ISSN: 0067-2904

The Complex Dynamics in a Food Chain Involving Different Functional Responses

Safaa Jawad Ali[1*], Adwea Naji Atewi[1], Raid Kamel Naji[2]
[1]*Institute of Medical Technology/Al-Mansour, Middle Technical University, Baghdad, Iraq,*
[2]*Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq,*

Received: 10/5/2021 Accepted: 21/8/2021 Published: 30/4/2022

Abstract

A food chain model in which the top predator growing logistically has been proposed and studied. Two types of Holling's functional responses type IV and type II have been used in the first trophic level and second trophic level respectively, in addition to Leslie-Gower in the third level. The properties of the solution are discussed. Since the boundary dynamics are affecting the dynamical behavior of the whole dynamical system, the linearization technique is used to study the stability of the subsystem of the proposed model. The persistence conditions of the obtained subsystem of the food chain are established. Finally, the model is simulated numerically to understand the global dynamics of the food chain under study.

Keywords: Chaos, Stability, Holling type IV, Leslie-Gower, Persistence.

الديناميكيات المعقدة في السلسلة الغذائية التي تنطوي على استجابات وظيفية مختلفة

صفاء جواد علي[1*] ، عدوية ناجي عطيوي [1] ، رائد كامل ناجي[2]
[1]لمعهد الطبي التقني/ المنصور، الجامعة التقنية الوسطى، بغداد، العراق
[2] قسم الرياضيات، كلية العلوم، جامعة بغداد، العراق

الخلاصة

تم اقتراح ودراسة نموذج السلسلة الغذائية الذي ينمو فيه المفترس العلوي لوجستيًا. تم استخدام نوعين من الاستجابات الوظيفية لهولينج من النوع الرابع والنوع الثاني في المستوى الغذائي الأول والمستوى الغذائي الثاني على التوالي بالاضافة الى ليسلي–جاور في المستوى الغذائي الثالث. تمت مناقشة خصائص الحل. نظرًا لأن ديناميكيات المستويات الحدودية في الفضاء الثلاثي تؤثر على السلوك الديناميكي للنظام الديناميكي بأكملها لذا ستخدمت التقنية الخطية لدراسة استقرارية النظم الجزئية من النموذج المقترح. تم تحديد شروط أصرار النظم الجزئية من السلسلة الغذائية. أخيرًا ، تمت محاكاة النموذج عدديًا لفهم الديناميكية الكلية للسلسلة الغذائية قيد الدراسة.

1. Introduction

In ecological modeling, many researchers and scientists supported this field of study with brand new ideas. May [1] with Hasting and Powell [2] put a new base to propose and describe the new generation of complex ecological models. The complex behaviors of various

*Email: safajawad1970@gmail.com

ecological models involving different factors namely, predation, switching, and competition, is the most challenging task in such studies and thus it receives good attention from many scientists [3-10]. The functional response defined by Lotka and Volterra [11] for a predator-prey model is linear and unbounded while studying the complexity in model ecosystems needs reasonable functional responses that should be nonlinear and bounded [12]. In 1959 Holling [13] used a type II functional response. Collings [14] proposed a new function and called it Holling type IV response. This response function describes a situation in which the predator's per capita rate of predation decreases at sufficiently high prey densities. Moreover, both Holling type IV [5-7] and Leslie-Gower [15,16] functional responses are relatively less studied in population ecology. In their experiments about the kinetics of phenol oxidation Sokol and Howell [17] suggested a simplified Holling type IV function and found that it is simpler and better than the original function of Holling type IV. In this paper, the three species food chain model proposed by Alaoui [16] is modified so that it contains three different types of functional responses.

