Tawfiq and Hussein Iraqi Journal of Science, 2022, Vol. 63, No. 10, pp: 4388-4396
DOI: 10.24996/ijs.2022.63.10.25

ISSN: 0067-2904

Exact Solution for Systems of Nonlinear (2+1)D-Differential Equations

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Received: 21/5/2021 Accepted: 21/1/2022 Published: 30/10/2022

Abstract
The aim of this article is to present the exact analytical solution for models as system of (2+1) dimensional PDEs by using a reliable manner based on combined LA-transform with decomposition technique and the results have shown a high-precision, smooth and speed convergence to the exact solution compared with other classic methods. The suggested approach does not need any discretization of the domain or presents assumptions or neglect for a small parameter in the problem and does not need to convert the nonlinear terms into linear ones. The convergence of series solution has been shown with two illustrated examples such (2+1)D- Burger's system and (2+1)D- Boiti-Leon-Pempinelli (BLP) system.

Keywords: System (2+1) dimensional -PDEs; BLP System, Burger's system, Coupled Method, ADM.

الحل المضبوط لمنظومات معادلات تفاضلية غير خطية ذو البعد (2+1)

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الخلاصة
يهدف البحث لإيجاد الحل التحليلي المضبوط لنماذج كمنظومة معادلات تفاضلية جزئية ذو البعد (2+1) باستخدام اسلوب موثوق مبني على اقتران التحويل LA مع تقنية التحليل و النتائج اثبتت دقة عالية ضمن مجال املس وسرعة التقارب الى الحل المضبوط مقارنة مع الطرق التقليدية. الاسلوب المقترح لا يحتاج اي تجزئة للمجال او استخدام فرضيات او اهمال و حذف المعلمات في المسالة و لا يحتاج تحويل الحد غير الخطي الى خطي. تم توضيح تقارب متسلسلة الحل من خلال مثالين هما منظومة برگر ذات البعد (1+2) و منظومة BLP ذو البعد (1+2).

1. Introduction

Differential equations and (2+1)D-PDEs arise in all branches of science and engineering, hence the existence of suitable methods to find their solution are being a fundamental

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importance topic. As analytical solutions are only available in a few cases the construction of efficient approximate or numerical methods is essential [1-5].

Many researchers have paid attention in recent years to study the solutions of nonlinear systems of PDEs using various methods, namely ADM, HAM, HPM, DTM, Laplace Decomposition Method, Cubic Trigonometric B Spline Method, Parallel Processing Technique, VIM and Semi Analytic Technique [6-26]. In this article, we suggest an efficient reliable technique to solve a system of (2+1)D-PDEs based on combining two high performance methods such LA-Transform [27-30] and ADM [31-34] has been used to solve 2^{nd} order nonlinear (2+1)D- Burger's system also, to solve 3^{rd} order nonlinear (2+1)D- Boiti-Leon-Pempinelli system.

This paper has been arranged as follows: In section 2, we present LA- transformation. In section 3, the basic ideas of the suggested method will be given. Illustrative problems via two important models as a system of (2+1)D- Burger's equations and (2+1)D- Boiti-Leon-Pempinelli (BLP) has been presented to illustrate the efficiency and accuracy for the suggested method will be given in section 4. Finally, the conclusion is given in section 5.

2. LA- Transformation

Tawfiq and Jabber in 2018 [35] defined LA- transformation for a function $f(t)$ as follows:

$$\bar{f}(s) = \mathbb{T}\{f(t)\} = \int_0^\infty e^{-t} f\left(\frac{t}{s}\right) dt \ ,$$

Where s is a real number, for those values of s, the improper integral is convergent.

