R\beta O(X)$ and A_2 is a maximal regular β –open set. Then, $\beta Cl(A_1) = X$.

Remark 3.10. Next example shows that the Theorem 3.9 part i and ii are not necessary to be true if A_1 is not β-closed set.

Example 3. 11. Consider $X = \{u_1, u_2, u_3, u_4\}$ with a topology $\mathfrak{I} = \{\ \Phi, \{u_1\}, \{u_2\}, \{u_1, u_2\},\{u_1, u_3\}, \{u_1, u_2, u_3\}\ X\}$, $\beta O(X, \mathfrak{I}) = \{\Phi, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_4\}, \{u_2, u_4\}, \{u_1, u_2, u_3\}, \{u_1, u_3, u_4\}, \{u_1, u, u_4\}, X\}$, $R\beta O(X, \mathfrak{I}) = \{\ \Phi, \{u_2\}, \{u_1, u_3\}, \{u_2, u_4\}, \{u, u_3, u_4\}, X\}$.

i- Let $A_1 = \{u_1\},\ and\ A_2 = \{u_1, u_3\}$, then $\beta\text{Int}(A_1) = \{u_1\} \neq \Phi$.

ii- Let $A_1 = \{u_1, u, u_4\},\ and\ A_2 = \{u_2, u_4\}$, then $\beta\text{Int}(A_1) = A_1 \neq X$.

Theorem 3.12. Let U_1 be a non-empty proper regular β – open set, then the following statements are equivalent:

i. U_1 is a minimal regular β –open set,

ii. $U_1 \subseteq \beta\text{Cl}\,(U_2)$, where $U_2 \subseteq U_1$ and $\beta\text{Int}(U_2) \neq \Phi$,

iii. $U_1 = \beta\text{Cl}\,(U_2)$, where $\Phi \neq U_2 \subseteq U_1$.

Proof. (i→ii) Let $U_2 \subseteq U_1$ such that $\beta\text{Int}(U_2) \neq \Phi$, then $\beta\text{Int}(U_2) \subseteq \beta\text{Int}(\beta\text{Cl}(U_2)) \subseteq \beta\text{Int}(\beta\text{Cl}(U_1)) = U_1$, which is minimal regular β –open set. This means that $U_1 = \beta\text{Int}(\beta\text{Cl}(U_2)) \subseteq \beta\text{Cl}\,(U_2)$.

(ii→iii) Since $U_2 \subseteq U_1$,$U_1 \subseteq \beta Cl\,(U_2) \subseteq \beta Cl\,(U_1)$. Therefore, $U_1 = \beta Cl\,(U_1) = \beta Cl\,(U_2)$.

(iii→i) Let U_2 be a nonempty regular β – open set such that $U_2 \subseteq U_1$, then $\beta\text{Cl}(U_1) = U_1 = \beta\text{Cl}\,(U_2)$. This means that $U_2 = \beta\text{Int}\big(\beta\text{Cl}(U_2)\big) = \beta\text{Int}\big(\beta\text{Cl}(U_1)\big) = U_1$.

Therefore, U_1 is a minimal regular β –open set.

