DOI: 10.24996/ijs.2021.62.11.28

ISSN: 0067-2904

(α, β) – Derivations on Prime Inverse Semirings

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Received:11/5/2021 Accepted: 19/7/2021

Abstract

Let S be a prime inverse semiring with center $Z(S)$. The aim of this research is to prove some results on the prime inverse semiring with (α, β) – derivation that acts as a homomorphism or as an anti- homomorphism, where α, β are automorphisms on S.

Keywords: inverse semiring, prime inverse semiring, cancellative, left ideal , Jordan ideal, (α, β) – derivation.

المشتقات–(α, β) على أشباه الحلقات المعكوسة الاولية

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الخلاصة

لتكن S شبه حلقة اولية معكوسة مع المركز Z(S). الهدف من هذا البحث هو برهان بعض النتائج على شبه الحلقة الاولية المعكوسة S مع المشتقة–(α, β) التي تكون هومومورفزم او انتي هومومورفزم, عندما α, β تكون اوتومورفزم على.

1. Introduction

The concept of semiring was introduced by Vandiver in 1934 [1]. The algebraic structure of inverse semiring was introduced by Karevellas in1974 [2].

After that, many researchers studied classes of semiring, including Golan [3] and Fang [4]. Recently, many authors studied such semirings in various ways, describing the analysis of prime and semiprime semirings with various types of derivations [5-10].

A semiring $(S, +, \cdot)$ with commutative addition and absorbing zero 0 is called additively "inverse semiring" if, for every element $r \in S$, there exists a unique $r' \in S$ such that $r + r' + r = r$ and $r' + r + r' = r'$, as introduced by Bandlet and Petrich [11]. According to Karevellas [2], for all $a, b \in S$, we have $(ab)' = a'b = ab'$,$(a + b)' = a' + b'$, $a'b' = ab$, $(a')' = a$. A semiring S is additively left cancellative if $r + s = r + t$ then $s = t$, for all $r, s, t \in S$, S is additively right cancellative if $s + r = t + r$ then $s = t$, for all $r, s, t \in S$. Also, S is said to be additively cancellative if it is both additively left and right cancellative [12].

The additive inverse semiring that satisfies $r(s + s') = (s + s')r$, for all $r, s \in S$, i.e. $s + s' \in Z(S)$ for all $s \in S$, is called MA-semiring [13], which we adopt in this paper. A commutator in an inverse semirng is defined as $[r, s] = rs + s'r = rs + sr'$. We can define a Jordan product on S as follows: $a \circ b = ab + ba$, for all $a, b \in S$. A non-empty subset I of S

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is said to be a left ideal of S if for all $x, y \in I, s \in S$, then $x + y \in I$ and $sx \in I$ [14]. Also, we define the Jordan ideal in the setting of inverse semiring as the additive semigrup J of inverse semiring S, if for all $x \in J, r \in S$, we have $xr + rx \in J$ [10].

Recall that S is called a prime inverse semiring if $aSb = 0$, for all $a, b \in S$, implies that either $a = 0$ or $b = 0$. It is also called semiprime inverse semiring if $aSa = 0$, for all $a \in S$, implies that $a = 0$, or S has no non-zero nilpotetent ideal [9]. We call S as 2-tortion free if $2a = 0$, $a \in S$ implies that $a = 0$. An additive mapping $d: S \to S$ is called (α, β)-derivation of S if $d(rs) = d(r)\alpha(s) + \beta(r)d(s)$, for all r, $s \in S$, where α, β are automorphisms on S [8].

In this paper, we prove some results on prime inverse semiring with (α, β) – derivation that acts as a homomorphism or as an anti- homomorphism, and where S is a cancellative with $s + s' \in Z(S)$. We also extend some important previously found results [15, 16] on (α, β) – Derivations, where α, β are automorphisms on S.

2. Preliminaries

To prove the main Theorems in this paper, we need the following Lemmas.

Lemma 2.1 [9]

For all $r, s \in S$, if $r + s = 0$, then $r = s'$.

Lemma 2.2.

Let S be a cancellative prime inverse semiring. If $r = s$, where $r, s \in S$, then $r + s' = 0$.

Proof

If $r = s$, where $r, s \in S$, then by adding $(s + s')$ to the two sides, we get

$r + (s' + s) = s + (s' + s)$.

Since S is an inverse semiring, we get

$r + s' + s = s$

$r + s') + s = s$.(

Since S is a cancellative inverse semiring, we get $r + s' = 0$.

Lemma 2.3.

Let I be a non- zero left ideal on S, which is a semiprime inverse semiring .If $Iu = 0$ ($uI = 0$) for all $u \in S$, then $u = 0$.

Proof

If $uI = 0$ for all $u \in S$, then we have $uSI = 0$. Since S is a prime inverse semiring and I is a nonzero left ideal of S, therefore $u = 0$.

Now, we want to show that if $Iu = 0$, then $u = 0$. Suppose that $u \neq 0$.

Define K by

$$K = \{r \in S | Ir = 0\}.$$

Since $0 \neq u \in K$, it is clear that K is a nonzero right ideal of S, such that $IK = \{0\}$.

On the other side, $K \cap I$ is a right ideal of I and

$$(K \cap I)(K \cap I) \subset IK = \{0\},$$

That is, $(K \cap I)^2 = \{0\}$.