The domain of this transformation is wider than in the Laplace transform (L.T) this feature makes the LAT more widely used in problems, and has some interesting properties which make it more appropriate or suitable than the Laplace Transform. Such as: the duality between LAT with LT, therefore, the LAT can solve all the problems which can be solved by LT; the unit step function in the t-domain is transformed to unity in the u-domain; The integration and differentiation in the t-domain are equivalent to division and multiplication of the transformed function F(u) by u in the u-domain moreover, linearity property that is for any constant $a \in$R, $\mathbb{T}\{a\} = a\mathbb{T}\{1\} = a$, and hence, $\mathbb{T}^{-1}\{a\} = a$, that means there is no any problem when we deal with the constant term (the constant with respect to the parameter u). For more details see [35].

3. Suggested Approach for Solving Nonlinear (2+1)D- Differential Equations

Here, we suggest an efficient approach that is based on combining LA-transform with ADM and denoted by LATDM. Firstly, we write the form of the nonlinear system of (2+1)D-PDEs as follows:

$$\left.\begin{aligned} u_t(x,y,t) = g_1(x,y,t) + R_1(u,v) + N_1(u,v) \\ v_t(x,y,t) = g_2(x,y,t) + R_2(u,v) + N_2(u,v) \end{aligned}\right\}, \quad (1)$$

Subject to initial conditions: $u(x,y,0) = f_1(x,y), \ \ v(x,y,0) = f_2(x,y)$

where u_t and v_t are considered, without loss of generality, the first derivative for u and v w.r.t. (t) respectively, R_1 and R_2 represent the linear operators part, N_1 and N_2 represent the nonlinear operators part and g_1 and g_2 are the inhomogeneous source.

The implementation of the suggested approach is started by taking LA-transform (denoted by $\mathbb{T}$) on both sides of the equations in (1) to get

$$\left.\begin{aligned} \mathbb{T}\{u_t(x,y,t)\} = \mathbb{T}\{g_1(x,y,t)\} + \mathbb{T}\{R_1 + N_1\} \\ \mathbb{T}\{v_t(x,y,t)\} = \mathbb{T}\{g_2(x,y,t)\} + \mathbb{T}\{R_2 + N_2\} \end{aligned}\right\}, \quad (2)$$

using the property of derivative for LA-transform we get the following:

$$\left.\begin{array}{l} s\mathbb{T}\{u(x,y,t)\} - sf_1(x,y) = z_1(x,y,s) + \mathbb{T}\{R_1 + N_1\} \\ s\mathbb{T}\{v(x,y,t)\} - sf_2(x,y) = z_2(x,y,s) + \mathbb{T}\{R_2 + N_2\} \end{array}\right\}, \qquad (3)$$

So,

$$\left.\begin{array}{l} \mathbb{T}\{u(x,y,t)\} = w_1(x,y,s) + \frac{1}{s}\mathbb{T}\{R_1 + N_1\} \\ \mathbb{T}\{v(x,y,t)\} = w_2(x,y,s) + \frac{1}{s}\mathbb{T}\{R_2 + N_2\} \end{array}\right\}, \qquad (4)$$

Hence, we used for the linear part the decomposition series and for the nonlinear part, we used the infinite series of Adomian polynomials. Where Adomian polynomials A_m are defined as follows:

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad m = 0,1,2,\dots \ . \qquad (5)$$

So, by linearity property of LA- transform, the solutions $u(x,y,t)$ and $v(x,y,t)$ are obtained easily by applying the inverse of LA- transform. Putting these components into the expansion that is given by:

$$\left.\begin{array}{l} u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = u_0 + u_1 + u_2 + \cdots \\ v(x,y,t) = \sum_{n=0}^{\infty} v_n(x,y,t) = v_0 + v_1 + v_2 + \cdots \end{array}\right\}, \qquad (6)$$

We get the required solution such that

$$u_0(x,y,t) = \mathbb{T}^{-1}\{w_1(x,y,s)\}, \quad u_{n+1} = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{R_1 + N_1\}\right\}$$

$$v_0(x,y,t) = \mathbb{T}^{-1}\{w_2(x,y,s)\}, \quad v_{n+1} = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{R_2 + N_2\}\right\}$$

4. Illustrative Problems

In this section, two important models as system of (2+1)D-PDEs have been presented to illustrate the efficiency and accuracy for the suggested method.

