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The Optimal Control Problem for Triple Nonlinear Parabolic Boundary Value Problem with State Vector Constraints

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Abstract
In this paper, the classical continuous triple optimal control problem (CCTOCP) for the triple nonlinear parabolic boundary value problem (TNLPBVP) with state vector constraints (SVCs) is studied. The solvability theorem for the classical continuous triple optimal control vector CCTOCV with the SVCs is stated and proved. This is done under suitable conditions. The mathematical formulation of the adjoint triple boundary value problem (ATHBVP) associated with TNLPBVP is discovered. The Fréchet derivative of the Hamiltonian is derived. Under suitable conditions, theorems of necessary and sufficient conditions for the optimality of the TNLPBVP with the SVCs are stated and proved.

Keywords: Classical Continuous Optimal Control, Nonlinear Triple Parabolic Boundary Value Problem, Fréchet Derivative, Necessary and Sufficient Optimality Conditions

مسألة السيطرة الامثلية لمسألة القيم الحدودية المكافئة غير الخطية الثلاثية مع قيود الحالة

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الخلاصة
في هذا البحث مسالة السيطرة الامثلية المستمرة التقليدية لمسالة القيم الحدودية المكافئة غير الخطية الثلاثية بوجود قيود متجه الحالة .تم ذكر نص وبرهان مبرهنة قابلية الحل لمتجه سيطرة امثلية مستمرة تقليدية مع قيود متجه الحالةو قد تم ذلك بوجود شروط مناسبة. تم ايجاد الصياغة الرياضية لمسالة القيم الابتدائية الثلاثية المصاحبة والمرتبطة بمسالة القيم الحدودية المكافئة غير الخطية . تم اشتقاق مشتقة فريشيه لدالة الهاملتون .بوجود شروط مناسبة تمت ذكر نص وبرهان مبرهنتي الشروط الامثلية الضرورية والكافية لمسالة القيم الحدودية المكافئة غير الخطية الثلاثية مع قيود متجه الحالة .

1. Introduction

The subject of optimal control problem is divided into two types, namely the relaxed and the classical optimal control problems. The first type is mostly studied in the last century, while the second one began to study at the beginning of this century. On other hand, both of them are studied for systems that are controlled by ordinary or partial differential equations. The optimal control problems play an important role in many fields of real-world problems,

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various examples of applications of these problems are investigated in economic growth [1], electric power [2], aircraft [3], medicine [4], and many other fields.

This role motivates many investigators in the recent years to be interest in studying the classical optimal control problems (COCTPs) that are controlled by nonlinear ordinary differential equations [5], or controlled by different types of nonlinear parabolic PDEs like " single" nonlinear parabolic PDEs (NLPPDEs) [6], or couple NLPPDEs (CNLPPDEs) [7], or triple linear PPDEs (TLPPDEs) [8]. Other researchers are interested to study the CCTOCP for CNLPPDEs and TLPPDEs, which involve the Neumann boundary conditions (NBCs) for more details see [9] and [10], respectively, while authors [11] dealt with the CCTOCP controlling by the TNLPBVP without SVCs.

All these investigations encourage us to seek about the CCTOCP that is controlled by the TNLPBVP with the SVCs. The solvability theorem for a CCTOCV with the SVCs is stated and proved under suitable conditions. The mathematical formulation for the ATHBVP associated with TNLPBVP is discovered. The Fréchet Derivative of the Hamiltonian is discussed. Under suitable conditions, the theorems of the necessary and sufficient conditions for the optimality of the TNLPBVP with the SVCs are stated and proved.

2. Problem Description

Let $I = (0, T)$, $T < \infty$, and $\Omega \subset \mathbb{R}^3$ be a bounded open region with Lipschitz boundary $\Gamma = \partial\Omega$, $Q = \Omega \times I, \Sigma = \Gamma \times I$ The CCTOCP consists of the TNPPDEs which represents by the following boundary value problem of the triple state vector equations TSVEs:

$$y_{1t} - \Delta y_1 + y_1 - y_2 - y_3 = f_1(x, t, y_1, u_1) \quad \text{in } Q \tag{1}$$
$$y_{2t} - \Delta y_2 + y_2 + y_3 + y_1 = f_2(x, t, y_2, u_2) \quad \text{in } Q \tag{2}$$
$$y_{3t} - \Delta y_3 + y_3 + y_1 - y_2 = f_3(x, t, y_3, u_3) \quad \text{in } Q \tag{3}$$
$$y_1(x, t) = 0 \quad \text{on } \Sigma \tag{4}$$
$$y_1(x, 0) = y_1^0(x) \quad \text{on } \Omega \tag{5}$$
$$y_2(x, t) = 0 \quad \text{on } \Sigma \tag{6}$$
$$y_2(x, 0) = y_2^0(x) \quad \text{on } \Omega \tag{7}$$
$$y_3(x, t) = 0 \quad \text{on } \Sigma \tag{8}$$
$$y_3(x, 0) = y_3^0(x), \quad \text{on } \Omega . \tag{9}$$

