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Solving Nonlinear Boundary Value Problem Arising of Natural Convection Porous Fin By Using the Haar Wavelet Collocation Method and Temimi and Ansari Method

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Abstract

In this article, the boundary value problem of convection propagation through the permeable fin in a natural convection environment is solved by the Haar wavelet collocation method (HWCM). We also compare the solutions with the application of a semi-analytical method , namely the Temimi and Ansari (TAM), that is characterized by accuracy and efficiency.The proposed method is also characterized by simplicity and efficiency. The possibility of applying the proposed method to many types of linear or nonlinear ordinary and partial differential equations.

Keywords: Boundary value problem, convection propagation, Haar Wavelet Collocation Method, Semi-Analytical Method.

حل مشكلة القيمة الحدودية غير الخطية الناشئة عن الحمل الحراري الطبيعي المسامي بواسطة طريقة تجميع المويجات هار و طريقة التميمي و الانصاري

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الخلاصة

في هذا البحث ، يتم حل مشكلة القيمة الحدودية لانتشار الحمل الحراري من خلال الزعنفة المنفذة في بيئة الحمل الحراري الطبيعي ، بواسطة طريقة Haar wavelet التجميعية (HWCM) ، كما نقارن الحلول بتطبيق أسلوب شبه تحليلي ألا وهو التميمي والأنصاري (TAM) الذي يتميز بالدقة والكفاءة ، كما تتميز الطريقة المقترحة بالبساطة والكفاءة .إمكانية تطبيق الطريقة المقترحة على أنواع عديدة من المعادلات التفاضلية العادية والجزئية الخطية وغير الخطية..

Introduction

Most of the phenomena are represented as non-linear equations, and there is difficulty in finding an exact solution to these problems, so researchers resort to finding an approximate solution analytically or numerically. Many efforts have been made to solve nonlinear differential equations by variety of methods such as Homotopy perturbation method (HPM)

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[1,2], semi-analytical method with Laplace transform [3], Adomian decomposition method (ADM), variation iteration method (VIM) and finite difference method (FDM) [4].
Many numerical and analytical methods have been developed such as finite-difference methods, weighted residuals methods, Adomian decomposition method and variational iteration method. The wavelet approach has lately increased in the domain of numerical approximation. Various types of wavelets and approximate functions have been used in the numerical solution of differential equations one of them the Haar wavelet method is used due to their beneficial properties such as ease of application, orthogonality and effectiveness. Chen and Hsiao [5] created an integration matrix based on the Haar wavelet. The Haar wavelet collocation method (HWCM) is widely used in finding an approximate solution due to its inertia and effectiveness as in [6], [7], and [8]. The method of the wavelets can be numerically considered binoculars in the signal's processing and images, the representation of the waveform basis that arises from a set of wavelet coefficients organized at different levels of accuracy. Each coefficient is associated with an accuracy level and a point in the time field. The Haar wavelet coefficients are got by converting the differential equation and its boundary conditions into a system of algebraic equations, which eventually solve the proposed problem in this paper with semi-analytically method namely the Temimi and Ansari method (TAM), which is used to solve linear and non-linear equations [9] and [10]. The wavelet coefficients were calculated by the numerical method was calculated using MATHEMATICA 11 SOFTWARE. This method is helpful to solve the approximate solutions of linear and nonlinear boundary value problems without large computational work.

Problem formulation

Take a straight, heat-conducting fin that depends on temperature and cross-sectional area A and $(A = wt)$, where the fin is attached to the base surface. The temperature at the base surface is Tb and the ambient fluid temperature is Ta, knowing that the fin tip is isolated. So the linear equation for energy balance is:

$$p(X) - p(X + \triangle X) = \dot{a} c_q T - T_\infty \tag{1}$$

Fluid flow rate can be written through porous materials as follows:

$$\dot{a} = q v_w \triangle XW \tag{2}$$

by using the Darcy's model, we get:

$$v_w = \frac{gk\beta}{v} T - T_\infty \tag{3}$$

By substituting equations (2) and (3) into equation (1), we have:

$$\frac{p(X) - p(X + \triangle X)}{\triangle X} = \frac{q c_q g k \beta W}{v} (T - T_\infty)^2 \tag{4}$$