Since I is a semiprime inverse semiring, then we get $(K \cap I) = \{0\}$. Then, we have $KI \subset K \cap I = \{0\}$. Since S is a prime inverse semiring and I is a nonzero left ideal of S, we obtain $K = \{0\}$. Thus, we get $u = 0$.

Lemma 2.4.

Let I be a non- zero left ideal on S, which is a semiprime as an inverse semiring. If $d(I) = 0$, then $d = 0$ on S.

Proof

By the assumption, $d(I) = 0$, then for all $x \in I$ and $s \in S$:

$$0 = d(sx) = d(s)\alpha(x) + \beta(s)d(x)$$
$$= d(s)\alpha(x)$$

Since α is an automorphism on *S*, we get $\alpha^{-1}(d(s))I = 0$ for all $s \in S$. By Lemma 2.3, we get $\alpha^{-1}(d(s)) = 0$, for all $s \in S$. Again, since α is an automorphism on *S*, we get $d(s) = 0$ for all $s \in S$, therefore $d = 0$ on S.

Lemma 2.5. [17]

Let *J* be a nonzero Jordan ideal of *S*, then $2(rs + s'r)J \subseteq J$ and $2J(rs + s'r) \subseteq J$.

Lemma 2.6.

Let *J* be a nonzero Jordan ideal of *S*. If $u \in S$ and $uJ = 0$ or ($Ju = 0$), then $u = 0$.

Proof

Let $uJ = 0$. Since J is a nonzero Jordan ideal of S, we have $x \circ s \in J$ for all $x \in J$ and $s \in S$.

$$0 = u(x \circ s) = u(xs + sx) = uxs + usx$$

By assumption, we get

$usx = 0$ for all $x \in J$ and $s \in S$.

That is, $uSx = 0$ for all $x \in J$.

Since *S* is a prime inverse semiring and *J* is a nonzero Jordan ideal of *S*, we get $u = 0$.

Using the similar way, we can show that if $Ju = 0$ then $u = 0$.

Lemma 2.7.

Let S be 2-tortion free and J is a nonzero Jordan ideal of S. If $u, v \in S$ such that $uJv = 0$, then either $u = 0$ or $v = 0$.

Proof

By the assumption, we have $uJv = 0$. By Lemma 2.5, we get $2(rs + s'r)J \subseteq J$ for all $r, s \in S$. Then we have $2u(rs + s'r)xv = 0$, for all $x \in J$ and $r, s \in S$. This implies that, since *S* is 2-tortion free, we get

$$u(rs + s'r)xv = 0 \text{, for all } x \in J \text{ and } r, s \in S.$$

By replacing *s* by *su* in the above equation, we get

$$\begin{aligned} 0 &= u(r(su) + (su)'r)xv \\ &= ursuxv + usu'rxv \end{aligned}$$

Since *S* is an inverse semirings, we get

$$\begin{aligned} 0 &= ur(s + s' + s)uxv + usu'(r + r' + r)xv \\ &= ursuxv + ur(s + s')uxv + usu'rxv + usu'(r + r')xv \end{aligned}$$

Since *S* is an additiveiy inverse semiring, this gives

$$\begin{aligned} 0 &= ursuxv + u(s + s')ruxv + usu'rxv + us(r + r')u'xv \\ &= ursuxv + usruxv + us'ruxv + usu'rxv + usru'xv + usruxv \\ &= us(ru + u'r)xv + ursuxv + us'ruxv + usruxv + us'ruxv \\ &= us(ru + u'r)xv + ursuxv + us'ruxv \end{aligned}$$

$= us(ru + u'r)xv + u(rs + s'r)uxv$ for all $x \in J$ and $r, s \in S$

By using $uJv = 0$, we get $us(ru + u'r)xv = 0$ for all $x \in J$ and $r, s \in S$. And hence, we get, $uS(ru + u'r)xv = 0$. Since *S* is a prime inverse semiring, we have either $u = 0$ or $(ru + u'r)xv = 0$, for all $x \in J$ and $r \in S$.

If $(ru + u'r)xv = 0$, for all $x \in J$ and $r \in S$, then we have $ruxv + u'rxv = 0$, for all $x \in J$ and $r \in S$. By the hypothesis we get $ur'xv = 0$ for all $x \in J$ and $r \in S$, that is, $uSxv = 0$, for all $x \in J$. Again, since *S* is a prime inverse semiring, we get either $u = 0$ or $xv = 0$, for all $x \in J$. If $xv = 0$, for all $x \in J$, that is $Jv = 0$, then by Lemma 2.6, we get $v = 0$.

Lemma 2.8.

Let *S* be 2-tortion free and *J* is a nonzero Jordan Ideal of *S*, If *J* is commutative, then *J*⊆Z(S).

Proof

By Lemma 2.5, we have $2(rs + s'r)J \subseteq J$ for all $r, s \in S$. And since J is a commutative

Jordan ideal of S, we have $2(rs+s'r)xy+2y'(rs+s'r)x=0$, for all $x,y\in J$ and $r,s\in S$. Since *S is 2*-tortion free, we get

$$(rs+s'r)xy+y'(rs+s'r)x=0 \text{, for all } x,y\in J \text{ and } r,s\in S. \quad (1)$$

By Lemma 2.1, we get

$(rs+s'r)xy=y(rs+s'r)x$, for all $x,y\in J$ and $r,s\in S$.