Where $x = (x_1, x_2)$, $\vec{y} = (y_1, y_2, y_3) = (y_1(x, t), y_2(x, t), y_3(x, t)) \in \left(H_2(Q)\right)^3$is the triple state vector (TSVS) that corresponds to the CCTCV $\vec{u} = (u_1, u_2, u_3)$, $= (u_1(x, t), u_2(x, t), u_3(x, t)) \in (L^2(Q))^3$ and $(f_1, f_2, f_3) \in (L^2(Q))^3$,($f_i = f_i(x, t, y_i, u_i)$) is vector of given function defined on $(Q \times \mathbb{R} \times U_1) \times (Q \times \mathbb{R} \times U_2) \times (Q \times \mathbb{R} \times U_3)$ with $U_1 \times U_2 \times U_3 = \vec{U} \subset \mathbb{R}^3$, and let $\vec{W} = W_1 \times W_2 \times W_3$, $W_i \subset L^2(Q)$, i = 1,2,3, s.t.
$\vec{W} = \left\{\vec{w} \in \left(L^2(Q)\right)^3 | \vec{w} \in \vec{U} \ a.e. \ in \ Q \right\}$ with $\vec{U}$ is convex set.

The cost function is

$$G_0(\vec{u}) = \sum_{i=1}^{3} \int_Q g_{0i}(x, t, y_i, u_i) dxdt \tag{10}$$

The SVCs on the TSV and the CCTCV are

$$G_1(\vec{u}) = \sum_{i=1}^{3} \int_Q g_{1i}(x, t, y_i, u_i) dxdt = 0 , \tag{11}$$
$$G_2(\vec{u}) = \sum_{i=1}^{3} \int_Q g_{2i}(x, t, y_i, u_i) dxdt \leq 0, \tag{12}$$

The set of admissible CCTCV (ADCCTCV) is
$\vec{W}_A = \{\vec{u} \in \vec{W} | G_1(\vec{u}) = 0, G_2(\vec{u}) \leq 0 \}$

The CCTOCV is to find $\vec{u} \in \vec{W}_A$ $s.t. G_0(\vec{u}) = min_{\vec{w} \in \vec{W}_A} G_0(\vec{w})$.

Let $\vec{V} = V_1 \times V_2 \times V_3 = \{\vec{v} \in (H^1(\Omega))^3 \text{with } v_1 = v_2 = v_3 = 0 \ on \ \partial\Omega\}$.

The notations $(\vec{v}, \vec{v})$ and $\|\vec{v}\|_0$ are referred to the inner product and the norm in $(L^2(\Omega))^3$. The notation $\|\vec{v}\|_0$ is the inner product in $(L^2(Q))^3$, and $(\vec{v}, \vec{v})_1 = (v_1, v_1)_1 + (v_2, v_2)_1 + (v_3, v_3)_1$

represents the inner product in $\vec{V}$,while $\vec{V}^*$ is the dual of $\vec{V}$.
The weak form of the TSVEs (1-9) when $\vec{y} \in H_0^1(\Omega))^3$ is given by

$$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) - (y_3, v_1) = (f_1, v_1) \tag{13a}$$
$$(y_1^0, v_1) = (y_1(0), v_1), \quad \forall v_1 \in V \tag{13b}$$
$$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_1) + (y_3, v_2) + (y_1, v_2) = (f_2, v_2) \tag{14a}$$
$$(y_2^0, v_2) = (y_2(0), v_2), \quad \forall v_2 \in V \tag{14b}$$
$$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) + (y_3, v_3) + (y_1, v_3) - (y_2, v_3) = (f_3, v_3) \tag{15a}$$
$$(y_2^0, v_3) = (y_3(0), v_3), \quad \forall v_3 \in V \tag{15b}$$

To study the existence of a CCTOCV, we need the following assumptions, theorem and lemma.

Assumptions (A):

(i) Let f_i be Carathéodory type (CAT) on Q × (ℝ × ℝ) that satisfies

$|f_i(x,t,y_i,u_i)| \leq \eta_i(x,t) + c_i|y_i| + \acute{c}_i|u_i|$

where $(x,t) \in Q, y_i, u_i \in \mathbb{R}, c_i, \acute{c}_i > 0$ *and* $\eta_i \in L^2(Q)$ $\forall$ i =1,2,3

(ii) f_i is Lip w.r.t. y_i, i.e. $|f_i(x,t,y_i,u_i) - f_i(x,t,\bar{y}_i,u_i)| \leq L_i|y_i - \bar{y}_i|$,

where $(x,t) \in Q, y_i, \bar{y}_i, u_i \in \mathbb{R}$ and $L_i > 0$, $\forall$ i = 1,2,3.

Theorem (2.1)[11]: Existence and Uniqueness Of The Weak Form: With Assumptions (A) for each $\vec{u} \in (L^2(\Omega))^3$, the weak form of TSVEs (13-15) has a unique solution $\vec{y} = (y_1, y_2, y_3), \vec{y} \in (L^2(I,V))^3, s.t\ \vec{y}_t = (y_{1t}, y_{2t}, y_{3t}) \in (L^2(I,V^*))^3$.

Assumptions (B): Consider g_{li} ($\forall i = 1,2,3, \forall l = 0,1,2$) is of CAT on $Q \times (\mathbb{R} \times \mathbb{R})$ with:
$|g_{li}(x,t,y_i,u_i)| \leq \eta_{li}(x,t) + c_{li1}(y_i)^2 + c_{li2}(u_i)^2$, where $y_i, u_i \in \mathbb{R}$ with $\eta_{li} \in L^1(Q)$

Lemma (2.1): If Assumptions (B) are held, then $\vec{u} \mapsto G_l(\vec{u})$ fo*r all* $l = 0,1,2$ is continuous functional on $(L^2(Q))^3$.