When $\triangle X \longrightarrow 0$ equation (4) will be:

$$\frac{dp}{dX} = \frac{q c_q g k \beta W}{v} (T - T_\infty)^2 \tag{5}$$

From conductions Fourier Law, we get the following:

$$p = -\delta_{effec} A \frac{dT}{dX} \tag{6}$$

Where $-\delta_{effec}$ is the effective thermal conductivity of the porous fin given by $\delta_{effec} = \sigma\delta_f + (1 - \sigma)\delta_s$. Then by substituting equation (6) into equation (5) yields:

$$\frac{d^2T}{dX^2} - \frac{q c_q g k \beta W}{v} (T - T_\infty)^2 = 0 \tag{7}$$

Then by applying the equation of energy balance at steady-state conduction, where $x = X/L \; and$ Where L represents the dimensions of the fin then we have:

$$\frac{d^2\theta}{dx^2} - S_h \theta(x)^2 = 0 \tag{8}$$

$$\theta(1) = 1, \theta'(0) = 0 \tag{9}$$

Where $\theta = \frac{T - T_\infty}{T_b - T_\infty}$ and $S_h = \frac{Da\, Ra}{\delta_r} (\frac{L}{t})^2$

Haar wavelets collocation and prepared matrix of integration

Each Haar wavelet comprises a pair of fixed levels of an adverse sign through its subinterval and it is zero elsewhere.

The Haar wavelet family for x is defined $x \in [0,1)$ as follows [5, 6]:

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\varrho_1, \varrho_2) \\ -1 & for \quad x \in [\varrho_3, \varrho_4) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Here $\varrho_1 = \frac{k}{m}$, $\varrho_2 = \frac{k+0.5}{m}$, $\varrho_3 = \frac{k+0.5}{m}$ and $\varrho_4 = \frac{k+1}{m}$

$m = 2^j$, for $j = 1,2, \ldots, J$, where J is the resolution. The index i is calculated by $i = m + k + 1$ where $i \geq 2,\ 0 \leq k \leq m - 1$, where k is the translation parameter .The maximal value of i is $W = 2^{J+1}$. The collocation points are $x_\ell = \frac{\ell - 0.5}{2m}$, $\ell = 1,2, \ldots, 2m$

For instance, $if\ J = 2\ then\ W = 8$, then we have H(8,8) =

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{11}$$

The operational matrix p_i is derived from Haar wavelets integral,

$$P_i(x) = \int_0^x H_i(v)dv \tag{12}$$

and,

$$Q_i(x) = \int_0^x P_i(v)dv \tag{13}$$

We now calculate the integral of the Haar wavelet equation (10) and they are given by:

$$P_i(x) = \begin{cases} x - \varrho_1 & \text{for } x \in [\varrho_1, \varrho_2) \\ \varrho_4 - x & \text{for } x \in [\varrho_3, \varrho_4) \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

$$Q_i(x) = \begin{cases} \frac{1}{2}(x - \varrho_1)^2 & \text{for } x \in [\varrho_1, \varrho_2) \\ \frac{1}{4m^2} - \frac{1}{2}(\varrho_4 - x)^2 & \text{for } x \in [\varrho_3, \varrho_4) \\ \frac{1}{4m^2} & \text{for } x \in [\varrho_4, 1) \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

For instance, if $\mathrm{J} = 2$ then $\mathrm{W} = 8$, from (14)

$$P = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 & -4 & 4 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

and from (15) we get

$$Q = \frac{1}{512}\begin{bmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \tag{17}$$

1.1. Solution Procedure

Consider the second-order boundary value problem

$$\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} + \theta(x) = 0, \tag{18}$$

with boundary conditions $\theta(0) = \alpha, \theta'(1) = \beta, x \in [0,1)$ (19)

where α *and* β are real constant .

