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Mixed Optimal Control Vector for a Boundary Value Problem of Couple Nonlinear Elliptic Equations

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Abstract

In this research, we study the classical continuous Mixed optimal control vector problem dominated by couple nonlinear elliptic PDEs. The existence theorem for the unique state vector solution of the considered couple nonlinear elliptic PDEs for a given continuous classical mixed control vector is stated and proved by applying the Minty-Browder theorem under suitable conditions. Under suitable conditions, the existence theorem of a classical continuous mixed optimal control vector associated with the considered couple nonlinear elliptic PDEs is stated and proved.

Keywords: State Vector Solution, Mixed Optimal Control Vector, Couple Nonlinear Elliptic Boundary Value Problem.

متجه سيطرة امثلية مختلطة لمسالة قيمة حدوية لزوج من المعادلات اللاخطية الاهليجية

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الخلاصة

في هذا البحث تمت دراسة مسالة السيطرة الامثلية المختلطة المستمرة مسيطرة بزوج من المعادلات التفاضلية غير الخطية الاهليجية. تم ببرهان مبرهنة وجود ووحدانية الحل لمتجه الحالة لزوج من المعادلات التفاضلية غير الخطية الاهليجية عندما يكون متجهة السيطرة التقليدية المختلطة معلوما" باستخدام مبرهنة منتي – بروادر بوجود شروط مناسبة. كذلك بوجود شروط معينة تم وبرهان مبرهنة وجود سيطرة امثلية تقليدية مختلطة مستمرة لمسالة السيطرة الامثلية التقليدية المختلطة المستمرة والمسيطرة بزوج من المعادلات التفاضلية غير الخطية الاهيليجية.

1. Introduction

The subject of control theory has wide applications for many real life problems, in particular in science and engineering, for example, robotics [1], electric power [2], civil

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engineering [3], Aeronautics and Astronautics [4], medicine [5], economic [6], heat conduction [7] and biology [8].
In the field of mathematical sciences, optimal control problems are usually formulated in general either as ODEs [9] or as PDEs [10]. In the last two decades, many authors are interesting to investigate the continuous classical optimal control problems expended to deal with more general types of PDEs as the studies of the coupled nonlinear of elliptic, parabolic and hyperbolic PDEs [11-13]. While, others interested in studying these three kinds of PDEs, which involve the boundary optimal control, see [14-16]. All these studies and the study of the classical continuous mixed optimal control vector problem dominated by parabolic PDEs [17] encourage us to interest about such problem.
This work is concerned at first, with the proof of the existence and uniqueness theorem of the state vector solution of the coupled nonlinear elliptic PDEs for a given control vector using the Minty- Browder theorem under suitable conditions. Second, the continuity of the Lipschitz operator between the state vector solutions and their corresponding control vector is proved. Finally, the existence theorem of a mixed optimal control vector dominating for the considered PDEs is developed and proved.

2. Description of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Then the problem is considered as:
The state vector equations are assumed as follows:

$$A_1 y_1 + a_0(x)y_1 - b(x)y_2 + f_1(x,\vec{y},u_1) = f_2(x,u_1), \text{ in } \Omega \quad (1)$$

$$A_2 y_2 + b_0(x)y_2 + b(x)y_1 + h_1(x,\vec{y}) = h_2(x), \text{ in } \Omega \quad (2)$$

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial y_1}{\partial n} = 0 \text{ , in } \Gamma \quad (3)$$

$$\sum_{i,j=1}^{n} b_{ij}\frac{\partial y_2}{\partial n} = u_2 \text{ , in } \Gamma \quad (4)$$