Theorem 3.13. Let A_1 be a nonempty proper regular β –open set. Then, the next pair of conditions are equivalent:
i- A_1 is a minimal regular β –open set,
ii- $A_1 = \beta\text{Int}(A_2)$, where A_2 is β –closed set, $A_2 \subseteq A_1$ and $\beta\text{Int}(A_2) \neq \Phi$.
Proof. (i) →(ii) Let A_1 be a minimal regular β –open set and A_2 is β –closed set such that $A_2 \subseteq A_1$, then $\beta\text{Int}(A_2) \subseteq \beta\text{Int}(A_1) = A_1$. Since, by theorem 2.21 $\beta\text{Int}(A_2)$ is regular β – open set and A_1 is a minimal regular β –open set. Therefore, $A_1 = \beta\text{Int}(A_2)$.
(ii) → (i) Let A_2 be a nonempty regular β –open set such that $A_2 \subseteq A_1$.So, $A_1 = \beta\text{Int}(A_2) = A_2$
.This means that, $A_2 = A_1$. Therefore, A_1 is a minimal regular β –open set.
Remark 3.14. Next example shows that we cannot remove the condition $\beta\text{Int}(A_2) \neq \Phi$ in theorem 3.13.
Example 3. 15. Consider $X = \{u_1, u_2, u_3, u_4\}$ with a topology $\mathfrak{I} = \{\Phi, \{u_1\}, \{u_3\}, \{u_1, u_3\}, \{u_1, u_2\}, \{u_3, u_4\}, \{u_1, u_2, u_3\}, \{u_1, u_3, u_4\}, X\}$, $\beta O(X, \mathfrak{I}) = \{\Phi, \{u_1\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_3, u_4\}, \{u_1, u_2, u_3\}, \{u_1, u_3, u_4\}, X\}$, $R\beta O(X, \mathfrak{I}) = \{\Phi, \{u_1, u_2\}, \{u_3, u_4\}, X\}$. Let $A_2 = \{u_2\}$, $A_1 = \{u_1, u_2\}$ *and* $\beta\text{Int}(A_2) = \Phi$, then $\beta\text{Int}(A_2) \neq A_1$.
Theorem 3.16. If A_1 is proper β-closed set includes a maximal regular β –open set A_2, then $\beta\text{Int}(A_1) = A_2$.
Proof. The proof is similar to theorem 2.26. Hence it is omitted.
Theorem 3.17. i- If A_1 is a nonempty proper regular β – open set such that no existing proper β – open (β –closed) set includes A_1 then A_1 is maximal regular β – open set.
ii- If A_1 is a nonempty proper regular β –open set such that no existing proper β – open (β – closed) set included in A_1 then A_1 is minimal regular β –open set.
Proof. i- Let A_2 be a regular β –open set such that $A_1 \subseteq A_2$.So that A_2 is β –open (β –closed) set then $A_2 = A_1$ or $A_2 = X$.Therefore, A_1 is maximal regular β –open set.
ii- Let A_2 be a regular β –open set such that $A_2 \subseteq A_1$.So that A_2 is β –open (β –closed) set then $A_2 = A_1$ or $A_2 = \Phi$.Therefore, A_1 is minimal regular β –open set.
Theorem 3.18. i- If the union of a maximal regular β –open set and regular β –open set is not equal to X, then the regular β –open set is a subset of the maximal regular β –open set .
ii- If the union of each pair of a maximal regular β –open set is not equal to X, then it is equal to each other.
Proof. i- Let A_1 be a maximal regular β –open set and A_2 be a regular β –open set such that $A_1 \cup A_2 \neq X$, then we must to prove that $A_2 \subseteq A_1$.Since $A_1 \subseteq A_1 \cup A_2$and A_1 is a maximal regular β –open set then by definition 3.1 $A_1 \cup A_2 = X$ or $A_1 \cup A_2 = A_1$, but $A_1 \cup A_2 \neq X$ then $A_1 \cup A_2 = A_1$, which means that $A_2 \subseteq A_1$.
ii- If A_1 *and* A_2 are a maximal regular β –open sets such that $A_1 \cup A_2 \neq X$, then $A_2 \subset A_1$ *and*$A_1 \subset A_2$ by (i). Therefore $A_2 = A_1$
Theorem 3.19. If U_1 is minimal regular β –open set and U_2 is maximal regular β –open set in a space X such that $U_1 \nsubseteq U_2$, then $U_1 = X \setminus U_2$.
Proof. *Let* U_1 be a minimal regular β –open set ,then $X \setminus U_1$ is maximal regular β –open set, but U_2 is maximal regular β –open set .Therefore, either $U_2 = X \setminus U_1$ or $U_2 \cup X \setminus U_1 = X$. If $U_2 \cup X \setminus U_1 = X$ then $U_1 \subseteq U_2$ which is opposite to each other .Hence, $U_2 = X \setminus U_1$ .Means that, $U_1 = X \setminus U_2$.
Remark 3.20. i- The class $M_iR\beta O(X)$ ($M_aR\beta O(X)$) and $M_i\beta O(X)$ ($M_a\beta O(X)$) are incomparable.
ii- Next example shows that the class $M_iR\beta O(X)$ ($M_aR\beta O(X)$) and $M_iRO(X)$ ($M_aRO(X)$) are incomparable.
Example3.21. In example 3.11,