Example 1: Consider the following nonlinear (2+1)D-PDE (Burger's equations),[36]

$u_t + uu_x + vu_y = u_{xx} + u_{yy}$

$v_t + uv_x + vv_y = v_{xx} + v_{yy}$, with initial conditions

$u(x,y,0) = x + y$

$v(x,y,0) = x - y$

By taking LA-transformation on both sides above equations

$\mathbb{T}\{u_t\} = \mathbb{T}\{u_{xx} + u_{yy} - uu_x - vu_y\}$

$\mathbb{T}\{v_t\} = \mathbb{T}\{v_{xx} + v_{yy} - uv_x + vv_y\}$

$s\mathbb{T}\{u\} - su(x,y,0) = \mathbb{T}\{u_{xx} + u_{yy} - uu_x - vu_y\}$

$s\mathbb{T}\{v\} - sv(x,y,0) = \mathbb{T}\{v_{xx} + v_{yy} - uv_x + vv_y\}$

$u = \mathbb{T}^{-1}\left\{u(x,y,0) + \frac{1}{s}\mathbb{T}\{u_{xx} + u_{yy} - uu_x - vu_y\}\right\}$

$v = \mathbb{T}^{-1}\left\{v(x,y,0) + \frac{1}{s}\mathbb{T}\{v_{xx} + v_{yy} - uv_x + vv_y\}\right\}$

By using decomposition procedure

$u_0 = \mathbb{T}^{-1}\{u(x,y,0)\} = x + y$

$v_0 = \mathbb{T}^{-1}\{v(x,y,0)\} = x - y$

$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{u_{0xx} + u_{0yy} - A_0 - B_0\}\right\}$

$v_1 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{v_{0xx} + v_{0yy} - C_0 - D_0\}\right\}$

$$A_0 = F(u_0, v_0) = u_0 u_{0x} = x + y$$
$$B_0 = F(u_0, v_0) = v_0 u_{0y} = x - y$$
$$C_0 = F(u_0, v_0) = u_0 v_{0x} = x + y$$
$$D_0 = F(u_0, v_0) = v_0 v_{0x} = -x + y$$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{0 + 0 - x - y - x + y\}\right\} = -2xt$$
$$v_1 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{0 + 0 - x - y + x - y\}\right\} = -2yt$$

$$A_1 = u_0 u_{1x} + u_{0x} u_1$$
$$A_1 = -4xt - 2yt$$

$$B_1 = v_0 u_{1y} + u_{0y} v_1$$
$$B_1 = -2yt$$

$$C_1 = u_0 v_{1x} + v_{0x} u_1$$
$$C_1 = -2xt$$

$$D_1 = v_0 v_{1y} + v_{0y} v_1$$
$$D_1 = -2xt + 4yt$$

$$u_2 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{u_{1xx} + u_{1yy} - A_1 - B_1\}\right\}$$
$$u_2 = \mathbb{T}^{-1}\left\{\frac{1}{s^2}(4x + 4y)\right\} = 2(x + y)t^2$$

$$v_2 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{v_{1xx} + v_{1yy} - C_1 - D_1\}\right\}$$
$$v_2 = \mathbb{T}^{-1}\left\{\frac{1}{s^2}(4x - 4y)\right\} = 2(x - y)t^2$$

$$A_2 = \frac{1}{2}[u_0(2u_{2x}) + u_{1x}u_1 + u_{0x}(2u_2) + u_1 u_{1x}]$$
$$A_2 = 8xt^2 + 4yt^2$$

$$B_2 = \frac{1}{2}[v_0(2u_{2y}) + u_{1y}v_1 + u_{0y}(2v_2) + v_1 u_{1y}]$$
$$B_2 = 4xt^2 - 4yt^2$$