with $A_1 y_1 = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial y_1}{\partial x_i}\right)$, $A_2 y_2 = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(b_{ij}(x)\frac{\partial y_2}{\partial x_i}\right)$, where $a_0(x)$, $b_0(x)$, $b(x)$, $a_{ij}(x)$, $b_{ij}(x) \in C^\infty(\Omega)$ and $\vec{u} = (u_1,u_2) = \big(u_1(x),u_2(x)\big) \in L^2(\Omega)\times L^2(\Gamma)$ is the mixed control vector, $\vec{y} = (y_1,y_2) = \big(y_1(x),y_2(x)\big) \in \big(H^1(\Omega)\big)^2$ is its state vector solution and $(f_1,h_1) = \big(f_1(x,\vec{y},u_1),h_1(x,\vec{y})\big) \in \big(L^2(\Omega)\big)^2$, $(f_2,h_2) = \big(f_2(x,u_1),h_2(x)\big) \in \big(L^2(\Omega)\big)^2$ are a vector of given functions for all $x \in \Omega$.
The set of admissible control $\vec{W} \subset L^2(\Omega)\times L^2(\Gamma)$ is
$\vec{W} = \{\vec{u} \in L^2(\Omega)\times L^2(\Gamma) | \vec{u} \in U_1 \times U_2 = \vec{U} \subset \mathbb{R}^2 \text{ a.e. in } \Omega\times\Gamma\}$, where $\vec{U}$ is a convex set.
The cost functional has to be minimized is given by

$$G_0(\vec{u}) = \iint_\Omega [g_{01}(x,\vec{y},u_1)]dx_1dx_2 + \int_\Gamma [g_{02}(x,u_2)]d\gamma \quad (5)$$

The mixed optimal control vector problem is to minimize (5) subject to $\vec{u} \in \vec{W}$.
Let $\vec{V} = V\times V = H^1(\Omega)\times H^1(\Omega)$. We denote $(v,v)_\Omega((v,v)_\Gamma)$ and $\|v\|_{L^2(\Omega)}(\|v\|_{L^2(\Gamma)})$ the inner product and the norm in $L^2(\Omega)\big(L^2(\Gamma)\big)$, respectively. The $(v,v)$and $\|v\|_{H^1(\Omega)}$ are the inner product and the norm in $H^1(\Omega)$, respectively. While, the inner product and the norm in $L^2(\Omega)\times L^2(\Omega)$ are denoted by $(\vec{v},\vec{v})_\Omega = \sum_{i=1}^{2}(v_i,v_i)$ and $\|\vec{v}\|_{(L^2(\Omega))^2} = \sum_{i=1}^{2}\|v_i\|_{L^2(\Omega)}$, respectively. We also denote the inner product and the norm in $\vec{V}$ by $(\vec{v},\vec{v}) = \sum_{i=1}^{2}(v_i,v_i)$ and $\|\vec{v}\|_{(H^1(\Omega))^2} = \sum_{i=1}^{2}\|v_i\|_{H^1(\Omega)}$, respectively.

3. Existence of the unique state vector solution:

The weak form of (1- 4) is obtained by multiplying both sides of (1- 2) by $v_1, v_2 \in V$, integrating both sides and then using Green's theorem in Hilbert space for the terms which have the second derivatives, once gets.

$$a_1(y_1,v_1) + (a_0y_1,v_1)_\Omega - (by_2,v_1)_\Omega + (f_1(\vec{y},u_1),v_1)_\Omega = (f_2(u_1),v_1)_\Omega, \forall v_1 \in V_1 \quad (6)$$

and

$a_2(y_2, v_2) + (b_0 y_2, v_2)_\Omega + (b y_1, v_2)_\Omega + (h_1(\vec{y}), v_2)_\Omega = (h_2, v_2)_\Omega + (u_2, v_2)_\Gamma, \forall v_2 \in V_2$ (7)
We add these two equations to get the following
$a(\vec{y}, \vec{v}) + (f_1(\vec{y}, u_1), v_1)_\Omega + (h_1(\vec{y}), v_2)_\Omega$
$= (f_2(u_1), v_1)_\Omega + (h_2, v_2)_\Omega + (u_2, v_2)_\Gamma, \quad \forall (v_1, v_2) \in \vec{V}$ (8)
where $a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + (a_0 y_1, v_1)_\Omega - (b y_2, v_1)_\Omega + a_2(y_2, v_2) + (b_0 y_2, v_2)_\Omega + (b y_1, v_2)_\Omega$
with $a_1(y_1, v_1) = \sum_{i,j=1}^n a_{ij} \frac{\partial y_1}{\partial x_i} . \frac{\partial v_1}{\partial x_j}$, $a_2(y_2, v_2) = \sum_{i,j=1}^n b_{ij} \frac{\partial y_2}{\partial x_i} . \frac{\partial v_2}{\partial x_j}$,
$a_i(y_i, y_i) \geq c_i \|y_i\|^2_{H^1(\Omega)}$, where$c_i > 0$, $i = 1,2$
$|a_i(y_i, v_i)| \leq \bar{c}_i \|y_i\|_{H^1(\Omega)} \|v_i\|_{H^1(\Omega)}$, where $\bar{c}_i > 0$, $i = 1,2$.
The following assumptions are useful to prove the existence theorem of a unique solution of