i-The set $\{u_1, u_3\}$ is minimal regular β –open but not minimal β –open and the set $\{u_1\}$ is minimal β –open but not minimal regular β –open.
ii- The set $\{u_2, u_4\}$ is maximal regular β –open but not maximal β –open and the set $\{u_1, u_2, u_3\}$ is maximal β –open but not maximal regular β –open.
Example3.22. Consider $X = \{u_1, u_2, u_3, u_4\}$ with a topology $\mathfrak{I} = \{ \Phi, \{u_1\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}, X\}$, RβO(X, $\mathfrak{I}$) = $\{\Phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_4\}, \{u_2, u_3\}, \{ u_3, u_4\}, \{u_2, u_4\}, \{ u_2, u_3, u_4\}, \{u_1, u_3, u_4\}, \{u_1, u_2, u_4 \}, X\}$, RO(X, $\mathfrak{I}$) = $\{ \varphi, \{u_1\}, \{u_2, u_3\}, X\}$, Then:
i-The set $\{u_2\}$ is minimal regular β –open but not minimal regular open and the set $\{u_2, u_3\}$ is minimal regular open but not minimal regular β –open.
ii- The set $\{u_2, u_3, u_4\}$ is maximal regular β –open but not maximal regular open and the set $\{u_2, u_3\}$ is maximal regular open but not maximal regular β –open.
Theorem 3.23. Let X be a $T_{\beta\min}$ space. Then, $A_1 \in M_iR\beta O(X)$ if and only if $A_1 \in M_i\beta O(X)$
Proof. Let A_1 be a minimal regular β –open set and X be a $T_{\beta\min}$ space. Then, A_1 is β –open set. By definition 2.11, *have* $A_1 \in M_i\beta O(X)$. Conversely, Let A_1 is a minimal β –open set and A_2 is β –open set . From definition 2.11, we have A_2 is a minimal β –open set, and by theorem 2.16,we get $A_1 \cap A_2 = \Phi$, then by theorem 2.17 and Theorem 2.15, $A_1 \cup A_2 = X. Hence,\ A_2 = X \backslash A_1\ and\ A_1$ is β– closed set. By Theorem 2.13, A_1 is regular β –open set. Therefore, A_1 is a minimal regular β –open set.
Theorem 3.24. Let X be a $T_{\beta\min}$ space. Then, A_1 is a maximal regular β –open set if and only if $A_1$is a maximal β –closed set.
Proof. Let A_1 be a maximal regular β –open set and X be a $T_{\beta\min}$ space. Then A_1 is β –closed set. Assume that, A_1 is not maximal β –closed set. So that there exists an β –closed subset A_2 of X with $A_2 \neq X$ such that $A_1 \subset A_2$. Thus ${A_2}^c \subset {A_1}^c$. Hence ${A_1}^c$ is a proper β –open set which is no minimal and this opposite of being X is $T_{\beta\min}$ space. Hence, A_1 is a maximal β –closed set. Conversely, let A_1 is a maximal β –closed set then by Theorem 2.14, ${A_1}^c$ is minimal β –open set. By theorem 3.23, ${A_1}^c$ is a minimal regular β –open set. Hence A_1 is a maximal regular β –open set.
Remark 3.25. Next example indicates that we cannot remove the condition X be a $T_{\beta\min}$ space in theorem 3.23 and Theorem 3.24.
Example 3.26. In example 3.11, the space X is not $T_{\beta\min}$, the set $\{a_1, a_3\}$ is minimal regular β –open but not minimal β –open and the set $\{a_1\}$ *is* minimal β – open but not minimal regular β –open. So, the set $\{a_2, a_4\}$ is maximal regular β –open but not maximal β –closed and the set $\{a_2, a_3, a_4\}$ is maximal β –closed but not maximal regular β –open.
Theorem 3.27. If any regular β –open set is maximal (minimal) β –open set then it is maximal (minimal) regular β –open set.
Proof. Let A_1 be a regular β –open set and maximal (minimal) β –open set in a space X. We want to prove that A_1 is a maximal (minimal) regular β –open set. Assume that A_1 is not maximal(not minimal) regular β –open set, then $A_1 \neq X$ ($A_1 \neq \Phi$) and there exist regular β –open set A_2 such that $A_1 \subset A_2$ ($A_2 \subset A_1$) $A_1 \neq A_2$. By Theorem 2.13, A_2 is β –open set .So, A_2 includes (included in) A_1, which is opposite to each other. Hence, A_1 is a maximal (minimal) regular β –open set.
Theorem 3.28. If any regular open set is maximal (minimal) regular β –open set then it is maximal (minimal) regular open set.
Proof. Let A_1 be a regular open set and maximal (minimal) regular β –open set in a space X. We want to prove that A_1 is a maximal (minimal) regular open set. Assume that A_1 is not maximal (not minimal) regular open set, then $A_1 \neq X$ ($A_1 \neq \Phi$) and there exist regular open