2. The Mathematical Model

Consider a three-species food chain model consisting of the prey that denoted to their density at time t by x(t), the intermediate predator that denoted to their density at time t by y(t), and the top predator, which denoted to their density at time t by z(t). It is assumed that the intermediate predator preys at the lower level according to Holling type IV response, and the top predator preys upon the intermediate predator at the second level according to Holling type II and growing according to modified Leslie-Gower response. The dynamics of the above food chain model can be represented by the following

$$\begin{aligned}\frac{dx}{dt} &= ax - bx^2 - \frac{w_0 xy}{h_1+x^2} = F_1(x,y,z),\\ \frac{dy}{dt} &= \frac{w_1 xy}{h_1+x^2} - dy - \frac{w_2 yz}{h_2+y} = F_2(x,y,z),\\ \frac{dz}{dt} &= c_3 z^2 - \frac{w_3 z^2}{h_3+y} = F_3(x,y,z),\end{aligned} \tag{1}$$

with $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. Obviously, system (1) is continuous and have continuous partial derivatives on the positive octant $R_+^3 = \{(x,y,z) \in \mathrm{x}(0) \geq 0, \mathrm{y}(0) \geq 0, \mathrm{z}(0) \geq 0\}$, and hence the solution of the system (1) exists and is unique. Here the positive constants a, b, d, h_j; $j = 0,1,2,3,$ and w_k; $k = 0,1,2,3,$ can be described as: a is the growth rate of the prey x, b represents the intraspecific competition of prey x, w_k's are the maximum values attainable by each per capita rate, h_1 measures the extent to which the environment provides protection to the prey x and y respectively, d represent the death rate of the intermediate predator. While h_2 is the value of y at which the per capita removal rate of y becomes $\frac{w_2}{2}$, the constant c_3 represents the growth rate of z by sexual reproduction, however h_3 represents the residual loss in z population due to serve scarcity of its favorite food y. Moreover, it is easy to verify that model (1) is uniformly bounded.

3. Analysis of the subsystem

To study the dynamical behavior of the model (1), it is important to study their subsystem in the $xy-$plane. Many characteristics of the model (1) (such as persistence) depend on the dynamical behavior of their subsystem in the $xy-$plane, see [17,19]. Now in the absence of the top predator z, the system (1) reduces to the following subsystem.

$$\begin{aligned}\frac{dx}{dt} &= ax - bx^2 - \frac{w_0 xy}{h_1+x^2},\\ \frac{dy}{dt} &= \frac{w_1 xy}{h_1+x^2} - dy.\end{aligned} \tag{2}$$

Now, it is easy to verify that the next condition shows that the system (2) is a persistence Kolmogorov system

$$\frac{w_1 x}{h_1+x^2} > d \tag{3}$$

In addition to the above it is observed that, the Kolomogrov model (2) has the following nonnegative equilibrium points. The equilibrium points $e_{10} = (0,0)$ and $e_{11} = (\frac{a}{b}, 0)$ always exist. However, the positive equilibrium point, say $e_{12} = (\bar{x}, \bar{y})$ in the $Int(R_+^2)$ of the $xy-$plane can be determined by solving the first two equations of the model (2), such that

$$dx^2 - w_1 x + dh_1 = 0, \tag{4}$$

which gives

$$\bar{x} = \frac{w_1}{2d} \mp \frac{\sqrt{{w_1}^2 - 4d^2 h_1}}{2d}, \tag{5a}$$

and then it is obtain that

$$\bar{y} = \frac{(a-b\bar{x})(h_1+\bar{x}^2)}{w_0}. \tag{5b}$$

Clearly, $\bar{y} > 0$ provided that $\bar{x} < \frac{a}{b}$. Therefore, there are many cases about the existence of the positive equilibrium point, these are given below.

Case 1. If $w_1^2 - 4d^2 h_1 < 0$, then there is no positive roots for Eq.(4). This implies that the specialist predator goes extinct too, and the system (2) have just two equilibrium points $e_{10} = (0,0)$ and $e_{11} = (\frac{a}{b}, 0)$.