Assumptions (A):

a) $a(\vec{y}, \vec{v})$ is coercive, i.e. $\frac{a(\vec{y}, \vec{y})}{\|\vec{y}\|_{(H^1(\Omega))^2}} \geq c \|\vec{y}\|_{(H^1(\Omega))^2} > 0$, $\forall \vec{y} \in \vec{V}$.

b) $|a(\vec{y}, \vec{v})| \leq \ell_1 \|\vec{y}\|_{(H^1(\Omega))^2} \|\vec{v}\|_{(H^1(\Omega))^2}$, $\ell_1 > 0$, $\forall \vec{y}, \vec{v} \in \vec{V}$.

c) The functions $f_1(x, \vec{y}, u_1)$ and $h_1(x, \vec{y})$ are of Carathéodory type on $\Omega \times \mathbb{R}^2 \times U_1$ and $\Omega \times \mathbb{R}^2$ respectively. and the following conditions for $\phi_1, \phi_2 \in L^2(\Omega)$ and $\bar{c}_1, \bar{\bar{c}}_1, \bar{c}_2 \geq 0$ are satisfied:

$|f_1(x, \vec{y}, u_1)| \leq \phi_1(x) + \bar{c}_1 |\vec{y}| + \bar{\bar{c}}_1 |u_1|, \quad |h_1(x, \vec{y})| \leq \phi_2(x) + \bar{c}_2 |\vec{y}|$.

d) $f_1(x, \vec{y}, u_1)$ and $h_1(x, \vec{y})$ are monotonic functions for all $x \in \Omega$ w.r.t $(\vec{y}, u_1)$ and $\vec{y}$ respectively., with

$f_1(x, 0, u_1) = 0, \forall (x, u_1) \in \Omega \times U_1$, $h_1(x, 0) = 0, \forall x \in \Omega$.

e) The functions $f_2(x, u_1)$ *and* $h_2(x)$ are of "Carathéodory type" on $\Omega \times U_1$ and satisfy for $\phi_3, \phi_4 \in L^2(\Omega)$, and $\bar{c}_1 \geq 0$

$|f_2(x, u_1)| \leq \phi_3(x) + \bar{c}_1 |u_1|, \forall (x, u_1) \in \Omega \times U_1$ and $|h_2(x)| \leq \phi_4(x), \forall x \in \Omega$.

Proposition 3.1[16]: Let $f: \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of "Carathéodory type ", let F be a functional such that $F(y) = \int_\Omega f(x, y(x)) dx$, where Ω is a measurable subset of $\mathbb{R}^n$, and suppose that
$\|f(x, y)\| \leq \zeta(x) + \eta(x) \|y\|^\alpha$, $\forall (x, y) \in \Omega \times \mathbb{R}^n, y \in L^P(\Omega \times \mathbb{R}^n)$
where $\zeta \in L^1(\Omega \times \mathbb{R}), \eta \in L^{\frac{P}{P-\alpha}}(\Omega \times \mathbb{R})$ and $\alpha \in [1, P]$, if $P \in [1, \infty)$ and $\eta \equiv 0$, if $P = \infty$, then F is continuous on $L^P(\Omega \times \mathbb{R}^n)$.

Theorem 3.1 (Minty-Browder) [15]: Let V be a reflexive Banach space, and $A: V \longrightarrow V^*$ be a continuous nonlinear map such that
$\langle A v_1 - A v_2, v_1 - v_2 \rangle > 0, \forall v_1, v_2 \in V, v_1 \neq v_2$ and $\lim_{\|v\|_{H^1(\Omega)} \to \infty} \frac{\langle A v, v \rangle}{\|v\|_{H^1(\Omega)}} = \infty$.
Then for all $f \in V^*$ there exists a unique solution $y \in V$ of the equation $Ay = f$.