Proof: The requirement result is obtained ($\forall l = 0,1,2$) directly from the assumptions(B) and Lemma 4.1 in [11].

Theorem (2.2)[11]: Consider the set $\overrightarrow{W}_A \neq \emptyset$, the functions f_i,for all $i = 1,2,3$, has the form $f_i(x,t,y_i,u_i) = f_{i1}(x,t,y_i) + f_{i2}(x,t)u_i$

With $|f_{i1}(x,t,y_i)| \leq \eta_i(x,t) + c_i|y_i|$, where $\eta_i \in L^2(Q)$ and $|f_{i2}(x,t)| \leq k_i$,

If for all $i = 1,2,3$, g_{0i} is convex w.r.t. u_i for fixed (x,t,y_i). Then there exists a CCTOCV.

Assumptions (C): $g_{l_i y_i}$ and $g_{l_i u_i}$ are of CAT on $Q \times \mathbb{R} \times \mathbb{R}$ for $l = 0,1,2$, *and* $i = 1,2,3$ and satisfy

$\left|g_{l_i y_i}(x,t,y_i,u_i)\right| \leq \eta_{l_{i5}}(x,t) + c_{l_{i5}}|y_i| + c'_{l_{i5}}|u_i|$, for $(x,t) \in Q, y_i, u_i \in \mathbb{R}, \eta_{l_{i5}}, \in L^2(Q)$

$\left|g_{l_i u_i}(x,t,y_i,u_i)\right| \leq \eta_{l_{i6}}(x,t) + c_{l_{i6}}|y_i| + c'_{l_{i6}}|u_i|$, for $(x,t) \in Q, y_i, u_i \in \mathbb{R}, \eta_{l_{i6}} \in L^2(Q)$.

Theorem (2.3)[11]: In addition to assumptions (A), if $\vec{y}$ and $\vec{y} + \overrightarrow{\delta y}$ are the TSVS corresponding to the CCTCV $\vec{u}$, $\vec{u} + \overrightarrow{\delta u} \in (L^2(Q))^3$, respectively. Then

$$\left\|\overrightarrow{\delta y}\right\|_{L^\infty(I,L^2(\Omega))} \leq M \left\|\overrightarrow{\delta u}\right\|_Q, \ \left\|\overrightarrow{\delta y}\right\|_{L^2(Q)} \leq M \left\|\overrightarrow{\delta u}\right\|_Q, \ \left\|\overrightarrow{\delta y}\right\|_{L^2(I,V)} \leq M \left\|\overrightarrow{\delta u}\right\|_Q.$$

***Theorem* (2.4) (*The* TKL *Theorem*) [7]:**

Let U be a nonempty convex subset of a vector space X, K be nonempty convex positive cone in a normed space Z, and $W = \{u \in U | G_1(u) = 0, G_1(u) \in -K\}$.

The functional $G_0: U \to \mathbb{R}, G_1: U \to \mathbb{R}^m, G_2: U \to Z$ are $(m+1) -$ locally continuous at $u \in U$, and have $(m+1) -$ derivatives at u where $m \neq 0$. If $m = 0$, then we assume that $DG_l(u), l = 0,1,2$, are K -linear at the point u. If $G_0(u)$has a minimum at u in W, then it satisfies the following ***KUTULA*** conditions for all $w \in W$**:**

There exists $\lambda_0 \in \mathbb{R}, \lambda_1 \in \mathbb{R}^m, \lambda_2 \in \mathbb{Z}^*$, with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2}|\lambda_l| = 1$ such that

$\lambda_0 DG_0(u, w-u) + \lambda_1^T DG_1(u, w-u) + \langle \lambda_2, DG_2(u, w-u)\rangle \geq 0$

$\langle \lambda_2, G_2(u)\rangle = 0.$

Main Results

3. Existence of the CCTOCV and the Fréchet Derivative

This section deals with the existence of the CCTOCV and the derivation of the Fréchet Derivative under some suitable Assumptions after the TAHBVP is defined.

Theorem (3.1): Consider the set $\overrightarrow{W}_A \neq \emptyset$, the functions f_i for all $i = 1,2,3$, has the form

$$f_i(x,t,y_i,u_i) = f_{i1}(x,t,y_i) + f_{i2}(x,t)u_i$$

With $|f_{i1}(x,t,y_i)| \leq \eta_i(x,t) + c_i|y_i|$ and $|f_{i2}(x,t)| \leq k_i$,
where $\eta_i \in L^2(Q)$. If for all $= 1,2,3$, g_{1i} is independent of u_i, g_{0i} and g_{2i} are convex w.r.t. u_i for fixed (x,t,y_i). Then there exists a CCTOCV.

Proof: From the assumptions on W_i and g_{1i} $for\ all\ i = 1,2,3$ with using lemma 2.1and theorem 2.2, one can get that there exists a CCTOCV with the SVCs.