Case 2. If $w_1^2 - 4d^2 h_1 = 0$, then there is one positive root for Eq.(4), that is given by $\bar{x} = \frac{w_1}{2d}$, and then substitute $\bar{x}$ in the first equation of the system (2) gives the positive equilibrium point, say $e_{12} = (\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{w_1}{2d} \; ; \; \bar{y} = \frac{(a-b\bar{x})(h_1+\bar{x}^2)}{w_0}. \tag{6a}$$

This exists provided that

$$\frac{w_1}{2d} < \frac{a}{b}. \tag{6b}$$

Case 3. If $w_1^2 - 4d^2 h_1 > 0$, then there are two solutions

$$\bar{x}_1 = \frac{w_1}{2d} + \frac{\sqrt{{w_1}^2 - 4d^2 h_1}}{2d}, \bar{y}_1 = \frac{(a-b\bar{x}_1)(h_1+\bar{x}_1^2)}{w_0}. \tag{7a}$$

$$\bar{x}_2 = \frac{w_1}{2d} - \frac{\sqrt{{w_1}^2 - 4d^2 h_1}}{2d}, \bar{y}_2 = \frac{(a-b\bar{x}_2)(h_1+\bar{x}_2^2)}{w_0} \tag{7b}$$

Therefore, if $\bar{x}_1 < \frac{a}{b}$ or $\bar{x}_2 < \frac{a}{b}$, then model (2) has two positive equilibrium points that are represented by $e_{13} = (\bar{x}_1, \bar{y}_1)$ and $e_{14} = (\bar{x}_2, \bar{y}_2)$.

In addition to that, the system (2) has a unique positive equilibrium if $\bar{x}_2 < \frac{a}{b} < \bar{x}_1$, so in addition to $e_{10} = (0,0)$ and $e_{11} = (\frac{a}{b}, 0)$, the model has three equilibrium points.

In this paper, it is assumed that the system (2) has at most three equilibrium points e_{10}, e_{11} and the positive equilibrium point $e_{12} = (\bar{x}, \bar{y})$. Further, the stability analysis of the Kolmogorov model (2) is carried out and according to the following Jacobian's matrices of e_{10}, e_{11} and e_{12}, respectively the following results are obtained:

$$V(e_{10}) = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}.$$

$$V(e_{11}) = \begin{bmatrix} -b & -\frac{w_1}{h_1+1} \\ 0 & \frac{w_1}{h_1+1} - d \end{bmatrix}.$$

$$V(e_{12}) = \begin{bmatrix} \bar{x}\left(-b + \frac{2\bar{x}(a-b\bar{x})}{h_1+\bar{x}^2}\right) & -\frac{w_0\bar{x}}{h_1+\bar{x}^2} \\ \frac{w_1\bar{y}(h_1-\bar{x}^2)}{(h_1+\bar{x}^2)^2} & 0 \end{bmatrix}.$$

Here the equilibrium points $e_{10} = (0,0)$ is a saddle point, while $e_{11} = \left(\frac{a}{b}, 0\right)$ is unstable saddle point if condition (3) holds, and it is a locally asymptotically stable if

$$\frac{w_1}{h_1+1} < d, \tag{8}$$

On the other hand, the positive planar equilibrium point $e_{12} = (\bar{x}, \bar{y})$ is a locally asymptotically stable in the $Int(R_+^2)$ of the $xy-$plane if the following condition holds

$$\frac{2\bar{x}(a-b\bar{x})}{h_1+\bar{x}^2} < b. \tag{9}$$

while e_{12} is unstable saddle if the opposite of condition (9) holds. Moreover, the global stability of $e_{12} = (\bar{x}, \bar{y})$ is discussed in the following theorem.

Theorem 1 If the unique positive planar equilibrium point $e_{12} = (\bar{x}, \bar{y})$ is a locally asymptotically stable in the $Int(R_+^2)$ of the $xy-$plane, then it is a globally asymptotically stable.