Hence, by equation (1) we obtain

$y(rs+s'r)x+y'(rs+s'r)x=0$, for all $x,y\in J$ and $r,s\in S$,

Then we have

$((rs+s'r)y+y'(rs+s'r))x=0$, for all $x,y\in J$ and $r,s\in S$.

This gives

$((rs+s'r)y+y'(rs+s'r))J=0$, for all $y\in J$ and $r,s\in S$.

By Lemma 2.6, we get

$(rs+s'r)y+y'(rs+s'r)=0$, for all $y\in J$ and $r,s\in S$.

By replacing *s* by rs in the above equation, we get

$$\begin{aligned}0&=(rrs+rs'r)y+y'(rrs+rs'r)\\&=r(rs+s'r)y+y'r(rs+s'r)\end{aligned}$$

Since *J* is commutative, we get

$$ry(rs+s'r)+y'r(rs+s'r)=0 \text{ for all } y\in J \text{ and } r,s\in S.$$

Hence we get,

$$(ry+y'r)(rs+s'r)=0 \text{, for all } y\in J \text{ and } r,s\in S. \quad (2)$$

By further replacing *s* by *sy* in equation (2), we have

$(ry+y'r)(rsy+sy'r)=0$, for all $y\in J$ and $r,s\in S$.

Since *S* is an inverse semiring, we have

$$\begin{aligned}0&=(ry+y'r)(r(s+s'+s)y+sy'(r+r'+r))\\&=(ry+y'r)(rsy+r(s+s')y+sy'r+sy'(r+r'))\end{aligned}$$

Since *S* is additiveiy inverse semiring, we get

$$\begin{aligned}0&=(ry+y'r)(rsy+(s+s')ry+sy'r+s(r+r')y')\\&=(ry+y'r)(rsy+s'ry+sry+sy'r+sry+s'ry)\\&=(ry+y'r)(s(ry+y'r)+rsy+s'ry)\\&=(ry+y'r)(s(ry+y'r)+(rs+s'r)y)\\&=(ry+y'r)s(ry+y'r)+(ry+y'r)(rs+s'r)y\end{aligned}$$

By using equation (2) we get, $(ry+y'r)s(ry+y'r)=0$, for all $y\in J$ and $r,s\in S$. And hence we get $(ry+y'r)S(ry+y'r)=0$, for all $y\in J$ and $r\in S$.

Since *S* is a prime inverse semiring, we get

$ry+y'r=0$, for all $y\in J$ and $r\in S$. Hence, we get J⊆Z(S).

Proposition 2.9

Let *S* be 2-tortion free and *J* is a nonzero Jordan ideal and an inverse subsemiring on *S*. If $d(J)=0$, then $d=0$ or $J\subseteq Z(S)$.

Proof

By the assumption,

$$d(J)=0 \quad (3)$$

This yields

$$\begin{aligned}0&=d(xs+sx) \text{ for all } x\in J,s\in S\\&=d(x)\alpha(s)+\beta(x)d(s)+d(s)\alpha(x)+\beta(s)d(x)\end{aligned}$$

By using equation (3), we get

$$\beta(x)d(s)+d(s)\alpha(x)=0 \text{ for all } x\in J,s\in S. \quad (4)$$

By Lemma 2.1, we get

$$\beta(x)d(s)=d(s)\alpha(x)' \text{ for all } x\in J,s\in S. \quad (5)$$

By replacing *s* by *rs*, where $r\in S$, in equation (4), we get

$$0 = \beta(x)d(rs) + d(rs)\alpha(x)$$
$$= \beta(x)d(r)\alpha(s) + \beta(x)\beta(r)d(s) + d(r)\alpha(s)\alpha(x) + \beta(r)d(s)\alpha(x)$$

By using equation (5), we get

$$d(r)\alpha(x)'\alpha(s) + \beta(x)\beta(r)d(s) + d(r)\alpha(s)\alpha(x) + \beta(r)'\beta(x)d(s) = 0$$

Hence,

$$d(r)\big(\alpha(s)\alpha(x) + \alpha(x)'\alpha(s)\big) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)d(s) = 0 \tag{6}$$

By replacing *s* by *sy*, $s \in S$ in equation (6), we get

$$0 = d(r)\big(\alpha(sy)\alpha(x) + \alpha(x)'\alpha(sy)\big) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)d(sy)$$
$$= d(r)\big(\alpha(s)\alpha(y)\alpha(x) + \alpha(x)'\alpha(s)\alpha(y)\big) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)\big(d(s)\alpha(y) + \beta(s)d(y)\big)$$

By using equations (3) and (5), we get

$$d(r)\big(\alpha(s)\alpha(y)\alpha(x) + \alpha(x)'\alpha(s)\alpha(y)\big) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)d(s)\alpha(y) = 0$$

Since *S* is an inverse semiring, then we have

$$d(r)\alpha(s)\alpha(y + y' + y)\alpha(x) + d(r)\alpha(x)'\alpha(s + s' + s)\alpha(y) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)d(s)\alpha(y) = 0$$