Theorem (3.2): Neglecting the indicator l in g_l and G_l. In addition to assumptions A,B and C , if the TAHBVP associated with the TNLHBVP (1-9) are defined as:

$$-z_{1t} - \Delta z_1 + z_1 + z_2 + z_3 = z_1 f_{1_{y_1}}(x,t,y_1,u_1) + g_{1_{y_1}}(x,t,y_1,u_1) \qquad (16)$$

$$-z_{2t} - \Delta z_2 + z_2 - z_1 - z_3 = z_2 f_{2_{y_2}}(x,t,y_2,u_2) + g_{2_{y_2}}(x,t,y_2,u_2) \qquad (17)$$

$$-z_{3t} - \Delta z_3 + z_3 - z_1 + z_2 = z_3 f_{3_{y_3}}(x,t,y_3,u_3) + g_{3_{y_3}}(x,t,y_3,u_3) \qquad (18)$$

$$z_1(x,t) = 0,\ z_2(x,t) = 0, \quad \text{and} \quad z_3(x,t) = 0 \text{ on } \Sigma, \qquad (19)$$

$$z_1(T) = 0, z_2(T) = 0, \text{ and } z_3(T) = 0, \quad \text{on } \Gamma \qquad (20)$$

Then the Hamiltonian which is defined by:$H(x,t,\vec{y},\vec{z},\vec{u}) = \sum_{i=1}^{3} z_i\, f_i(x,t,y_i,u_i) + g_i(x,t,y_i,u_i)$ has the following Fréchet Derivative,

$$\acute{G}(\vec{u})\overrightarrow{\delta u} = \int_Q \begin{pmatrix} z_1 f_{1u_1} + g_{1u_1} \\ z_2 f_{2u_2} + g_{2u_2} \\ z_3 f_{3u_3} + g_{3u_3} \end{pmatrix} \cdot \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix} dx$$

Proof: Let $\vec{u}$ is a CCTCV, and $\vec{y}$ be its TSVS, and

$$G(\vec{u}) = \int_Q g_1(x,t,y_1,u_1)dxdt + \int_Q g_2(x,t,y_2,u_2)dxdt + \int_Q g_3(x,t,y_3,u_3)\,dxdt.$$

From the Assumptions on g_l ($l = 1,2,3$), the definition of the Fréchet Derivative, the result of Theorem 2.3, and then using the inequality of Minkowiski (INMK), we have

$$G(\vec{u} + \overrightarrow{\delta u}) - G(\vec{u}) = \int_Q (g_{1_{y_1}}\delta y_1 + g_{1_{u_1}}\delta u_1)\,dxdt + \int_Q (g_{2_{y_2}}\delta y_2 + g_{2_{u_2}}\delta u_2)\,dxdt + \int_Q (g_{3_{y_3}}\delta y_3 + g_{3_{u_3}}\delta u_3)\,dxdt + +\varepsilon_1(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_0 \qquad (21)$$

where $\varepsilon_1(\overrightarrow{\delta u}) \longrightarrow 0$, and $\|\overrightarrow{\delta u}\|_0 \longrightarrow 0$ as $\overrightarrow{\delta u} \longrightarrow 0$.

On the other hand, the weak form of the TAHBVP (with $v_1, v_2, v_3 \in V$) is

$$-\langle z_{1t}, v_1\rangle + (\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) + (z_3, v_1) = (z_1 f_{1_{y_1}}, v_1) + (g_{1_{y_1}}, v_1) \qquad (22)$$

$$-\langle z_{2t}, v_2\rangle + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) - (z_3, v_2) = (z_2 f_{2_{y_2}}, v_2) + (g_{2_{y_2}}, v_2) \qquad (23)$$

$$-\langle z_{3t}, v_3\rangle + (\nabla z_3, \nabla v_3) + (z_3, v_3) - (z_1, v_3) + (z_2, v_3) = (z_3 f_{3_{y_3}}, v_3) + (g_{3_{y_3}}, v_3) \qquad (24)$$

Substituting $v_i = \delta y_i$, $\forall i = 1,2,3$ in (22-24) respectively, integrating from 0 to T . Finally using integration by parts (IBPs) for each 1^{st} term to obtain

$$\int_0^T \langle \delta y_{1t}, z_1\rangle dt + \int_0^T [(\nabla z_1, \nabla \delta y_1) + (z_1, \delta y_1) + (z_2, \delta y_1) + (z_3, \delta y_1)]dt = \int_0^T \left[(z_1 f_{1_{y_1}}, \delta y_1) + (g_{1_{y_1}}, \delta y_1)\right] dt \qquad (25)$$

$$\int_0^T \langle \delta y_{2t}, z_2\rangle dt + \int_0^T [(\nabla z_2, \nabla \delta y_2) + (z_2, \delta y_2) - (z_1, \delta y_2) - (z_3, \delta y_2)]dt = \int_0^T \left[(z_2 f_{2y_2}, \delta y_2) + (g_{2y_2}, \delta y_2)\right] dt \qquad (26)$$

$$\int_0^T \langle \delta y_{3t}, z_3\rangle dt + \int_0^T [(\nabla z_3, \nabla \delta y_3) + (z_3, \delta y_3) - (z_1, \delta y_3) + (z_2, \delta y_3)\ ]dt = \int_0^T \left[(z_3 f_{3y_3}, \delta y_3) + (g_{3y_3}, \delta y_3)\right] dt \qquad (27)$$