Theorem 3.2: In addition to assumptions A-(a and d). If one of the functions f_1 or h_1 in assumptions A(d) is strictly a monotone function, Then for each given $\vec{u} \in \vec{W}$, the (8) has a unique solution $\vec{y} \in \vec{V}$.

Proof: Let $\bar{A}: \vec{V} \longrightarrow \vec{V}^*$, then (8) can rewrite as follows:

$\langle \bar{A}(\vec{y}), \vec{v} \rangle = (\vec{F}(\vec{u}), \vec{v})$ (9)

where $\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, \vec{y}, u_1), v_1)_\Omega + (h_1(x, \vec{y}), v_2)_\Omega$, and
$(\vec{F}(\vec{u}), \vec{v}) = (f_2(x, u_1), v_1)_\Omega + (h_2(x), v_2)_\Omega + (u_2, v_2)_\Gamma$
i) From assumptions A-(a and d), $\bar{A}$ is coercive.
ii) From assumptions A-(b and c) with applying Proposition 3.1, the mapping$\vec{y} \longmapsto \langle \bar{A}(\vec{y}), \vec{v} \rangle$ is continuous w.r.t. $\vec{y}$.
iii) From assumptions A-(a and d) with applying part (i), $\bar{A}$ is strictly monotone w.r.t. $\vec{y}$.

The uniqueness of the state vector solution $\vec{y} \in \vec{V}$ of (9) is obtained from applying Theorem (3.1).

4. Existence of the continuous classical mixed control vector

This section deals with the state and proof of the existence theorem will be done under suitable assumptions. The following lemmas and assumptions are necessary for the proof of the existence theorem.

Lemma 4.1: In addition to the assumptions (A), if the functionsf_1, f_2 are Lipschitz w.r.t. u_1 and h_1 is Lipschitz w.r.t. $\vec{y}$, the function h_2 is bounded. Then the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ from $\vec{W}$ to $L^2(\Omega) \times L^2(\Gamma)$, is Lipschitz continuous, i.e.

$\|\overrightarrow{\Delta y}\|_{(L^2(\Omega))^2} \leq L\|\overrightarrow{\Delta u}\|_{L^2(\Omega)\times L^2(\Gamma)}$, with $L > 0$.

Proof: Let $\vec{u}, \vec{u}' \in \vec{W}$ are two given mixed control vectors, then by Theorem 3.1. $\vec{y}$ and$\vec{y}'$ are the state vector solutions of (8). Subtracting these two weak forms one from the other, setting $\overrightarrow{\Delta y} = \vec{y}' - \vec{y}$ and $\overrightarrow{\Delta u} = \vec{u}' - \vec{u}$, with $\vec{v} = \overrightarrow{\Delta y}$, then adding the obtained equations to get

$$\begin{aligned} &a_1(\Delta y_1, \Delta y_1) + (a_0 \Delta y_1, \Delta y_1)_\Omega + a_2(\Delta y_2, \Delta y_2) + (b_0 \Delta y_2, \Delta y_2)_\Omega \\ &+(f_1(x, \vec{y} + \Delta\vec{y}, u_1 + \Delta u_1) - f_1(x, \vec{y}, u_1), \Delta\vec{y})_\Omega + (h_1(x, \vec{y} + \Delta\vec{y}) - h_1(x, \vec{y}), \Delta\vec{y})_\Omega \\ &= (f_2(x, u_1 + \Delta u_1) - f_2(x, u_1), \Delta y_1)_\Omega + (\Delta u_2, \Delta y_2)_\Gamma \end{aligned} \tag{10}$$

Applying assumptions A-(a, d) in (10), then taking the absolute value for both sides to get

$$c\|\overrightarrow{\Delta y}\|^2_{(H^1(\Omega))^2} \leq |(f_2(x, u_1 + \Delta u_1) - f_2(x, u_1), \Delta y_1)_\Omega| + |(\Delta u_2, \Delta y_2)_\Gamma| \tag{11}$$