That is,

$$d(r)\alpha(s)\alpha(y)\alpha(x) + d(r)\alpha(s)\alpha(y' + y)\alpha(x) + d(r)\alpha(x)'\alpha(s)\alpha(y) + d(r)\alpha(x)'\alpha(s' + s)\alpha(y) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)d(s)\alpha(y) = 0$$

By adding $d(r)\alpha(s)\alpha(x)'\alpha(y) + d(r)\alpha(s)\alpha(x)\alpha(y)$ to the above equation, on the two sides, we get

$$d(r)\alpha(s)\alpha(y)\alpha(x) + d(r)\alpha(s)\alpha(x)'\alpha(y) + d(r)\alpha(s)\alpha(y' + y)\alpha(x) + d(r)\alpha(s)\alpha(x)\alpha(y) + d(r)\alpha(x)'\alpha(s)\alpha(y) + d(r)\alpha(x)'\alpha(s' + s)\alpha(y) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)d(s)\alpha(y) = d(r)\alpha(s)\alpha(x)\alpha(y) + d(r)\alpha(s)\alpha(x)'\alpha(y)$$

Since *S* is an additively inverse semiring, we get

$d(r)\alpha(s)(\alpha(y)\alpha(x) + \alpha(x)'\alpha(y)) + d(r)(\alpha(s)\alpha(x) + \alpha(x)'\alpha(s))\alpha(y) + \big(\beta(x)\beta(r) + \beta(r)'\beta(x)\big)d(s)\alpha(y) + d(r)\alpha(s)\alpha(x)\alpha(y' + y) + d(r)\alpha(s' + s)\alpha(x)'\alpha(y) = d(r)\alpha(s)\alpha(x)\alpha(y) + d(r)\alpha(s)'\alpha(x)\alpha(y)$ for all $x, y \in J$ and $r, s \in S$

By using equation (6), we get

$$d(r)\alpha(s)(\alpha(y)\alpha(x) + d(r)\alpha(s)\alpha(x)'\alpha(y)) + d(r)\alpha(s)\alpha(x)\alpha(y) + d(r)\alpha(s)\alpha(x)'\alpha(y) + d(r)\alpha(s)\alpha(x)\alpha(y) + d(r)\alpha(s)\alpha(x)'\alpha(y) = d(r)\alpha(s)\alpha(x)'\alpha(y) + d(r)\alpha(s)\alpha(x)\alpha(y)$$

Since *S* is an inverse semiring, and using the canccelative low, we get

$d(r)\alpha(s)(\alpha(y)\alpha(x) + \alpha(x)'\alpha(y)) = 0$ for all $x, y \in J$ and $r, s \in S$.

And hence,

$\alpha^{-1}(d(r))S(yx + x'y) = 0$ for all $x, y \in J$ and $r \in S$.

Since *S* is a prime inverse semiring and *α* is an automorphism on *S*, then we have either $d(r) = 0$ or $yx + x'y = 0$ for all $x, y \in J$ and $r \in S$. If $yx + x'y = 0$ for all $x, y \in J$, then it follows that *J* is commutative . By using Lemma 2.8, we get $J \subseteq Z(S)$.

3.MAIN RUSELTS

Theorem 3.1

Let *I* be a non- zero left ideal on *S*, which is a semiprime as an inverse semiring. If $d(I)u = 0$, ($ud(I) = 0$), then $u = 0$, for all $u \in S$.

Proof

$$d(x)u = 0 \text{ , for all } x \in I,\ u \in S. \tag{7}$$

By replacing *x* by *sx*, $s \in S$ in equation (7), we have

$$0 = d(sx)u = \big(d(s)\alpha(x) + \beta(s)d(x)\big)u$$
$$= d(s)\alpha(x)u + \beta(s)d(x)u$$

By using equation (7), we get

$$d(s)\alpha(x)u = 0, \text{ for all } x \in I,\ u, s \in S \quad (8)$$

By replacing x by rx, $r \in S$ in equation (7), we get

$$0 = d(s)\alpha(rx)u$$
$$= d(s)\alpha(r)\alpha(x)u$$

Therefore

$d(s)S\alpha(x)u = 0$ for all $x \in I,\ u, s \in S$.

Since S is a prime inverse semiring, we get either $d(s) = 0$ or $\alpha(x)u = 0$ for all $x \in I$, $u, s \in S$. Since d is a nonzero (α,β)- derivation on S, then we have $\alpha(x)\, u = 0$ for all $x \in I$. Therefore , $I\,\alpha^{-1}(u) = 0$, for all $u \in S$. By Lemma 2.3 and since α is an automorphism on S, we get $u = 0$.

If

$$u\, d(I) = 0 \quad (9)$$

$0 = u\, d(x\, y) = u\, (d(x)\, \alpha(y) + \beta(x)\, d(y))$

$= u\, d(x)\, \alpha(y) + u\, \beta(x)\, d(y)$

By using equation (9), we get $u\,\beta(x)\, d(y) = 0$, for all $x, y \in I, u \in S$. Therefore,
$\beta^{-1}(u)\ I\,\beta^{-1}(d(y)) = 0$, for all $y \in I, u \in S$. Since I is a left ideal of S , we get $\beta^{-1}(u)\, S\, I\, \beta^{-1}(d(y)) = 0$, for all $y \in I, u \in S$. Since S is a prime inverse semiring,
we get either $\beta^{-1}(u) = 0$ or $I\,\beta^{-1}(d(y)) = 0$, for all $y \in I, u \in S$.
If $\beta^{-1}(u) = 0$, for all $u \in S$, then since β is an automorphism on S, then we have $u = 0$.
If $I\,\beta^{-1}(d(y)) = 0$, for all $y \in I$, then by Lemma 2.3 and since β is an automorphism on S, we get $d(I) = 0$. And then by Lemma2.4, we have $d = 0$ on S. But this is a contradiction, since d is a nonzero (α,β)- derivation on S. This yields that $u = 0$.