$$C_2 = \frac{1}{2}[u_0(2v_{2x}) + v_{1x}u_1 + v_{0x}(2u_2) + u_1 v_{1x}]$$
$$C_2 = 4xt^2 + 4yt^2$$

$$D_2 = \frac{1}{2}[v_0(2v_{2y}) + v_{1y}v_1 + v_{0y}(2v_2) + v_1 v_{1y}]$$
$$D_2 = -4xt^2 + 8yt^2$$

$$u_3 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{u_{2xx} + u_{2yy} - A_2 - B_2\}\right\}$$
$$u_3 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{0 + 0 - 8xt^2 - 4yt^2 - 4xt^2 + 4yt^2\}\right\}$$
$$u_3 = \mathbb{T}^{-1}\left\{\frac{2}{s^3}(-12x)\right\} = -4xt^3$$

$v_3 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{v_{2xx} + v_{2yy} - C_2 - D_2\}\right\}$
$v_3 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{-4xt^2 - 4yt^2 + 4xt^2 - 8yt^2\}\right\}$
$v_3 = \mathbb{T}^{-1}\left\{\frac{2}{s^3}(-12y)\right\} = -4yt^3$

$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_3 + \cdots$
$u = x + y - 2xt + 2xt^2 + 2yt^2 - 4xt^3 + \cdots$
$u = x(1 + 2t^2 + \cdots) - 2xt(1 + 2t^2 + \cdots) + y(1 + 2t^2 + \cdots)$
$u = \frac{x}{1-2t^2} - \frac{2xt}{1-2t^2} + \frac{y}{1-2t^2}$
$u = \frac{x+y-2xt}{1-2t^2}$

$v = \sum_{n=0}^{\infty} v_n = v_0 + v_1 + v_3 + \cdots$
$v = x - y - 2yt + 2xt^2 - 2yt^2 - 4yt^3 + \cdots$
$v = x(1 + 2t^2 + \cdots) - y(1 + 2t^2 + \cdots) - 2yt(1 + 2t^2 + \cdots)$
$v = \frac{x}{1-2t^2} - \frac{y}{1-2t^2} - \frac{2yt}{1-2t^2}$
$v = \frac{x-y-2yt}{1-2t^2}$
This is close to the exact solution.

Suleman et al [36] solved the system by using the Elzaki Homotopy perturbation method and get analytic solution but herein we get the exact an analytic solution with easy implementation and fewer steps.

Example 2: Consider the following nonlinear (2+1)D- Boiti-Leon-Pempinelli system (BLP) [30, 37]

$$\left.\begin{aligned} &u_{ty} - 2u_y u_x - 2uu_{xy} + u_{xxy} - 2v_{xxx} = 0 \\ &v_t - v_{xx} - 2uv_x = 0 \end{aligned}\right\} \quad (9)$$

With initial conditions ,
$u_y(x, y, 0) = sech^2(x + y)$, $v(x, y, 0) = tanh(x + y)$
To solve (9) by using the L.A. method with D. method,
we take the L.A. transform to both sides of (9)
$\mathbb{T}\{u_{ty}\} = \mathbb{T}\{2u_y u_x + 2uu_{xy} - u_{xxy} + 2v_{xxx}\}$
$\mathbb{T}\{v_t\} = \mathbb{T}\{v_{xx} + 2uv_x\}$
$s\mathbb{T}\{u_y(x, y, t)\} - su_y(x, y, 0) = \mathbb{T}\{2u_y u_x + 2uu_{xy} - u_{xxy} + 2v_{xxx}\}$
$s\mathbb{T}\{v(x, y, t)\} - sv(x, y, 0) = \mathbb{T}\{v_{xx} + 2uv_x\}$
$u_y = \mathbb{T}^{-1}\left\{u_y(x, y, 0) + \frac{1}{s}\mathbb{T}\{2u_y u_x + 2uu_{xy} - u_{xxy} + 2v_{xxx}\}\right\}$
$v = \mathbb{T}^{-1}\left\{v(x, y, 0) + \frac{1}{s}\mathbb{T}\{v_{xx} + 2uv_x\}\right\}$
By using decomposition procedure
$u_{ny} = \mathbb{T}^{-1}\left\{sech^2(x + y) + \frac{1}{s}\mathbb{T}\{2A_n + 2B_n - u_{nxxy} + 2v_{nxxx}\}\right\}$
$v_n = \mathbb{T}^{-1}\left\{tanh(x + y) + \frac{1}{s}\mathbb{T}\{v_{nxx} + 2c_n\}\right\}$
$u_{0y} = sech^2(x + y)$
$u_{0y} = sech^2(x + y)$, integrating w.r.t (y)