set A_2 such that $A_1 \subset A_2$ ($A_2 \subset A_1$) and $A_1 \neq A_2$. By theorem 2.22, A_2 is regular β –open set. So, A_2 includes (included in) A_1, which is opposite to each other. Hence A_1 is a maximal (minimal) regular open set.

Theorem 3.29. If any regular β –open set is maximal (minimal) β –closed set then it is maximal (minimal) regular β –open set.

Proof. Let A_1 be a regular β –open set and maximal (minimal) β –closed set in a space X. We want to prove that A_1 is a maximal (minimal) regular β –open set. Assume that A_1 is not maximal (minimal) regular β –open set, then $A_1 \neq X$ (resp. $A_1 \neq \Phi$) and there exist regular β –open set A_2 such that $A_1 \subset A_2$ ($\subset A_1$) $A_1 \neq A_2$. By theorem 2.13,A_2 is β –closed set .So, A_2 includes (included in) A_1, which is opposite to each other. Hence A_1 is a maximal (minimal) regular β –closed set.

Theorem 3.30. If any regular closed set is maximal (minimal) regular β –open set then it is maximal (minimal) regular open set.

Proof. Let A_1 be a regular closed set and maximal (minimal) regular β –open set in a space X. We want to prove that A_1 is a maximal (minimal) regular closed set. Assume that A_1 is not maximal (not minimal) regular closed set, then $A_1 \neq X$ ($A_1 \neq \Phi$) and there exist regular closed set A_2 such that $A_1 \subset A_2$($A_2 \subset A_1$) $A_1 \neq A_2$. By theorem 2.22, A_2 is regular β –open set. So, A_2 includes (included in) A_1, which is opposite to each other. Hence $A_1$is a maximal (minimal) regular closed set.

Theorem 3.31. Let $A_1 \in M_a R\beta O(X)$ such that $a_1 \notin A_1$. Then, $A_1 \cup A_2 = \mathrm{X}$ for each $A_2 \in R\beta O(X, a_1)$ ..

Proof. Since, $a_1 \notin A_1$. We have $A_2 \not\subset A_1$ for any regular β –open set A_2 including a_1 . By theorem 3.19 $A_1 \cup A_2 = \mathrm{X}$.