Theorem 3.2

Let S be a Cancellative prime inverse semiring. If d acts as a homomorphism on I, then $d = 0$ on S.

Proof

Since d acts as a homomorphism on I, then we have

$$d\,(xy) = d(x)d(y) \text{ for all } x, y \in I. \quad (10)$$

And since d is a (α,β)- derivation on S, we get

$$d(xy) = d(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in I. \quad (11)$$

By replacing y by yz, $z \in I$ in equation (10), we get

$$d(xyz) = d(xy)d(z)$$
$$= (d(x)\alpha(y) + \beta(x)d(y))d(z)$$
$$= d(x)\alpha(y)d(z) + \beta(x)d(y)d(z) \text{ for all } x, y, z \in I. \quad (12)$$

Again, by replacing y by yz , $z \in I$ in equation (11), we get

$$d(xyz) = d(x)\alpha(yz) + \beta(x)d(yz)$$
$$= d(x)\alpha(y)\alpha(z) + \beta(x)d(y)d(z) \text{ for all } x, y, z \in I. \quad (13)$$

By the equivalence between equations (12) and (13), we get

$d(x)\alpha(y)d(z) = d(x)\alpha(y)\alpha(z)$ for all $x, y, z \in I$.

By Lemma 2.2, we get

$d(x)\alpha(y)d(z) + d(x)\alpha(y)\alpha(z)' = 0$ for all $x, y, z \in I$.

Therefore

$d(x)\alpha(y)(d(z) + \alpha(z)') = 0$ for all $x, y, z \in I$.

And hence

$\alpha^{-1}d(x))\ I\ \alpha^{-1}(d(z) + \alpha(z)') = 0$, for all $x, z \in I$.

Since I is left ideal on S, we get

$\alpha^{-1}d(x))\, SI\alpha^{-1}(\, d(z) + \alpha(z)') = 0$, for all $x\,,\ z\ \in I$.
And since *S* is a prime inverse semiring, we get
either $\alpha^{-1}(d(x)) = 0$ or $I\alpha^{-1}(d(z) + \alpha(z)') = 0$, for all $x\,, z\ \in I$.
If $\alpha^{-1}(d(x)) = 0$, for all $x \in I$. Since α is an automorphism on *S*, then by Lemma 2.4, we get *d* = 0 on *S*.
If $I\alpha^{-1}(d(z) + \alpha(z)') = 0$, for all $z\ \in I$, then by Lemma 2.3, we obtain
$\alpha^{-1}(d(z) + \alpha(z)') = 0$, for all $z\ \in I$. Since α is an automorphism on *S*, we have $(z) + \alpha(z)' = 0$, for all $z\ \in I$. By Lemma 2.1, we get

$$d(z) = \alpha(z) \ \text{, for all } z\ \in I\,. \tag{14}$$

By replacing *z* by *zy* in equation (14), we get

$$d(zy) = \alpha(zy)$$

$d(z)\alpha(y) + \beta(z)d(y) = \alpha(z)\alpha(y)$, for all $y\,, z\ \in I$. By using equation (14) with the Cancellative low, we get $\beta(z)d(y) = 0$, for all $y\,, z\ \in I$.
Therefore, $\beta^{-1}((d(y)) = 0$, for all $y \in I$. Since β is an automorphism on *S* and by Lemma 2.3, we get $d(y) = 0$, for all $y \in I$. By Lemma 2.4, we get *d* = 0 on *S* .

Theorem 3.3

Let *I* be a nonzero left ideal of *S*m which is a semiprime as inverse semiring and *d* : S→ S is a (*α,β*)- derivation on *S*, where α , β are automorphisms on *S*. If *d* acts as an anti-homomorphism on *I*, then *d* = 0 on *S*.

Proof

Since *d* acts as an anti- homomorphism on *I*, then we have

$$d(xy) = d(y)d(x) \quad \text{for all } x, y\ \in I. \tag{15}$$

And since *d* is a (*α,β*)- derivation on *S*, we get

$$d(xy) = d(x)\alpha(y) + \beta(x)d(y) \ \text{, for all } x, y\ \in I. \tag{16}$$

By replacing *y* by *xy* in equation (15), we get

$$\begin{aligned} d(x(xy)) &= d(xy)d(x) \\ &= (d(x)\alpha(y) + \beta(x)d(y))d(x) \\ &= d(x)\alpha(y)d(x) + \beta(x)d(y)d(x) \quad \text{for all } x, y\ \in I. \end{aligned} \tag{17}$$

And by replacing *y* by *xy* in equation (16), we get

$$\begin{aligned} d(x(xy)) &= d(x)\alpha(xy) + \beta(x)d(xy) \\ &= d(x)\alpha(x)\alpha(y) + \beta(x)d(y)d(x) \ \text{, for all } x, y\ \in I. \end{aligned} \tag{18}$$