The highest derivative can be written as the Haar wavelet as follows :

$$\frac{d^2\theta}{dx^2} = \sum_{i=1}^{m} A_i h_i(x) \tag{20}$$

Where A_i's are the Haar coefficients, that are determined by Integrating equation (20) with substitution the boundary condition equation (19) we get,

$$\frac{d\,\theta}{dx} = \theta'(0) + \sum_{i=1}^{m} A_i P_i(x) \ , \tag{21}$$

We note that there are some boundary conditions missing in the equation, which can be found as follows.

Put x=1 and substitute in (21) we get ,

$$\theta'(1) = \theta'(0) + \sum_{i=1}^{m} A_i P_i(1), \tag{22}$$

substitute with the equation (19) we get,

$$\theta'(0) = \beta - \sum_{i=1}^{m} A_i P_i(1) \tag{23}$$

$P_i(1)$ is got through equation (14),So the equation (21) is as follows:

$$\frac{d\,\theta}{dx} = \beta - \sum_{i=1}^{m} A_i P_i(1) + \sum_{i=1}^{m} A_i P_i(x) \tag{24}$$

The integrals are repeated several times according to the order of the differential equation with the substitution of the boundary conditions (19) until we reach an equation that is clear of derivatives, which represents the approximate solution of the equation (18)

$$\theta(x) = \alpha + (\beta - \sum_{i=1}^{m} A_i P_i(1))x + \sum_{i=1}^{m} A_i Q_i(x) \tag{25}$$

Substituting equations (20)–(22) in (18), we get the following non linear system :

$$\sum_{i=1}^{m} A_i h_i(x) + \beta - \sum_{i=1}^{m} A_i P_i(1) + \sum_{i=1}^{m} A_i P_i(x) + \alpha + (\beta - \sum_{i=1}^{m} A_i P_i(1))x + \sum_{i=1}^{m} A_i Q_i(x) = 0 \tag{26}$$

By dissolving the system $2M \times 2M$ using the numerical method by the Mathematica Software 11 , where A_i's are the unknown Haar coefficients $i = 1, 2, \dots\dots\dots, 2M$, that are collected. Finally, the approximate solution can be handled.

Application of Haar wavelet collocation method

In this section, HWCM is applied , which is discussed in the previous section, to solve equation (8) with its boundary conditions (9).

The highest derivative of equation (8) can be written as the Haar wavelet as :

$$\frac{d^2\theta}{dx^2} = \sum_{i=1}^{m} A_i h_i(x) \tag{27}$$

Where A_i's are the Haar coefficients, that can be determined by Integrating equation (27) with substitution the boundary condition equation (9) we get,

$$\frac{d\,\theta}{dx} = \sum_{i=1}^{m} A_i P_i(x) \ , \tag{28}$$

Integrate equation (28) end using equation (9), we have

$$\theta(x) = \theta(0) + \sum_{i=1}^{m} A_i Q_i(x) \tag{29}$$

We note that there are some boundary conditions missing in the equation, which can be found as follows.
Put x=1 and substitute in equation (29) we get ,

$$\theta(1) = \theta(0) + \sum_{i=1}^{m} A_i Q_i(1), \quad (30)$$

$Q_i(1)$ is obtained through equation (15),that are equal to $\{\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{16}, 0,0,0, \frac{1}{64}\}$, and $\theta(1)$ is obtained through equation (9), we have

$$\theta(0) = 1 - \frac{A_0}{2} - \frac{A_1}{4} - \frac{A_3}{16} - \frac{A_7}{64} \quad (31)$$

Now the equation 29 will be as form

$$\theta(x) = 1 - \frac{A_0}{2} - \frac{A_1}{4} - \frac{A_3}{16} - \frac{A_7}{64} + \sum_{i=1}^{m} A_i Q_i(x) \quad (32)$$

Substituting equations (27) and (32) in equation (8), we get

$$\sum_{i=1}^{m} A_i h_i(x) + S_h(1 - \frac{A_0}{2} - \frac{A_1}{4} - \frac{A_3}{16} - \frac{A_7}{64} + \sum_{i=1}^{m} A_i Q_i(x))^2 = 0 \quad (33)$$