Theorem 3.32. If the intersection of regular open set and minimal regular β –open set is not equal to φ. Then the intersection of these is equal to minimal regular β –open set.

Proof. Let U_1 be a regular open set and U_2 is minimal regular β –open set such that $U_1 \cap U_2 \neq \Phi$, by theorem 2.19, $U_1 \cap U_2$ is regular β –open set. We want to prove that $U_1 \cap U_2$ is minimal regular β –open set. Assume that $U_1 \cap U_2$ is not minimal regular β –open set, then there exists nonempty regular β –open set U_3 such that $U_3 \subset U_1 \cap U_2$and $U_3 \neq U_1 \cap U_2$.Hence $U_3 \subset U_1$ and $U_3 \subset U_2$.This means that U_3 is regular β –open set included in U_2 and $U_3 \neq U_2$, which is opposite to each other. Therefore $U_1 \cap U_2$ is minimal regular β –open set.

Theorem 3.33. If the union of regular closed set and maximal regular β –open set is not equal to X.Then the union of these is equal to maximal regular β –open set.

Proof. Let U_1 be a regular closed set and U_2 be a maximal regular β –open set such that $U_1 \cup U_2 \neq X$, then $X \backslash U_1$ is regular open set and $X \backslash U_2$ is minimal regular β –open set. By theorem 3.32 $(X \backslash U_1) \cap (X \backslash U_2) = X \backslash (U_1 \cup U_2)$ is minimal regular β –open set. This means that $U_1 \cup U_2$ is maximal regular β –open set.

Theorem 3.34. Let U_1be a maximal regular β –open set such that Cl (U_1)$\neq X$, then Cl (U_1) is maximal regular closed set.

Proof. If U_1 is maximal regular β –open set by theorem 2.18, Cl (U_1) is regular closed set. Since Cl (U_1)$\neq X$ then Cl (U_1) is proper regular closed set. We want to prove that Cl(U_1) is maximal regular closed set. Assume that Cl(U_1) is not maximal regular closed set, then there exists proper regular closed set U_2 such that Cl(U_1) $\subset U_2$.Since $U_1 \subset$Cl(U_1) $\subset U_2$, then $U_1 \subset U_2$.So by theorem 2.22 U_2 is regular β –open set including U_1, which is opposite to each other. Hence Cl(U_1) is maximal regular closed set.

Theorem 3.35. Let A_1be a minimal regular β –open set such that Int (A_1) $\neq \Phi$ then Int (A_1) is minimal regular open set.

Proof. If A_1 is minimal regular β –open set by theorem 2.18, Int(A_1) is regular open set. Since Int (A_1) $\neq \Phi$ then Int(A_1) is proper regular open set. We need to prove that Int (A_1) is minimal regular open set. Assume that Int(A_1) is not minimal regular open set, then there exist nonempty regular open set A_2 such that $A_2 \subset$ Int (A_1) .Since $A_2 \subset$ Int (A_1) $\subset A_1$, then $A_2 \subset A_1$.So by theorem 2.22 A_2 is regular β –open set included in A_1, which is opposite to each other. Hence Int (A_1) is minimal regular open set.

Remark 3.36. The following example shows that the converse part of Theorem 3.35 is not true in general.

Example 3.37. : Consider $X = \{u_1, u_2, u_3, u_4\}$ and let $\mathfrak{I} = \{ \Phi, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_3, u_4\}, \{u_1, u_3, u_4\}, \{u_2, u_3, u_4\}, X\}$, then $RO(X) = \{ \Phi, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_3, u_4\}, \{u_1, u_3, u_4,, \{u_2, u_3, u_4, X\}$, $R\beta O(X)$=$\{ \Phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}, \{u_1, u_2\}, \{u_3, u_4\}, \{u_1, u_3\}$, $\{u_2, u_4\}, \{u_1, u_4\}, \{u_2, u_3\}, \{u_1, u_3, u_4\}, \{u_2, u_3, u_4\}, \{u_1, u_2, u_4\}, \{u_1, u_2, u_3\}, X\}$. Let G=$\{u_1, u_3\}$, then Int(G)= $\{u_1\}$ is minimal regular open but G is not minimal regular β –open set.