Proof. Let

$$K(x,y) = \frac{1}{xy}, \quad k_1(x,y) = x\left[a - bx - \frac{w_0 y}{h_1+x^2}\right],$$

and

$$k_2(x,y) = y\left[\frac{w_1 x}{h_1+x^2} - d\right].$$

Obviously, the function $K(x,y) > 0$ be C^1 in the $Int(R_+^2)$ of the $xy-$plane. Also we have that

$$\nabla(x,y) = \frac{\partial}{\partial x}(k_1, K) + \frac{\partial}{\partial y}(k_2, K) = \frac{1}{\bar{y}}\left[-b + \frac{2w_0\bar{x}\bar{y}}{(h_1+\bar{x}^2)^2}\right],$$

Hence

$$\nabla(x,y) = \frac{1}{\bar{y}}\left[-b + \frac{2\bar{x}(a-b\bar{x})}{h_1+\bar{x}^2}\right].$$

Clearly if condition (9) holds, $\nabla(x,y)$ does not change sign and is not identically zero in the $Int(R_+^2)$ of the $xy-$plane. Then by using Dulic-Bendixons criterion there is no closed curve in the $Int(R_+^2)$. Since the Kolmogorov model (2) has a unique equilibrium point e_{12} in the $Int(R_+^2)$ of the $xy-$plane, hence according to Poincare-Bendixon theorem $e_{12} = (\bar{x}, \bar{y})$ is a globally asymptotically stable in the $Int(R_+^2)$ of the $xy-$plane.

4. Persistence of the subsystem

In the next theorem the conditions of persistence of model (2) is established. Mathematically persistence of the model means that if all the variables are initially positive then the solution of the model does not have omega limit sets on the boundary planes for all the time.

Theorem 2 The model (2) is uniformly persistence provided that condition (3) holds.

Proof. Consider the following function $\sigma(x,y) = x^{s_1}y^{s_2}$, where s_1 and s_2 are undetermined positive constants. Obviously $\sigma(x,y)$ is a positive function in the $Int(R_+^2)$ and $\sigma(x,y) \to 0$ if $x \to 0$ or $y \to 0$. Now since

$$\psi(x,y) = \frac{\acute{\sigma}(x,y)}{\sigma(x,y)} = s_1\frac{\acute{x}}{x} + s_2\frac{\acute{y}}{y}.$$

Therefore

$$\psi(x,y) = \frac{\acute{\sigma}(x,y)}{\sigma(x,y)} = s_1\left[(a-bx) - \frac{w_0 y}{h_1+x^2}\right] + s_2\left[\frac{w_1 x}{h_1+x^2} - d\right].$$

Recall that, since $e_{12} = (\bar{x}, \bar{y})$ is globally asymptotically stable in the $Int(R_+^2)$ of the $xy-$plane. Therefore, there are no periodic orbits in this boundary plane. So to prove that σ is

persistence function, in the sense of Gard [20], and hence model (2) is uniform persists, it is enough to show that there are no omega limit sets on the boundary planes of R_+^2 or equivalently the following conditions should be satisfied:

$$\Psi(e_{10}) = s_1 a - s_2 d > 0\,,$$
$$\Psi(e_{11}) = s_2\left[\frac{w_1 x}{h_1+x^2} - d\right] > 0,$$

Note that by choosing $s_1 > 0$ sufficiently large value and keeping s_2 fixed at small positive value then $\Psi(e_{10}) > 0$ holds. Also, due to the Kolmogorov condition (3) the inequality $\Psi(e_{11}) > 0$ is satisfied for any positive value of s_2. Therefore σ represents persistence function and hence system (2) is uniformly persists.

Now, before go further, we have to mention that the function σ will be an extinction function, in the sense of Gard [18], if condition (3) violated, then the solution of model (2) approaches to an omega limit point on the boundary planes of the R_+^2 and hence model (2) does not persist.

5. Numerical exploration

The food chain model (1) is simulated numerically to study the global dynamics of it by using six order Runge-Kutta method. For the following set of fixed parameters values

$$a = 0.45, b = 0.075, h_1 = h_2 = 10, h_3 = 20, w_0 = 1, w_1 = 2,$$
$$w_2 = 0.405, w_4 = 1, d = 0.15, c_3 = 0.047 \qquad (10)$$