Now, substituting $y_i = \delta y_i$ and $v_i = z_i$ ($\forall i = 1,2,3$) in ((13)a-(15)a), IBS w.r.t. t from 0 to T, they become

$\int_0^T \langle \delta y_{1t}, z_1 \rangle dt + \int_0^T [(\nabla \delta y_1, \nabla z_1) + (\delta y_1, z_1) + (\delta y_2, z_1) + (\delta y_3, z_1)] dt =$
$$\int_0^T (f_1(y_1 + \delta y_1, u_1 + \delta u_1), z_1) dt - \int_0^T (f_1(y_1, u_1), z_1) dt \qquad (28)$$
$\int_0^T \langle \delta y_{2t}, z_2 \rangle dt + \int_0^T [(\nabla \delta y_2, \nabla z_2) + (\delta y_2, z_2) - (\delta y_1, z_2) - (\delta y_3, z_2)] dt =$
$$\int_0^T (f_2(y_2 + \delta y_2, u_2 + \delta u_2), z_2) dt - \int_0^T (f_2(y_2, u_2), z_2) dt \qquad (29)$$
$\int_0^T \langle \delta y_{3t}, z_3 \rangle dt + \int_0^T [(\nabla \delta y_3, \nabla z_3) + (\delta y_3, z_3) - (\delta y_1, z_3) + (\delta y_2, z_3)] dt =$
$$\int_0^T (f_3(y_3 + \delta y_3, u_3 + \delta u_3), z_3) dt - \int_0^T (f_3(y_3, u_3), z_3) dt \qquad (30)$$
Using the assumptions on f_i (for $i = 1,23$) , the Fréchet Derivative of them are exist, then from the result of Theorem 3.2 in [11], and the INMK, the following are yielded

$\int_0^T \langle \delta y_{1t}, z_1 \rangle dt + \int_0^T [(\nabla \delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) - (\delta y_3, z_2)] dt =$
$$\int_0^T \left(f_{1y_1} \delta y_1 + f_{1u_1} \delta u_1, z_1\right) dt + \varepsilon_2\left(\overrightarrow{\delta u}\right) \left\|\overrightarrow{\delta u}\right\|_Q \qquad (31)$$
$\int_0^T \langle \delta y_{2t}, z_2 \rangle dt + \int_0^T [(\nabla \delta y_2, \nabla z_2) + (\delta y_2, z_2) - (\delta y_1, z_2) + (\delta y_3, z_2)] dt =$
$$\int_0^T \left(f_{2y_2} \delta y_2 + f_{2u_2} \delta u_2, z_2\right) dt + \varepsilon_3\left(\overrightarrow{\delta u}\right) \left\|\overrightarrow{\delta u}\right\|_Q \qquad (32)$$
$\int_0^T \langle \delta y_{3t}, z_3 \rangle dt + \int_0^T [(\nabla \delta y_3, \nabla z_3) + (\delta y_3, z_3) - (\delta y_1, z_3) + (\delta y_2, z_3)] dt =$
$$\int_0^T \left(f_{3y_3} \delta y_3 + f_{3u_3} \delta u_3, z_3\right) dt + \varepsilon_4\left(\overrightarrow{\delta u}\right) \left\|\overrightarrow{\delta u}\right\|_Q \qquad (33)$$
where $\varepsilon_i\left(\overrightarrow{\delta u}\right) \longrightarrow 0$, $(i = 2,3,4)$ and $\left\|\overrightarrow{\delta u}\right\|_0 \longrightarrow 0$ as $\overrightarrow{\delta u} \longrightarrow 0$.

Subtracting ((31)-(33)) from ((28) - (30)) respectively, then add the obtained equations to get
$$\int_0^T \left[\left(f_{1u_1} \delta u_1, z_1\right) + \left(f_{2u_2} \delta u_2, z_2\right) + (f_{3u_3} \delta u_2, z_2)\right] dt + \varepsilon_6\left(\overrightarrow{\delta u}\right) \left\|\overrightarrow{\delta u}\right\|_Q =$$
$$\int_0^T \left[\left(g_{1u_1} \delta y_1\right) + \left(g_{2u_2} \delta y_2\right) + (g_{3u_3} \delta y_3)\right] dt \qquad (34)$$
where $\varepsilon_5\left(\overrightarrow{\delta u}\right) = \varepsilon_2\left(\overrightarrow{\delta u}\right) + \varepsilon_3\left(\overrightarrow{\delta u}\right) + \varepsilon_4\left(\overrightarrow{\delta u}\right) \to 0, as \left\|\overrightarrow{\delta u}\right\|_Q \to 0$

Now, by substituting (34) in (21), one has
$$G\left(\vec{u} + \overrightarrow{\delta u}\right) - G(\vec{u}) = \int_Q \left[\left(z_1 f_{1u_1} + g_{1u_1}\right) \delta u_1 + \left(z_2 f_{2u_2} + g_{2u_2}\right) \delta u_2\right] dx dt$$
$$+ \int_Q (z_3 f_{3u_3} + g_{3u_3}) \delta u_3 dx dt + \varepsilon_6\left(\overrightarrow{\delta u}\right) \left\|\overrightarrow{\delta u}\right\|_Q \qquad (35)$$
Where $\varepsilon_6\left(\overrightarrow{\delta u}\right) = \varepsilon_1\left(\overrightarrow{\delta u}\right) + \varepsilon_5\left(\overrightarrow{\delta u}\right) \to 0, as \left\|\overrightarrow{\delta u}\right\|_Q \to 0$

Using the Fréchet Derivative of G and from (35), it yields to
$$\left(\acute{G}(\vec{u}), \overrightarrow{\delta u}\right) = \int_Q \begin{pmatrix} z_1 f_{1u_1} + g_{1u_1} \\ z_2 f_{2u_2} + g_{2u_2} \\ z_3 f_{3u_3} + g_{3u_3} \end{pmatrix} . \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix} dx \ .$$

4. The necessary and sufficient conditions:

In this section we state and prove of the necessary conditions theorem and sufficient conditions theorem under some additional assumptions.