Solving (33) by using numerical calculation that gets from Mathematica Software
and choose the value $S_h = 0.09$,that is proposed in [1]
we get the Haar coefficients
$A =$
$\{0.08502, -0.00182, -0.00044, -0.00139, -0.00011, -0.00033, -0.00057, -0.00082\}$
Substituting these coefficients in (32), we get the HWCM solution of (8).
In comparison with TAM, the numerical solution is achieved for (8) it is shown as follows:

Description of Semi-analytical method

We rewrite equation (8) as follows:

$$L\big(\theta(x)\big) + N\big(\theta(x)\big) + g(x) = 0 \quad (34)$$

With boundary conditions $B(\theta, \theta')$, (35)

The solution is to take the linear part with the source as follows:

$$L\big(\theta_0(x)\big) + g(x) = 0 \quad (36)$$

Then to find the next iteration we solve the following problem:

$$L\big(\theta_1(x)\big) + N\big(\theta_0(x)\big) + g(x) = 0 \quad (37)$$

Also to find the rest of the iterations

$$L\big(\theta_{n+1}(x)\big) + N\big(\theta_n(x)\big) + g(x) = 0 \quad (38)$$

It is important to clarify that each iteration is an amelioration over the previous iteration, and as we progress with iterations we get closer to solve the problem.

Application of TAM

In this section, we use the proposed method (TAM) to solve equation (8) with boundary conditions (9):

$$L\big(\theta(x)\big) = \frac{d^2\theta}{dx^2} \ , N\big(\theta(x)\big) = -S_h\theta(x)^2, g(x) = 0, \quad (39)$$

Thus,

$$L\big(\theta_0(x)\big) = 0, \text{ with } \theta_0(1) = 1, \theta'_0(0) = 0, \quad (40)$$

We can get the rest of the iterations from the following relationship:

$$L\big(\theta_{n+1}(x)\big) + N\big(\theta_n(x)\big) = 0, \text{ with } \theta_n(1) = 1, \theta'_n(0) = 0, \quad (41)$$

the initial iteration will be:

$$\theta''_0(x) = 0, \quad (42)$$

By integrating the equation (41) twice and substituting the boundary conditions, we get

$$\theta_0(x) = 1, \quad (43)$$

$$\theta''_1(x) = S_h\theta_0(x)^2, \text{ with } \theta_1(1) = 1, \theta'_1(0) = 0, \quad (44)$$

$$\theta_1(1) = \frac{1}{2}(2 - s + sx^2), \quad (45)$$

Likewise, other iterations can be achieved.

$$\theta''_2(x) = S_h \theta_1(x)^2, \text{ with } \theta_2(1) = 1, \theta'_2(0) = 0, \quad (46)$$

then we get:

$$\theta_2(x) = 0.000749\big(1273.333 + 50s - 11{S_h}^2 + 60x^2 - 60S_h x^2 + 15{S_h}^2 x^2 + 10S_h x^4 - 5{S_h}^2 x^4 + 1{S_h}^2 x^6\big), \quad (47)$$

and so on:

$$\theta_3(x) = 2.7815934 \times 10^{-10}(3.44517 \times 10^9 - 9475584.4444S_h + 1888917.3333{S_h}^2 + 69785.3131s^3 - 7577.76768 + 1.4754538 \times 10^8 x^2 + 1.158733 \times 10^7 s x^2 - 2321713.3334s^2x^2 - 100100s^3x^2 + 11011s^4x^2 + 2317466.6667x^4 - 2226466.6667sx^4 + 468346.6667s^2x^4 + 42770s^3x^4 - 5005s^4x^4 + 21840x^6 + 110817.7778sx^6 - 38422.2222s^2x^6 - 15288s^3x^6 + 2032.3333s^4x^6 + 3900sx^8 + 2426.6667s^2x^8 + 3250s^3x^8 - 559s^4x^8 + 444.8889s^2x^{10} - 444.8889s^3x^{10} + 111.2222s^4x^{10} + 27.5757s^3x^{12} - 13.7878s^4x^{12} + 1s^4x^{14}) \quad (48)$$

The solutions θ (x) are very long and for brevity they are not listed, knowing that the repetition of θ(x) was calculated to the eighth iteration.