By the equivalence of equations (17) and (18), we get

$$d(x)\alpha(y)d(x) = d(x)\alpha(x)\alpha(y) \quad \text{, for all } x, y\ \in I. \tag{19}$$

By replacing y by yz , $z \in I$ in equation (19), we get
$d(x)\alpha(y)\alpha(z)d(x) = d(x)\alpha(x)\alpha(y)\alpha(z)$, for all $x, y, z \in I$.
By using equation (19), we get
$d(x)\alpha(y)\alpha(z)d(x) = d(x)\alpha(y)d(x)\alpha(z)$, for all $x, y, z \in I$.
By using Lemma2.2, we get
$d(x)\alpha(y)\alpha(z)d(x) + d(x)\alpha(y)d(x)'\alpha(z) = 0$, for all $x, y, z \in I$.
$d(x)\alpha(y)(\alpha(z)d(x) + d(x)'\alpha(z)) = 0$, for all $x, y, z \in I$.
$\alpha^{-1}(d(x))I\alpha^{-1}(\alpha(z)d(x) + d(x)'\alpha(z)) = 0$, for all $x, z\ \in I$.
Since *I* is left ideal on *S* and S is a prime inverse semiring, we get either $d(x)) = 0$ or $I\ (\alpha(z)d(x) + d(x)'\alpha(z)) = 0$, for all $x\,, z\ \in I$.
If $d(x) = 0$, for all $x \in I$, it gives $d(I) = 0$. By Lemma 2.4, we get *d* = 0 on *S*.
If

$$\alpha(z)d(x) + d(x)'\alpha(z) = 0 \ \text{, for all } x\,, z\ \in I \tag{20}$$

By Lemma 2.1, we get $(z)d(x) = d(x)\alpha(z)$, for all $x\,, z\ \in I$.
By replacing *z* by *rz* , $r \in S$ in equation (20) , we have

$$\begin{aligned} 0 &= \alpha(rz)d(x) + d(x)'\alpha(rz) \\ &= \alpha(r)\alpha(z)d(x) + d(x)'\alpha(r)\alpha(z) \end{aligned}$$

$$= \alpha(r)d(x)\alpha(z) + d(x)'\alpha(r)\alpha(z)$$
$$= (\alpha(r)d(x) + d(x)'\alpha(r))\alpha(z)$$
$= (\alpha(r)d(x) + d(x)'\alpha(r))\alpha(I)$, for all $x \in I$, $r \in S$.
Therefore
$\alpha^{-1}\big(\alpha(r)d(x) + d(x)'\alpha(r)\big)I = 0$ for all $x \in I$, $r \in S$.
Since *I* is a nonzero left ideal on *S* and S is a prime inverse semiring, we get
which forces *d* to be a homomorphism of *S*. It follows that *d* = 0 on *S*, by Theorem 3. 2.

Theorem3.4.

Let *S* be 2-tortion free, *J* is a nonzero Jordan ideal and a sub inverse semiring of *S*, and $d: S \to S$ is a (*α, β*)-derivation on *S*, where *α, β* are automorphisms on *S*. If d acts as a homomorphism on *J*, then either $d = 0$ or $J \subseteq Z(S)$.

Proof

Assume that $J \nsubseteq Z(S)$.
If *d* acts as a homomorphism on *J*, then we have
$$d(xy) = d(x)d(y) \text{ for all } x, y \in J. \quad (21)$$
And since *d* is a (*α, β*)-derivation on *S*, then we have
$$d(xy) = d(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in J. \quad (22)$$
By the equivalence of equations (21), (22), we get
$$d(x)d(y) = d(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in J. \quad (23)$$
Now, by replacing *y* by *yb*, $b \in J$ in equation (23), we get
$$d(x)d(yb) = d(x)\alpha(yb) + \beta(x)d(yb)$$
$$d(x)d(y)\alpha(b) + d(x)\beta(y)d(b) = d(x)\alpha(y)\alpha(b) + \beta(x)d(y)\alpha(b) + \beta(x)\beta(y)\, d(b)$$
$$= \big(d(x)\alpha(y) + \beta(x)d(y)\big)\alpha(b) + \beta(x)\beta(y)\, d(b)$$
By using equation (23) and the cancellative low, we get
$d(x)\beta(y)d(b) = \beta(x)\beta(y)\, d(b)$ for all $x, y, b \in J$.
By Lemma 2.1, we get
$0 = d(x)\beta(y)d(b) + \beta(x)'\beta(y)\, d(b)$
$= (d(x) + \beta(x)')\beta(y))\, d(b)$ for all $x, y, b \in J$.
And hence, $\beta^{-1}(d(x) + \beta(x)')J\beta^{-1}(d(b) = 0$ for all $x, b \in J$. By Lemma2.7, we get either $d(b) = 0$ or $d(x) + \beta(x)' = 0$ for all $x, b \in J$. If $d(b) = 0$ for all $b \in J$, then by Proposition 2.9, we get $d = 0$ on *S*.
If $d(x) + \beta(x)' = 0$ for all $x \in J$, then by Lemma2.1, we get
$$d(x) = \beta(x) \text{ for all } x \in J. \quad (24)$$
Using equation (23) in equation (24), we obtain
$$d(x)\alpha(y) = 0 \text{ for all } x, y \in J. \quad (25)$$
Now, by replacing *y* by *yb*, we get
$d(x)\alpha(y)\alpha(b) = 0$ for all $x, y, b \in J$. That is, $\alpha^{-1}\big(d(x)\big)Jb = 0$ for all $x, b \in J$. By Lemma2.7, we get either $d(x) = 0$ or $b = 0$ for all $x, b \in J$. But *J* is a nonzero Jordan ideal on *S*, hence we get $d(x) = 0$ for all $x \in J$. By Proposition 2.9, we get $d = 0$ on *S*.