Theorem (4.1): The necessary conditions:

(i) With Assumptions (A), (B),(C), if $\vec{u} \in \overrightarrow{W}_A$is a CCTOCV, then there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ *with* $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$ such that the following Kuhn-Tucker-Lagrange (TKL) conditions hold:
$$\int_Q H_{\vec{u}}(x, t, \vec{y}, \vec{z}, \vec{u}) \overrightarrow{\delta u} \, dx dt \geq 0, , \forall \vec{w}, \overrightarrow{\delta u} = \vec{w} - \vec{u} \qquad (36a)$$
where $g_i = \sum_{l=0}^{2} \lambda_l g_{li}$and $z_i = \sum_{l=0}^{2} \lambda_l z_{li}$ $(\forall i = 1,2,3)$
$$\lambda_2 G_2(\vec{u}) = 0, \qquad (36b)$$
(ii) Minimum Weak form : (36a) is equivalent to the following minimum weak form:

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u}(t) = min_{\vec{w}\in\vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{w}\ a.e.$ on Q (37)

Proof: (i) For each $l = 0,1,2$, and from Lemma 2.1, the functional $G_l(\vec{u})$ is continuous and from theorem 3.2, the functional $\acute{G}_l$ is continuous w.r.t. $\vec{u}$ and linear in $\vec{u}$, then $\acute{G}_l$ is M −differential for every M, hence by utilizing theorem (2.4), there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$, such that (36 a and b) are held, by utilizing the result of theorem 3.2, then (36a) gives

$\sum_{i=1}^{3} \int_Q \left[(\lambda_0 z_{0i} + \lambda_1 z_{1i} + \lambda_2 z_{2i}) f_{iu_i}\right] \delta u_i dxdt$

$+ \sum_{i=1}^{3} \int_Q \left[(\lambda_0 g_{0iu_i} + \lambda_1 g_{1iu_i} + \lambda_2 g_{2iu_i})\right] \delta u_i dxdt \geq 0$

$\Rightarrow \sum_{i=1}^{3} \int_Q \left[(z_i f_{iu_i} + g_{iu_i})\right] \delta u_i dxdt \geq 0$, (38)

where $g_i = \sum_{l=0}^{2} \lambda_l g_{li}$, and $z_i = \sum_{l=1}^{3} \lambda_l z_{li}$, $\forall i = 1,2,3$

ii) Now, let $\{\vec{w}_k\}$ be a dense sequence in $\vec{W}$, and $q \subset Q$ be a measurable set "with *Lebesgue* measure μ" s.t. $\vec{w}(x,t) = \begin{cases} \vec{w}_k(x,t)\ , if\,(x,t) \in q \\ \vec{u}(x,t)\ ,\ if\,(x,t) \notin q \end{cases}$.

So (38) becomes

$$\int_q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\,(\vec{w}_k - \vec{u}) \geq 0, \forall q \quad (39)$$

Or becomes

$$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w}_k - \vec{u}) \geq 0,\ a.e.\,in\ Q \quad (40)$$

It means this inequality holds in $Q - Q_k$, with $\mu(Q_K) = 0$, $for\ all\ k$, thus it holds in $Q/\cup_k Q_k$, with $\mu(\cup_k Q_k) = 0$. From the density of $\{\vec{w}_k\}$ in $\vec{W}$, there exists $\vec{w} \in \vec{W}$ such that

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w} - \vec{u}) \geq 0$, a.e. in Q

$\Rightarrow H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = min_{\vec{w}\in\vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{w}$, a.e. in Q.

Conversely, let

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = min_{\vec{w}\in\vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{w}$, a.e. in Q

$\Rightarrow H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w} - \vec{u}) \geq 0, \forall \vec{w} \in \vec{W}$, a.e. in Q

$\Rightarrow \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\,\overrightarrow{\delta u}\ dxdt \geq 0, \forall \vec{w} \in \vec{W}$.

Theorem (4.2) : The Sufficient conditions:

Suppose that the Assumptions(A,B,C) are held, $f_i$and g_{1i} for each $i = 1,2,3$ that are affine w.r.t. $(y_i,u_i)$for each (x,t), and g_{0i} , g_{2i} are convex w.r.t. (y_i,u_i) for each (x,t). Then the NCOs in Theorem (4.1) with $\lambda_0 > 0$ are also SCOs.

Proof: Suppose $\vec{u} \in \vec{W}_A$ is satisfied the TKL condition, i.e.

$\int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\delta u}\ dxdt \geq 0, \forall \vec{w} \in \vec{W}$.