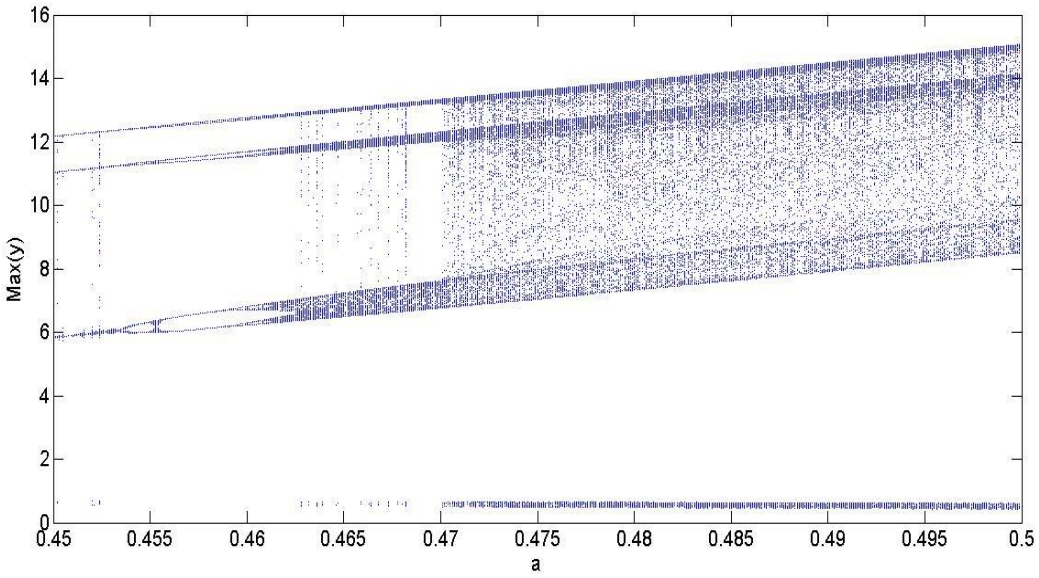

Figure 1-Bifurcation diagram of model (1) for data set (10) that shows the successive maxima of y as a function of $a \in (0.45,0.50)$ and $c_3 = 0.047$.

Bifurcation diagram and the typical 3D attractors of model (1) are plotted with their time series. Our target here is to investigate the behavior of model (1) depending upon the parameters a and c_3 with keeping other parameters of (10) fixed.

Now, the first case is by fixing $c_3 = 0.047$ and varying the value of a in the range $0.45 - 0.25$. It is observed for the value $a = 0.45$ the dynamics of model (1) with data (10) is chaotic as shown in Fig.1 and Fig.2. Decreasing the value of $a = 0.30$ change the behavior of model (1) to period-doubling as shown in Fig.3, and for the value $a = 0.25$ the model will be stable as shown in Fig.4.

The second case by fixing $a = 0.45$ and varying the value of c_3 in the range $0.041 - 0.049$. It is observed for data (10) that model (1) behavior is chaotic as it is blotted in Fig.2 and periodic with z approach to extinction as it is shown in Fig.5. Moreover, the stable case appears in model (1) for data (13) when $c_3 = 0.04$ and $a = 0.25$ with z approach to extinction as it is shown in Fig.6.