Theorem 3.38. αO(X) ∩ M$_i$RβO(X) ⊂ M$_i$RO(X).

Proof. Let $A_1 \in$ αO(X) ∩ M$_i$RβO(X) then $A_1 \in$ αO(X) ∩ RβO(X). By theorem 2.20 $A_1 \in$ RO(X). We have to prove that A_1 is minimal regular open set($A_1 \in$ M$_i$RO(X)). Assume that A_1 is not minimal regular open set ($A_1 \notin$ M$_i$RO(X)), then there exists nonempty regular open set A_2 such that $A_2 \subset A_1$ and $A_2 \neq A_1$.Since A_2 is regular β –open set included in A_1, which is opposite to each other .Therefore A_1 is minimal regular open set ($A_1 \in$ M$_i$RO(X)).

Theorem3.39. If A_1 is a minimal β –open set in a space X such that A_1 is not dense in X, then βCl(A_1) is minimal regular β –open set.

Proof. Let A_1 be a minimal β –open set from theorem 2.21, we have βCl(A_1) is regular β – open set. Since A_1 is nonempty β –open and $A_1 \subseteq$ βCl(A_1), then $\Phi \neq$ βCl(A_1). Now Cl(A_1) $\neq X$ because of A_1 is not dense in X. So that βCl(A_1) ⊆ Cl(A_1) and βCl(A_1) $\neq X$. That means βCl(A_1) is nonempty proper regular β –open set. We must to prove that βCl(A_1) is minimal regular β –open set. Let A_2 be a nonempty regular β –open set such that $A_2 \subseteq$ βCl(A_1). Then βCl(A_2) ⊆ βCl(A_1).If $A_1 \cap A_2 = \Phi$, then βCl(A_1) ∩ βCl(A_2) $= \Phi$ which is opposite to each other. Hence, $A_1 \cap A_2 \neq \Phi$ and by theorem 2.13 A_2 is β –open. So that from theorem 3.7(i) we get $A_1 \subset A_2$ and βCl(A_1) ⊆ βCl(A_2). This implies that βCl(A_1) = βCl(A_2). Since, A_2 is regular β –open set then from Theorem 2.13 we have A_2 is β –closed. This implies that βCl(A_2) = , *and* βCl(A_1) $= A_2$. Therefore, βCl(A_1) is minimal regular β –open set.

Theorem3.40. Let U_1, U_2 and U_3 be maximal regular β –open sets such that $U_1 \neq U_2$. If $U_1 \cap U_2 \subset U_3$, then either $U_1 = U_3$ *or* $U_2 = U_3$.

Proof. The proof is similar to theorem 2.24.So that it is omitted.

Theorem3.41. Let U_1, U_2 and U_3 be maximal regular β –open sets which are various from each ether. Then, $(U_1 \cap U_2) \not\subset (U_1 \cap U_3)$.

Proof. The proof is similar to theorem 2.5.

Theorem3.42. i- Let G and $\{G_\lambda\}_{\lambda \in \Lambda}$be minimal regular β –open sets. If $G \subset \bigcup_{\lambda\in\Lambda} G_\lambda$, then there exists $\lambda \in \Lambda$ such that $G = G_\lambda$.

ii- Let G and $\{G_\lambda\}_{\lambda \in \Lambda}$ be a minimal regular β –open sets. If $G \neq G_\lambda$ for each $\lambda \in \Lambda$, then $(\bigcup_{\lambda\in\Lambda} G_\lambda) \cap G = \Phi$.