$u_0 = tanh(x+y) + a(x)$, let $a(x) = -\frac{1}{2}$
$\Longrightarrow u_0 = -\frac{1}{2} + tanh(x+y)$
$\Longrightarrow v_0 = tanh(x+y)$
$u_{1y} = \mathbb{T}^{-1}\left\{\frac{1}{s}\,\mathbb{T}\{2A_0 + 2B_0 - u_{0xxy} + 2v_{0xxx}\}\right\}$
$v_1 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{v_{0xx} + 2c_0\}\right\}$
$A_0 = F(u_0) = u_{0y}u_{0x}$, $A_1 = u_{0y}u_{1x} + u_{0x}u_{1y}$
$B_0 = G(u_0) = u_0u_{0xy}$, $B_1 = u_0u_{1xy} + u_{0xy}u_1$
$c_0 = H(u_0, v_0) = u_0v_{0x}$, $c_1 = u_0v_{1x} + v_{0x}u_1$
$u_{0x} = sech^2(x+y)$
$u_{0xx} = -2sech^2(x+y)tanh(x+y)$
$u_{0xxy} = -2sech^4(x+y) + 4sech^2(x+y)tanh^2(x+y)$
$u_{0y}u_{0x} = sech^4(x+y)$
$u_0u_{0xy} = sech^2(x+y)tanh(x+y) - 2sech^2(x+y)tanh^2(x+y)$
$v_{0x} = sech^2(x+y)$
$v_{0xx} = -2sech^2(x+y)tanh(x+y)$
$v_{0xxx} = -2sech^4(x+y) + 4sech^2(x+y)tanh^2(x+y)$
$u_0v_{0x} = -\frac{1}{2}\,sech^2(x+y) + sech^2(x+y)tanh(x+y)$
$u_{1y} = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{2sech^4(x+y) + 2sech^2(x+y)tanh(x+y) - 4sech^2(x+y)tanh^2(x+y) + 2sech^4(x+y) - 4sech^2(x+y)tanh^2(x+y) + -4sech^4(x+y) + 8sech^2(x+y)tanh^2(x+y)\}\right\}$
$u_{1y} = 2tsech^2(x+y)tanh(x+y)$
$\Rightarrow\ u_1 = -tsech^2(x+y)$
$v_1 = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{-2sech^2(x+y)tanh(x+y) - \ sech^2(x+y) + 2sech^2(x+y)tanh(x+y)\}\right\}$
$\Rightarrow\ v_1 = -tsech^2(x+y)$
$u_{1x} = 2tsech^2(x+y)tanh(x+y)$
$u_{1xx} = 2tsech^4(x+y) - 4tsech^2(x+y)tanh^2(x+y)$
$u_{1xxx} = -16tsech^4(x+y)\tanh(x+y) + 8tsech^2(x+y)tanh^3(x+y)$

$v_{1x} = 2tsech^2(x+y)tanh(x+y)$
$v_{1xx} = 2tsech^4(x+y) - 4tsech^2(x+y)tanh^2(x+y)$
$v_{1xxx} = -16tsech^4(x+y)\tanh(x+y) + 8tsech^2(x+y)tanh^3(x+y)$