$\lambda_2 G_2(\vec{u}) = 0$

Let $G(\vec{u}) = \sum_{l=0}^{2} \lambda_l G_l\,(\vec{u})$, then from theorem 3.2

$$\begin{aligned}\acute{G}(\vec{u}).\overrightarrow{\delta u} &= \sum_{l=0}^{2} \lambda_l \acute{G}_l\,(\vec{u}).\overrightarrow{\delta u}\\ &= \lambda_0 \int_Q \sum_{i=1}^{3} (z_{0i} f_{iu_i} + g_{0iu_i})\,\delta u_i dxdt +\\ &\quad \lambda_1 \int_Q \sum_{i=1}^{3} (z_{1i} f_{iu_i} + g_{1iu_i})\,\delta u_i dxdt +\\ &\quad \lambda_2 \int_Q \sum_{i=1}^{3} (z_{2i} f_{iu_i} + g_{2iu_i})\,\delta u_i dxdt\\ &= \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\delta u}\ dxdt \geq 0\end{aligned}$$

Now, consider the first three functions in the R.H.S. of the TSVEs (1-3) are affine w.r.t. (y_i,u_i), $\forall (x,t) \in Q$, for $i = 1,2,3$ resp., i.e.

$f_i(x,t,y_i,u_i) = f_{i1}(x,t)y_i + f_{i2}(x,t)u_i + f_{i3}(x,t)$, $\forall i = 1,2,3$.

Let $\vec{u} = (u_1,u_2,u_3)$& $\vec{\bar{u}} = (\bar{u}_1,\bar{u}_2,\bar{u}_3)$ are two given CCTCVs and then by Theorem (2.1), $\vec{y} = \left(y_{u_1},y_{u_2},y_{u_3}\right) = (y_1,y_2,y_3)$& $\vec{\bar{y}}$=$(\bar{y}_{\bar{u}_1},\bar{y}_{\bar{u}_2},\bar{y}_{\bar{u}_3}) = (\bar{y}_1,\bar{y}_2,\bar{y}_3)$ are their corresponding