Proof. i- Let G and $\{G_\lambda\}_{\lambda \in \Lambda}$ be a minimal regular β –open sets with $\subset \bigcup_{\lambda\in\Lambda} G_\lambda$. we must to prove that $G \cap G_\lambda \neq \varphi$. Since if $G \cap G_\lambda = \varphi$, then $G_\lambda \subset X \backslash G$ and hence, $G \subset \bigcup_{\lambda\in\Lambda} G_\lambda \subset X \backslash G$ which is opposite to each other. Now as $G \cap G_\lambda \neq \varphi$, then $G \cap G_\lambda \subset G$ and $G \cap G_\lambda \subset G_\lambda$. Since $G \cap G_\lambda \subset G$ and this gives that G is minimal regular β –open, then by definition 3.1, $G \cap G_\lambda = G$ or $G \cap G_\lambda = \varphi$. But $G \cap G_\lambda \neq \varphi$, then $G \cap G_\lambda = G$ which means $G \subset G_\lambda$.Similarly $G \cap G_\lambda \subset G_\lambda$ gives that G_λ is minimal regular β –open, then by Definition

3.1, $G \cap G_\lambda = G_\lambda$ or $G \cap G_\lambda = \varphi$. But $G \cap G_\lambda \neq \varphi$ then $G \cap G_\lambda = G_\lambda$ which means $G_\lambda \subset G$. Therefoer, $G = G_\lambda$.
ii- Assume that $(\bigcup_{\lambda \in \Lambda} G_\lambda) \cap G \neq \varphi$ then there exists $\lambda \in \Lambda$ such that $G_\lambda \cap G \neq \varphi$. By theorem 3.7 (ii), we have $G = G_\lambda$ which is opposite to the fact $G \neq G_\lambda$. Hence, $(\bigcup_{\lambda \in \Lambda} G_\lambda) \cap G = \varphi$.

CONCLUSION

In this paper, maximal and minimal open sets via regular β-open sets are introduced. We also get several results that are presented to reveal many various properties of the minimal regular β –open and maximal regular β –open sets and their complements. The relation between the minimal regular β –open and maximal regular β –open are shown. Finally, we have discussed their relationship with various types of open sets such as minimal and maximal regular open sets.

References

[1] Abd El-Monsef,M., "β-open sets and β-continuous mappings", *Bull. Fac. Sci. Assiut Univ.,* no. 12, pp. 77-90, 1983.

[2] Andrijević, D., "Semi-preopen sets”. *Matematički Vesnik*, vol. 38, no. 93, pp.24-32, 1986.

[3] Anuradha, N. and Chacko, B., "On minimal regular open sets and maps in topological spaces", *J. Math. Sci*, vol. 4, pp.182-192, 2015.

[4] Hasan, H., "Maximal and Minimal α-Open Sets", *Int. J. Engineering Research and Applications.* Vol. 5, pp. 50- 53, 2015.

[5] Nakaoka, F. and Oda, N., "Some properties of maximal open sets ". *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 21, pp.1331-1340, 2003.

[6] Nakaoka, F. and Oda, N., " Several applications of minimal open sets", *International Journal of Mathematics and Mathematical Sciences,* vol. 8, no. 27, pp. 471-476, 2001.

[7] Nasser, F.M., "On Minimal and Maximal Regular Open Sets", Doctoral dissertation, The Islamic University-Gaza, Gaza: The Islamic University – Gaza, 2016.

[8] Njȧstad, O., "On some classes of nearly open sets”. *Pacific journal of mathematics*, vol. 15, no. 3, pp.961-970, 1956.

[9] Shakir, Q.R., "Minimal and Maximal Beta Open Sets", *Al-Nahrain Journal of Science*, vol. 17, no. 1, pp.160-166, 2015.

[10] Stone, M., " Applications of the theory of Boolean rings to general topology", *Transactions of the American Mathematical Society,* vol. 3, no. 41, pp. 375-481, 1937.

[11] Yunis, R., "Regular β -open setse," *Zanco journal of pure and applied Science",* vol. 3, no. 16, pp. 79-83, 2004.