Results and discussion

In this article, the HWCM such as a numerical technique to find the numerical solution to the non-linear fin problem is employed. For more accuracy, we compare all these results with the TAM, such as semi- analytical method to find the analytical solution, which are shown in Table 1.

Table 1- The results of Numerical methods and analytical method for $\theta(X)$ for $S_h = 0.09$

x	HWCM	TAM
1/16	0.958207	0.958252
3/16	0.959499	0.959543
5/16	0.962085	0.96213
7/16	0.965974	0.966019
9/16	0.971175	0.97122
11/16	0.977704	0.977748
13/16	0.985577	0.985621
15/16	0.994816	0.99486

Figure 1 represents a comparison between the semi-analytic solution of the TAM and the numerical solution of the HWCM, which shows that the results are identical.

In order to verify the extraction results more accurately, we can calculate the residual error of the Tamimi method, since the results are mostly the same for the two methods.

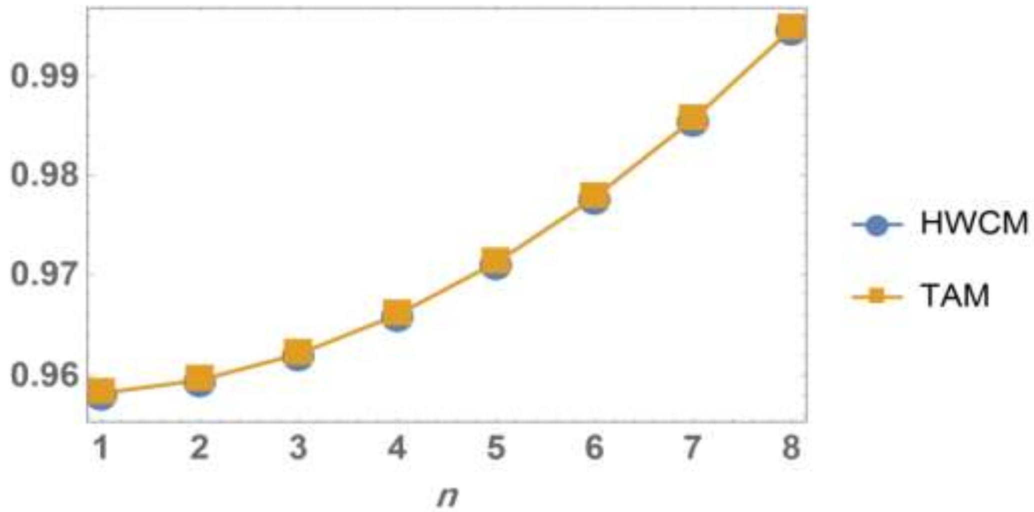
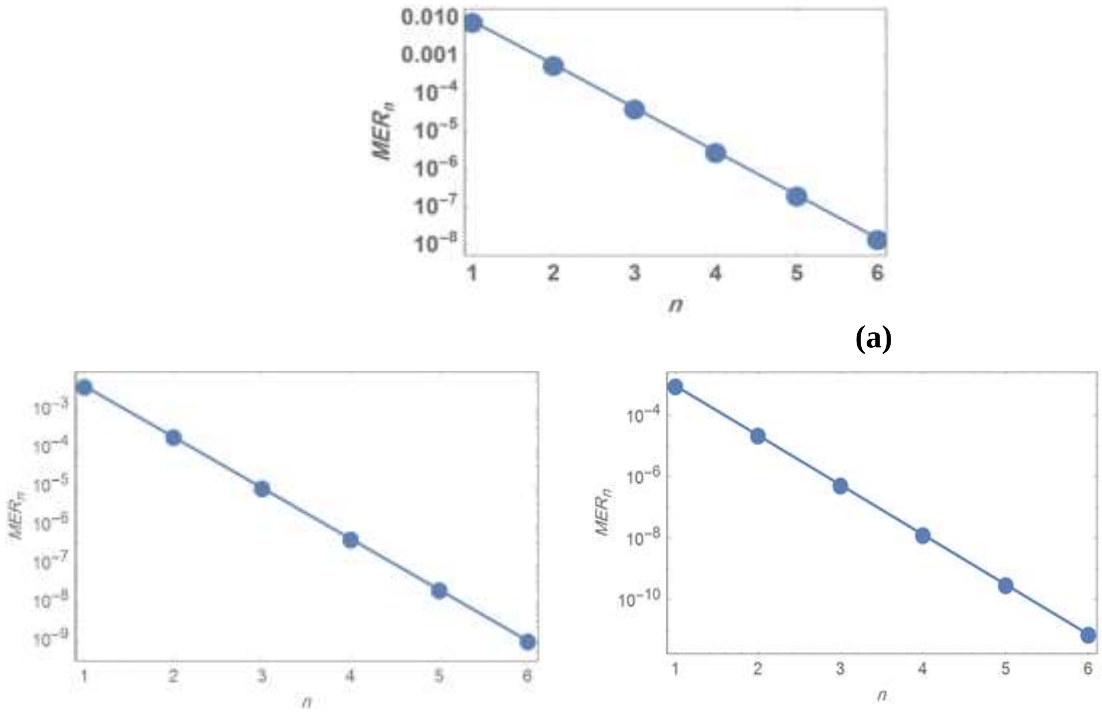