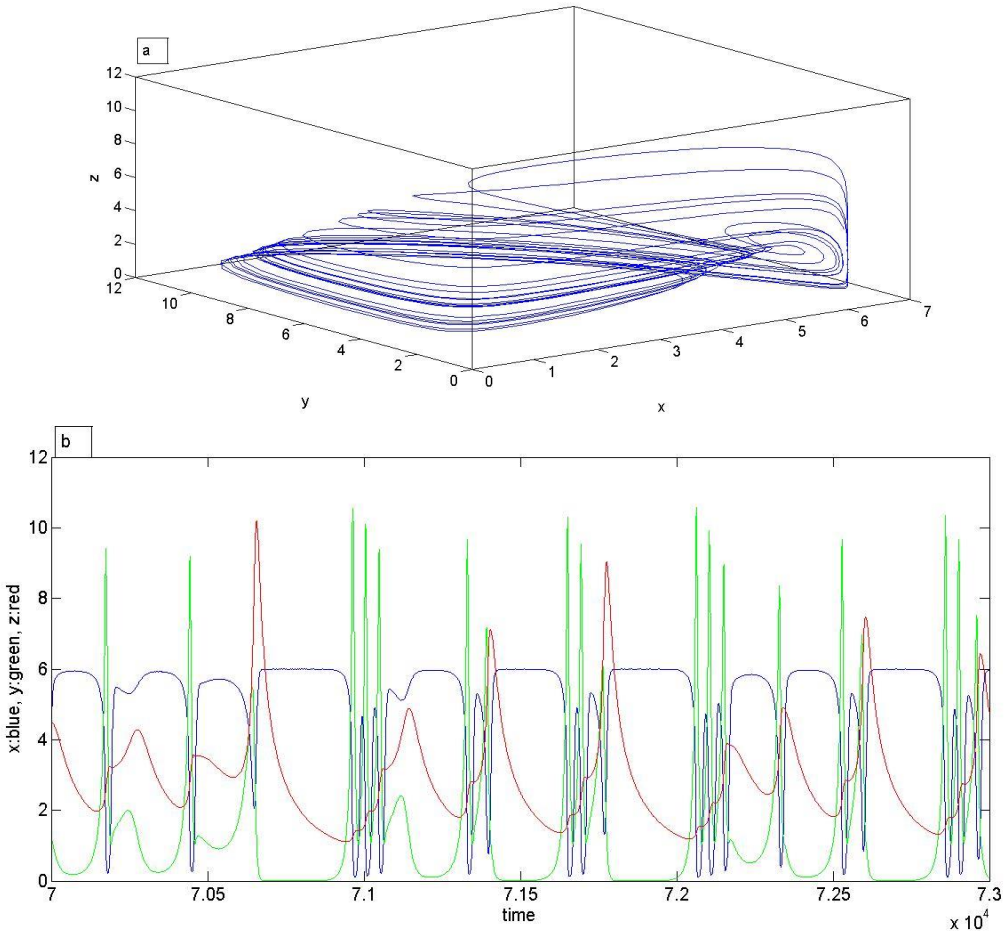

Figure 2-a) 3D chaotic attractor of model (1) for data set (10) with $a = 0.45$ and $c_3 = 0.047$. **b)** Time series of Fig.2a

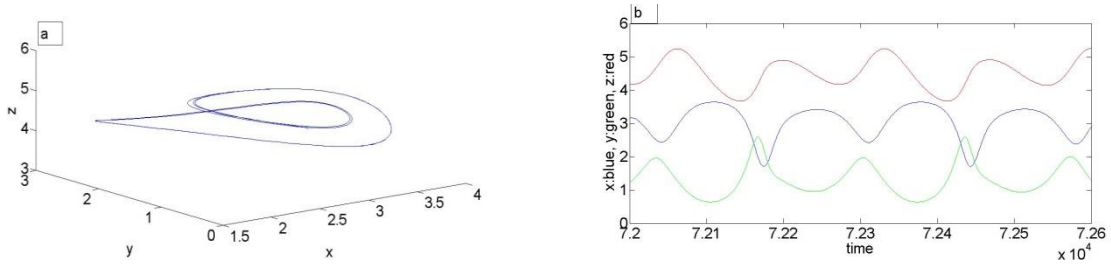

Figure 3- a) 3D of model (1) periodic attractor for data (10) with $a = 0.30$, **b)** Time series of Fig.3a