Theorem 3.5

Let *S* be 2-tortion free, *J* is a nonzero Jordan ideal and a sub inverse semiring of *S*, and $d: S \to S$ is a (*α, α*)-derivation on *S*, where *α* is an automorphism on *S*. If *d* acts as an anti-homomorphism on *J*, then either $d = 0$ or $J \sqsubseteq Z(S)$.

Proof: Suppose that $J \nsubseteq Z(S)$.
Since *d* acts as an anti- homomorphism on *J*, then we have
$$d(xy) = d(y)d(x) \text{ for all } x, y \in J. \quad (26)$$
And since d is a (*α, α*)-derivation on *S***,** we get
$$d(xy) = d(x)\alpha(y) + \alpha(x)d(y) \text{ for all } x, y \in J \quad (27)$$
By the equivalence of equations (26) and (27), we get

$$d(y)d(x) = d(x)\alpha(y) + \alpha(x)d(y) \text{ for all } x, y \in J. \quad (28)$$

Now, by replacing *x* by *xy* in equation (28), we get

$d(y)d(xy) = d(xy)\alpha(y) + \alpha(xy)d(y)$ for all $x, y \in J$.

$d(y)d(x)\alpha(y) + d(y)\alpha(x)d(y) = (d(x)\alpha(y) + \alpha(x)d(y))\alpha(y) + \alpha(x)\alpha(y)d(y)$

By using equation (28) with the cancellation low, we get

$$d(y)\alpha(x)d(y) = \alpha(x)\alpha(y)d(y) \text{ for all } x, y \in J. \quad (29)$$

Now, by replacing *x* by *bx*, in equation (29), we get

$d(y)\alpha(b)\alpha(x)d(y) = \alpha(b)\alpha(x)\alpha(y)d(y)$ for all $x, y, b \in J$.

Using equation (29) in the above equation, we get

$d(y)\alpha(b)\alpha(x)d(y) = \alpha(b)d(y)\alpha(x)d(y)$ for all $x, y, b \in J$.

By Lemma2.1, we get

$d(y)\alpha(b)\alpha(x)d(y) + \alpha(b)'d(y)\alpha(x)d(y) = 0$ for all $x, y, b \in J$.

And hence

$(d(y)\alpha(b) + \alpha(b)'d(y))\alpha(x)d(y) = 0$ for all $x, y, b \in J$.

Then

$\alpha^{-1}(d(y)\alpha(b) + \alpha(b)'d(y))J\alpha^{-1}(d(y)) = 0$ for all $x, y, b \in J$.

Since α is an automorphism on *S* and by Lemma 2.7, then we have either $d(y)\alpha(b) + \alpha(b)'d(y) = 0$ or $d(y) = 0$ for all $y, b \in J$. If $d(y) = 0$ for all $y \in J$ then by Proposition 2.9, we get $d = 0$ on *S*.

If

$$d(y)\alpha(b) + \alpha(b)'d(y) = 0 \text{ for all } y, b \in J. \quad (30)$$

By Lemma 2.1, we get

$$d(y)\alpha(b) = \alpha(b)d(y) \text{ for all } y, b \in J. \quad (31)$$

Now, by replacing *y* by *yb*, $b \in J$ in equation (30), we get

$$0 = d(yb)\alpha(b) + \alpha(b)'d(yb)$$

$$= \big(d(y)\alpha(b) + \alpha(y)d(b)\big)\alpha(b) + \alpha(b)'(d(y)\alpha(b) + \alpha(y)d(b))$$

$$= d(y)\alpha(b)\alpha(b) + \alpha(y)d(b)\alpha(b) + \alpha(b)'d(y)\alpha(b) + \alpha(b)'\alpha(y)d(b)$$

$$= d(y)\alpha(b)\alpha(b) + \alpha(y + y' + y)d(b)\alpha(b) + \alpha(b)'d(y)\alpha(b + b' + b) + \alpha(b)'\alpha(y)d(b)$$

$$= d(y)\alpha(b)\alpha(b) + \alpha(y)d(b)\alpha(b) + \alpha(y' + y)d(b)\alpha(b) + \alpha(b)'d(y)\alpha(b) + \alpha(b)'d(y)\alpha(b' + b) + \alpha(b)'\alpha(y)d(b)$$