$A_1 = 4tsech^4(x+y)\tanh(x+y)$
$B_1 = 2tsech^4(x+y)\tanh(x+y) - 4tsech^2(x+y)\tanh^3(x+y) - tsech^4(x+y) + 2tsech^2(x+y)\tanh^2(x+y) + 2tsech^4(x+y)\tanh(x+y)$
$c_1 = -tsech^2(x+y)tanh(x+y) + 2tsech^2(x+y)\tanh^2(x+y) - tsech^4(x+y)$

$u_{2y} = \mathbb{T}^{-1}\left\{\frac{1}{s}\mathbb{T}\{8tsech^4(x+y)\tanh(x+y) + 4tsech^4(x+y)\tanh(x+y) - 8tsech^2(x+y)\tanh^3(x+y) - 2tsech^4(x+y) + 4tsech^2(x+y)\tanh^2(x+y) + \right.$

$4tsech^4(x+y)\tanh(x+y)+16tsech^4(x+y)\tanh(x+y)-8tsech^2(x+y)tanh^3(x+y)-32tsech^4(x+y)\tanh(x+y)+16tsech^2(x+y)tanh^3(x+y)\}\Big\}$

$u_{2y}=\frac{t^2}{2!}\left(-2sech^4(x+y)+4sech^2(x+y)tanh^2(x+y)\right)$

$\Rightarrow u_2=\frac{t^2}{2!}\left(-2sech^2(x+y)tanh(x+y)\right)$

$v_2=\mathbb{T}^{-1}\big\{\mathbb{T}\{2tsech^4(x+y)-4tsech^2(x+y)tanh^2(x+y)-2tsech^2(x+y)tanh(x+y)+4tsech^2(x+y)tanh^2(x+y)-2tsech^4(x+y)\}\big\}$

$\Rightarrow v_2=\frac{t^2}{2!}\left(-2sech^2(x+y)tanh(x+y)\right)$

Thus , $u=\sum_{n=0}^{\infty}u_n$ and $v=\sum_{n=0}^{\infty}v_n$

$u=-\frac{1}{2}+tanh(x+y)-tsech^2(x+y)+\frac{t^2}{2!}\left(-2sech^2(x+y)tanh(x+y)\right)+\cdots$

$u=-\frac{1}{2}+\left[tanh(x+y-t)+t\frac{\partial}{\partial t}tanh(x+y-t)+\frac{t^2}{2!}\frac{\partial^2}{\partial t^2}tanh(x+y-t)+\cdots\right]_{t=0}$

$u=-\frac{1}{2}+\tanh(x+y-t)$, also

$v=tanh(x+y)-tsech^2(x+y)+\frac{t^2}{2!}\left(-2sech^2(x+y)tanh(x+y)\right)+\cdots$

$v=\left[tanh(x+y-t)+t\frac{\partial}{\partial t}tanh(x+y-t)+\frac{t^2}{2!}\frac{\partial^2}{\partial t^2}tanh(x+y-t)+\cdots\right]_{t=0}$

$v=\tanh(x+y-t)$. This is closed to the exact solution.

Wang in [37] studied the properties of this type of system, Ayati in [38] solved this system by using the modified simple equation method and he got the exact solution but this method generates some parameters that will be determined. Also, Qi and Zhang in [39] solved this system by using Traveling Wave with computer simulations to get the exact solution.

5. Conclusion

In this article, an efficient approach based on combining LA-Transform with the decomposition method is suggested to solve nonlinear system of (2+1)-D model problems to get the exact solution, where other directed methods are used to solve the same model problems without getting the exact solution. Also, the Elzaki Homotopy perturbation method, modified simple equation method and Traveling Wave have been used in this type of systems but the suggested method is easier to implement than other methods. Moreover, in LATDM the nonlinear terms are easier to compute than in ADM or its modifications. So, the suggested approach has high accuracy, easy implementation and rapid convergence to the exact solutions.

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