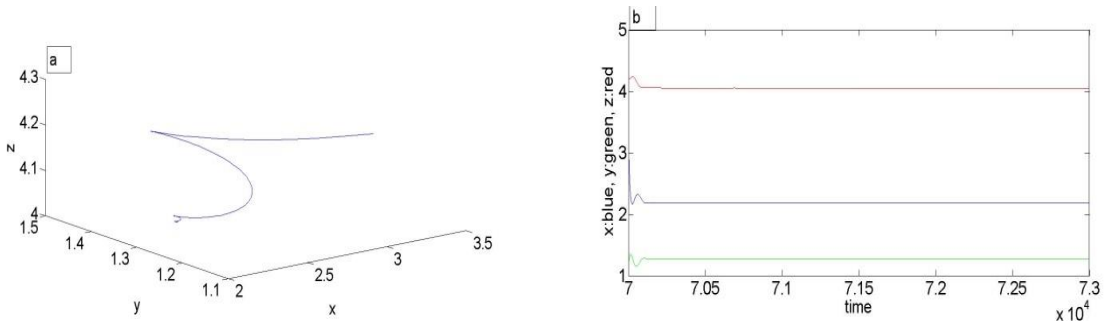

Figure 4-a) 3D of model (1) stable attractor for data (10) with $a = 0.25$, **b)** Time series of Fig.4a.

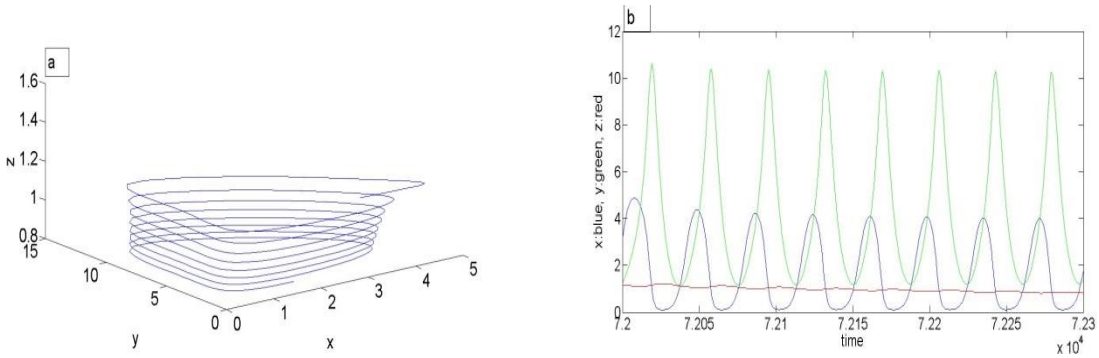

Figure 5- a) 3D of model (1) periodic attractor for data (10) with $a = 0.45,\ c_3 = 0.049,$ with z approach to extinction **b)** Time series of Fig.5a

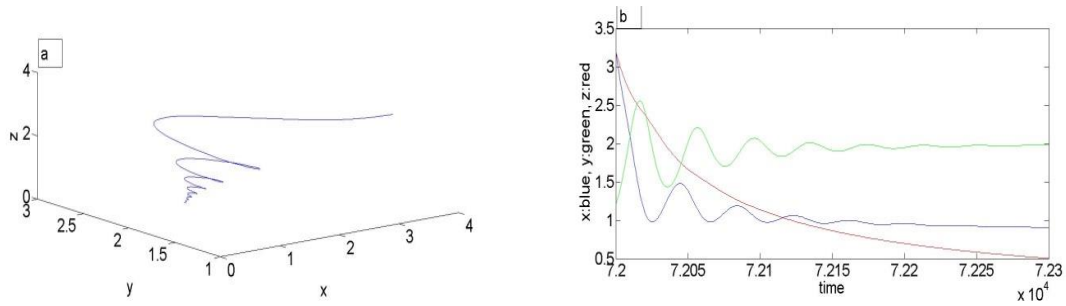

Figure 6- a) 3D of model (1) stable point for data (10) with $a = 0.25,\ c_3 = 0.040,$ with z approach to extinction **b)** Time series of Fig.6a

Conclusions

A food chain model with different functional responses including Holling type IV and type II in additional to Leslie-Gower is proposed and studied. In order to explain the dynamical behavior of the proposed food chain model (1) is local as well as global stability analyses are carried out for the subsystem. Persistence of the subsystem is discussed. Global stability for the food chain model (1) is analyzed numerically. According to our study above we obtained that the parameters a and c_3 are controlling parameters and they are responsible about the chaotic, periodic and asymptotic stable of Leslie-Gower food chain model (1) with simplified Holling type IV functional response.

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