Figure 1- The comparison between the TAM and the HWCM

$$ER_n = \frac{d^2\theta_n}{dx^2} - S_h\theta_n(x)^2 = 0,$$

and the following maximum error remainder parameter:

$$MER_n = \underset{0 \leq x \leq 1}{MAX} |ERp_n(t)|$$



(b) **(c)**

Figure-2 The maximum error remainder plots for the non-linear fin problem by using the TAM where
(a) $S_h = 0.09$ (b) $S_h = 0.06$ (c) $S_h = 0.03$.

Conclusions

In this paper, they successfully employed HWCM and TAM to get numerical solutions for nonlinear fin problems. The obtained results are assured of the reliability of the methods used. Also, they directly provide the solutions directly without using linearization, discretization or conditional hypotheses. Their high ability to perform numerical calculations and high accuracy distinguishes the methods used here. Its outstanding performance is distinguished the HWCM in terms of speed in the execution of calculations compared to the TAM.

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References

[1] I.R. Petroudi, D.D. Ganji, etal. "Semi-Analytical method for solving Non-Linear Equation Arising of Natural Convection Porous Fin," *Thermal Science*, vol.16, no. 5, pp. 1303-1308, 2012.

[2] D.D. Ganji, "The application of he's homotopy perturbation method to non-linear equation arising in heat transfer," *Physics Letters A*, vol. 355, no. 4-5, pp. 337-341, 2006.

[3] A.S. Mohammed, S.H. Abd Almjeed, "Semi-Analytical Method with Laplace Transform for Certain Types of Nonlinear Problems," In Proceedings of the Fifth International Scientific Conference of Al-Khwarizmi Society, Iraq, 2020 .

[4] M. A. Sabaa and M. A. Mohammed, "Approximate solutions of nonlinear smoking habit model," *Iraqi Journal of Science*, vol. 61, no. 2, pp. 435-443, 2020.
[5] C.F. Chen, C.H. Hsiao, "Haar wavelet method for solving lumped and distributed parameter systems," IEE Proceedings - Control Theory and Applications, vol. 144, no. 1, pp. 87 – 94,1997.
[6] U. Lepik, "Numerical solution of differential equations using Haar wavelets," *Mathematics and Computers in Simulation*, vol. 68, no. 2, pp. 127-143, 2005.
[7] U. Lepik, "Numerical solution of evolution equations by the Haar wavelet method," *Applied Mathematics and Computation*, vol. 158, no. 1, pp. 695-704, 2007.
[8] U. Lepik, "Application of the Haar wavelet transform to solving integral and differential equations," *The Proceedings of the Estonian Academy of Sciences*, vol. 56, no. 1, pp. 28–46, 2007.
[9] H. Temimi, A. Ansari, "A semi analytical iterative technique for solving nonlinear problems," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 203-210, 2011.
[10] S. M. Abed, M. A. AL-Jawary, "Efficient iterative methods for solving the sir epidemic model," *Iraqi Journal of Science*, vol. 62, no. 2, pp. 613-62, 2021.