Using the Lipschitz property, the Cauchy-Schwartz inequality and then the trace operator to get

$$c\|\overrightarrow{\Delta y}\|^2_{(H^1(\Omega))^2} \leq 2c_1\|\overrightarrow{\Delta u}\|_{L^2(\Omega)\times L^2(\Gamma)}\|\overrightarrow{\Delta y}\|_{(H^1(\Omega))^2}$$

$$\Rightarrow \|\overrightarrow{\Delta y}\|_{(H^1(\Omega))^2} \leq c_2\|\overrightarrow{\Delta u}\|_{L^2(\Omega)\times L^2(\Gamma)}, \text{ where } c_2 = \frac{2c_1}{c} \tag{12}$$

which gives the requirement result

$$\|\overrightarrow{\Delta y}\|_{(L^2(\Omega))^2} \leq L\|\overrightarrow{\Delta u}\|_{L^2(\Omega)\times L^2(\Gamma)}, \text{ where } L = cc_2 \tag{13}$$

Assumptions *(B):*

Assume that g_{01} and g_{02} are of CAT on $\Omega \times \mathbb{R}^2 \times U_1$ and $\Omega \times U_2$, respectively. And the following conditions are satisfied for $\vec{y} \in \mathbb{R}^2$, $\vec{u} \in \vec{U}$ with $\gamma_{01}, \gamma_{02} \in L^1(\Gamma)$ and $c_{01}, \bar{c}_{01}, c_{02} \geq 0$:

$|g_{01}(x, \vec{y}, u_1)| \leq \gamma_{01}(x) + c_{01}\vec{y}^2 + \bar{c}_{01}u_1^2$ and $|g_{02}(x, u_2)| \leq \gamma_{02}(x) + c_{02}u_2^2$

Lemma 4.2: With assumptions (B), the functional $\vec{u} \longmapsto G_0(\vec{u})$ which is defined on $L^2(\Omega) \times L^2(\Gamma)$ is continuous.

Proof: The functional $\iint_\Omega g_{01}(x, \vec{y}, u_1)dx_1dx_2$ and $\int_\Gamma g_{02}(x, u_2)d\gamma$ are continuous on $(L^2(\Omega))^2$ and $(L^2(\Gamma))^2$, respectively. (from Assumptions (B) and by using Proposition 3.1). Hence,

$G_0(\vec{u}) = \iint_\Omega g_{01}(x, \vec{y}, u_1)dx_1dx_2 + \int_\Gamma g_{02}(x, u_2)d\gamma$ is continuous on $L^2(\Omega) \times L^2(\Gamma)$.

Theorem 4.1: In addition to assumptions (A), assume that $\vec{W} \neq \emptyset$, f_1 andh_1 are independent functions of u_1 and u_2 , respectively. The functions f_2 and h_2 are bounded such that $f_1(x, \vec{y}, u_1) = f_1(x, \vec{y}), f_2(x, u_1) = f_2(x)u_1$

$|f_1(x, \vec{y})| \leq \phi_1(x) + \bar{c}_1|\vec{y}|$, where $\phi_1 \in L^2(\Omega)$, $\bar{c}_1 \geq 0$

$|h_1(x, \vec{y})| \leq \phi_2(x) + \bar{c}_2|\vec{y}|$, where $\phi_2 \in L^2(\Omega)$, $\bar{c}_2 \geq 0$

$|f_2(x)| \leq \kappa_1$, and $|h_2(x)| \leq \kappa_2$,

If G_0 is coercive, then there exists a mixed optimal control vector.

Proof: Since$\vec{U}$ is convex and bounded, then $\vec{W}$ is convex and bounded, and it is closed (from Egorov's theorem), hence $\vec{W}$ is weakly compact.

Since $\vec{W} \neq \emptyset$, then there exists $\vec{w} \in \vec{W}$, and a minimum sequence $\{\vec{u}_n\} = \{(u_{1n}, u_{2n})\} \in \vec{W}$ for each n, s. t.:

$\lim_{n\to\infty} G_0(\vec{u}_n) = \inf_{\vec{w}\in\vec{W}} G_0(\vec{w})$.