Since *S* is an additively inverse semiring, we get

$$0 = d(y)\alpha(b)\alpha(b) + \alpha(y)d(b)\alpha(b) + d(b)\alpha(y' + y)\alpha(b) + \alpha(b)'d(y)\alpha(b) + \alpha(b)'\alpha(b' + b)d(y) + \alpha(b)'\alpha(y)d(b)$$

$$= d(y)\alpha(b)\alpha(b) + \alpha(y)d(b)\alpha(b) + d(b)\alpha(y')\alpha(b) + d(b)\alpha(y)\alpha(b) + \alpha(b)'d(y)\alpha(b) + \alpha(b)'\alpha(b)d(y) + \alpha(b)\alpha(b)d(y) + \alpha(b)'\alpha(y)d(b)$$

And by using equation (31), we get

$$\alpha(b)\big(d(y)\alpha(b) + \alpha(b)'d(y)\big) + \big(d(y)\alpha(b) + \alpha(b)'d(y)\big)\alpha(b) + \alpha(y)\big(d(b)\alpha(b) + \alpha(b)'d(b)\big) + \big(\alpha(y)\alpha(b) + \alpha(b)'\alpha(y)\big)d(b) = 0$$

By using equation (30), we get

$$\alpha(y)\big(d(b)\alpha(b) + \alpha(b)'d(b)\big) + \big(\alpha(y)\alpha(b) + \alpha(b)'\alpha(y)\big)d(b) = 0 . \quad (32)$$

Now, by replacing *y* by *wy*, $w \in J$ in equation (32), we get

$$0 = \alpha(wy)\big(d(b)\alpha(b) + \alpha(b)'d(b)\big) + \big(\alpha(wy)\alpha(b) + \alpha(b)'\alpha(wy)\big)d(b)$$

$$= \alpha(w)\alpha(y)\big(d(b)\alpha(b) + \alpha(b)'d(b)\big) + \big(\alpha(wy)\alpha(b) + \alpha(b)'\alpha(wy)\big)d(b)$$

$$= \alpha(w)\alpha(y)d(b)\alpha(b) + \alpha(w)\alpha(y)\alpha(b)'d(b) + \alpha(w)\alpha(y)\alpha(b)d(b) + \alpha(b)'\alpha(w)\alpha(y)d(b)$$

$$= \alpha(w)\alpha(y)d(b)\alpha(b) + \alpha(w)\alpha(y + y' + y)\alpha(b)'d(b) + \alpha(w)\alpha(y)\alpha(b + b' + b)d(b) + \alpha(b)'\alpha(w)\alpha(y)d(b)$$

$$= \alpha(w)\alpha(y)d(b)\alpha(b) + \alpha(w)\alpha(y)\alpha(b)'d(b) + \alpha(w)\alpha(y' + y)\alpha(b)'d(b) + \alpha(w)\alpha(y)\alpha(b)d(b) + \alpha(w)\alpha(y)\alpha(b' + b)d(b) + \alpha(b)'\alpha(w)\alpha(y)d(b)$$

Since S is an additively inverse semiring, we get

$$0 = \alpha(w)\alpha(y)d(b)\alpha(b) + \alpha(w)\alpha(y)\alpha(b)'d(b) + \alpha(w)\alpha(b)'\alpha(y' + y)d(b) + \alpha(w)\alpha(y)\alpha(b)d(b) + \alpha(w)\alpha(b' + b)\alpha(y)d(b) + \alpha(b)'\alpha(w)\alpha(y)d(b)$$
$$= \alpha(w)\alpha(y)d(b)\alpha(b) + \alpha(w)\alpha(y)\alpha(b)'d(b) + \alpha(w)\alpha(b)\alpha(y)d(b) + \alpha(w)\alpha(b)'\alpha(y)d(b) + \alpha(w)\alpha(y)\alpha(b)d(b) + \alpha(w)\alpha(b)\alpha(y)d(b) + \alpha(w)\alpha(b')\alpha(y)d(b) + \alpha(b)'\alpha(w)\alpha(y)d(b)$$
$$= \alpha(w)\alpha(y)\big(d(b)\alpha(b) + \alpha(b)'d(b)\big) + \alpha(w)\big(\alpha(y)\alpha(b) + \alpha(b)'\alpha(y)\big)d(b)) + \alpha(w)\alpha(b)\alpha(y)d(b) + \alpha(b)'\alpha(w)\alpha(y)d(b)$$

Hence,

$\alpha(w)(\alpha(y)\big(d(b)\alpha(b) + \alpha(b)'d(b)\big) + \big(\alpha(y)\alpha(b) + \alpha(b)'\alpha(y)\big)d(b)) + (\alpha(w)\alpha(b) + \alpha(b)'\alpha(w))\alpha(y)d(b) = 0$, for all $y, b, w \in J$.

By using equation (32), we get

$\big(\alpha(w)\alpha(b) + \alpha(b)'\alpha(w)\big)\alpha(y)d(b) = 0$, for all $y, b, w \in J$.

Hence,

$(wb + b'w)y\alpha^{-1}\big(d(b)\big) = 0$, for all $y, b, w \in J$.

Therefore, $(wb + b'w)J\alpha^{-1}\big(d(b)\big) = 0$, for all $b, w \in J$. By Lemma 2.7, we get either $wb + b'w = 0$ or $d(b) = 0$, for all $b, w \in J$.

If $wb + b'w = 0$, for all $b, w \in J$, it follows that J is commutative. By Lemma 2.8, we get $J \subseteq Z(S)$, which is a contradiction. If $d(b) = 0$, for all $b \in J$, then by Proposition 2.9, we get $d = 0$ on S.

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