solutions, i.e. for the first components $y_1$and $\bar{y}_1$, we have
$y_{1t} - \Delta y_1 + y_1 - y_2 - y_3 = f_{11}(x,t)y_1 + f_{12}(x,t)u_1 + f_{13}(x,t)$
$y_1(x,0) = y_1^0(x)$
$\bar{y}_{1t} - \Delta\bar{y}_1 + \bar{y}_1 - \bar{y}_2 - \bar{y}_3 = f_{11}(x,t)\bar{y}_1 + f_{12}(x,t)\bar{u}_1 + f_{13}(x,t)$
$\bar{y}_1(x,0) = y_1^0(x)$
By multiplying the 1st equation and its initial condition by $\alpha \in [0,1]$, and the 2nd one and its initial condition by$(1-\alpha)$, then add the obtained equations and their initial conditions, yield to
$(\alpha y_1 + (1-\alpha)\bar{y}_1)_t - \Delta(\alpha y_1 + (1-\alpha)\bar{y}_1) + (\alpha y_1 + (1-\alpha)\bar{y}_1) - (\alpha y_2 + (1-\alpha)\bar{y}_2) - (\alpha y_3 + (1-\alpha)\bar{y}_3) =$
$f_{11}(x,t)(\alpha y_1 + (1-\alpha)\bar{y}_1) + f_{12}(x,t)(\alpha u_1 + (1-\alpha)\bar{u}_1) + f_{13}(x,t)$ (41a)
$\alpha y_1(x,0) + (1-\alpha)\bar{y}_1)(x,0) = y_1^0(x)$ (41b)
Using the same steps for the other two components to obtain
$(\alpha y_2 + (1-\alpha)\bar{y}_2)_t - \Delta(\alpha y_2 + (1-\alpha)\bar{y}_2) + (\alpha y_2 + (1-\alpha)\bar{y}_2) + (\alpha y_3 + (1-\alpha)\bar{y}_3) + (\alpha y_1 + (1-\alpha)\bar{y}_1) =$
$f_{21}(x,t)(\alpha y_2 + (1-\alpha)\bar{y}_2) + f_{22}(x,t)(\alpha u_2 + (1-\alpha)\bar{u}_2) + f_{23}(x,t)$ (42a)
$\alpha y_2(x,0) + (1-\alpha)\bar{y}_2)(x,0) = y_2^0(x)$ (42b)
$(\alpha y_3 + (1-\alpha)\bar{y}_3)_t - \Delta(\alpha y_3 + (1-\alpha)\bar{y}_3) + (\alpha y_3 + (1-\alpha)\bar{y}_3) + (\alpha y_1 + (1-\alpha)\bar{y}_1) - (\alpha y_2 + (1-\alpha)\bar{y}_2) =$
$f_{31}(x,t)(\alpha y_3 + (1-\alpha)\bar{y}_3) + f_{32}(x,t)(\alpha u_3 + (1-\alpha)\bar{u}_3) + f_{33}(x,t)$ (43a)
$\alpha y_3(x,0) + (1-\alpha)\bar{y}_3)(x,0) = y_3^0(x)$ (43b)
From equations (41-43)**,** we get that the CCTCV $\vec{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, with $\vec{\tilde{u}} = \alpha\vec{u} + (1-\alpha)\vec{\bar{u}}$ has the corresponding solutions, $\vec{\tilde{y}} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$, $\vec{\tilde{y}} = \alpha\vec{y} + (1-\alpha)\vec{\bar{y}}$,i.e.
$\tilde{y}_{1t} - \Delta\tilde{y}_1 + \tilde{y}_1 - \tilde{y}_2 - \tilde{y}_3 = f_{11}(x,t)\tilde{y}_1 + f_{12}(x,t)\tilde{u}_1 + f_{13}(x,t)$
$\tilde{y}_1(x,0) = y_1^0(x)$
$\tilde{y}_{2t} - \Delta\tilde{y}_2 + \tilde{y}_2 + \tilde{y}_3 + \tilde{y}_1 = f_{21}(x,t)\tilde{y}_2 + f_{22}(x,t)\tilde{u}_2 + f_{23}(x,t)$
$\tilde{y}_2(x,0) = y_2^0(x)$
$\tilde{y}_{3t} - \Delta\tilde{y}_3 + \tilde{y}_3 + \tilde{y}_1 - \tilde{y}_2 = f_{31}(x,t)\tilde{y}_3 + f_{32}(x,t)\tilde{u}_3 + f_{33}(x,t)$
$\tilde{y}_3(x,0) = y_3^0(x)$
Thus the operator $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is convex – linear (CL) w.r.t $(\vec{y},\vec{u})$ for each (x,t).
Also, since $g_{1i}(x,t,y_i,u_i)$ is affine w.r.t. (y_i,u_i) for each $i = 1,2,3, \forall(x,t) \in Q$,i.e.
$g_{1i}(x,t,y_1,u_1) = h_{1i}(x,t)y_i + h_{2i}(x,t)u_i + h_{3i}(x,t)$
Since $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is CL, then
$G_1(\vec{u} + (1-\alpha)\vec{\bar{u}})$
$= \sum_{i=1}^3 \left[\int_Q g_{1i}\big(x,t,y_{i(au_i+(1-\alpha),\bar{u}_i)}, au_i + (1-\alpha),\bar{u}_i\big)\,dxdt\right]$
$= \sum_{i=1}^3 \int_Q \big\{h_{1i}(x,t)y_{i(au_i+(1-\alpha),\bar{u}_i)} + h_{2i}(x,t)(au_i + (1-\alpha),\bar{u}_i + h_{3i}(x,t)\big\}\,dxdt$
$= \sum_{i=1}^3 \int_Q \{h_{1i}(x,t)(ay_i + (1-\alpha),\bar{y}_i) + h_{2i}(x,t)(au_i + (1-\alpha),\bar{u}_i + h_{3i}(x,t)\}\,dxdt$
$= \alpha\sum_{i=1}^3 \int_Q [h_{1i}(x,t)y_i + h_{2i}(x,t)u_i + h_{3i}(x,t)]\,dxdt +$
$(1-\alpha)\sum_{i=1}^3 \int_Q [h_{1i}(x,t)\bar{y}_i + h_{2i}(x,t)\bar{u}_i + h_{3i}(x,t)]\,dxdt$
$= \alpha G_1(\vec{u}) + (1-\alpha)G_1(\vec{\bar{u}})$
$G_1(\vec{u})$ is CL w.r.t $(\vec{y},\vec{u}), \forall(x,t) \in Q$.
Since g_{0i} &g_{2i},are convex w.r.t. $(y_i,u_i), \forall(x,t) \in Q$, then $G_0(\vec{u})$& $G_2(\vec{u})$ are convex w.r.t. $(\vec{y},\vec{u}), \forall(x,t) \in Q$ from the assumptions on the functions g_{0i} *and* g_{2i} and since the sum of integrals of convex function is also convex). Then $G(\vec{u})$ is convex w.r.t. $(\vec{y},\vec{u}), \forall(x,t) \in Q$, in the convex set $\vec{W}$, has a continuous Fréchet Derivative (by theorem 3.2) and satisfies $\acute{G}(\vec{u})\overrightarrow{\delta u} \geq 0$ this implies it has a minimum at $\vec{u}$, i.e.

$G(\vec{u}) \le G(\vec{W}), \forall \vec{w} \in \vec{W} \Rightarrow$

$\sum_{l=0}^{2} \lambda_l G_l(\vec{u}) \le \sum_{l=0}^{2} \lambda_l G_l(\vec{w})$, , $\forall \vec{w} \in \vec{W}$

Let $\vec{w} \in \vec{W}_A$, with $\lambda_2 \ge 0$, then from (36b), the above inequality led to

$\lambda_0 G_0(\vec{u}) \le \lambda_0 G_0(\vec{w})$, , $\forall \vec{w} \in \vec{W} \Rightarrow G_0(\vec{u}) \le G_0(\vec{w})$, , $\forall \vec{w} \in \vec{W}$. Therefore $\vec{u}$ is a CCTOCV.

5. Conclusions: The CCTOCP controlling by the TNLPBVP with the SVCs is studied. The existence theorem for the CCTOCV with the SVCs is stated and proved under suitable conditions. The mathematical formulation of the ATHBVP associated with the TNLPBVP is discovered. The Fréchet Derivative of the Hamiltonian is derived. The theorem of the NCOs for OP and the theorem of the SCOs for the OP of the TNLPBVP with the SVCs under suitable conditions are stated and prove.

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