But $\vec{W}$ is weakly compact, then there exists a subsequence of $\{\vec{u}_n\}$, say again $\{\vec{u}_n\}$ which converges weakly to some point $\vec{u}$ in $\vec{W}$, i.e. $\vec{u}_n \longrightarrow \vec{u}$ weakly in $\left(L^2(\Omega) \times L^2(\Gamma)\right)$.

Then from the proof of Theorem 3.2, corresponding to this sequence $\{\vec{u}_n\}$ there is a sequence of solution $\{\vec{y}_n\}$(with $\|\vec{y}_n\|_{\left(H^1(\Omega)\right)^2}$ is bounded for all$n$), of the sequence:

$$a_1(y_{1n}, v_1) + (a_0 y_{1n}, v_1)_\Omega - (b y_{2n}, v_1)_\Omega + a_2(y_{2n}, v_2) + (b_0 y_{2n}, v_2)_\Omega + (b y_{1n}, v_2)_\Omega$$
$$+(f_1(x, \vec{y}_n), v_1)_\Omega + (h_1(x, \vec{y}_n), v_2)_\Omega = (f_2(x) u_{1n}, v_1)_\Omega + (h_2(x), v_2)_\Omega + (u_{2n}, v_2)_\Gamma \qquad (14)$$

Then by Alaoglu theorem, there exists a subsequence of $\{\vec{y}_n\}$, say again $\{\vec{y}_n\}$ such that $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\vec{V}$.

To prove that (14) converges to

$$a_1(y_1, v_1) + (a_0 y_1, v_1)_\Omega - (b y_2, v_1)_\Omega + a_2(y_2, v_2) + (b_0 y_2, v_2)_\Omega + (b y_1, v_2)_\Omega$$
$$+(f_1(x, \vec{y}), v_1)_\Omega + (h_1(x, \vec{y}), v_2)_\Omega = (f_2(x) u_1, v_1)_\Omega + (h_2(x), v_2)_\Omega + (u_2, v_2)_\Gamma \qquad (15)$$

Let $(v_1, v_2) \in \left(C(\overline{\Omega})\right)^2$, and first for the left hand sides, we have $y_{in} \longrightarrow y_i$ weakly in V_i, i.e. $y_{in} \longrightarrow y_i$ weakly in $L^2(\Omega)$, $\forall i = 1,2$.

Then from the left hand sides of (14), (15) and by using Cauchy- Schwarz inequality, one has

$$|a_1(y_{1n}, v_1) + (a_0 y_{1n}, v_1)_\Omega - (b y_{2n}, v_1)_\Omega + a_2(y_{2n}, v_2) + (b_0 y_{2n}, v_2)_\Omega + (b y_{1n}, v_2)_\Omega$$
$$-a_1(y_1, v_1) - (a_0 y_1, v_1)_\Omega + (b y_2, v_1)_\Omega - a_2(y_2, v_2) - (b_0 y_2, v_2)_\Omega - (b y_1, v_2)_\Omega|$$
$$\leq c_1 \|y_{1n} - y_1\|_{H^1(\Omega)} \|v_1\|_{H^1(\Omega)} + c_2 \|y_{1n} - y_1\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} + c_3 \|y_{2n} - y_2\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)}$$
$$+c_4 \|y_{2n} - y_2\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} + c_5 \|y_{2n} - y_2\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)}$$
$$+c_6 \|y_{1n} - y_1\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)} \qquad (16)$$

From assumptions (B), and Proposition 3.1, the functional $\iint_\Omega f_1(x, \vec{y}_n) v_1 dx_1 dx_2$ and $\iint_\Omega h_1(x, \vec{y}_n) v_2 dx_1 dx_2$ are continuous w.r.t. $\vec{y}_n$. But $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(\Omega)\right)^2$, then by using the compactness theorem in [16], to get $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\left(L^2(\Omega)\right)^2$, then

$(f_1(x, \vec{y}_n), v_1)_\Omega + (h_1(x, \vec{y}_n), v_2)_\Omega \longrightarrow (f_1(x, \vec{y}), v_1)_\Omega + (h_1(x, \vec{y}), v_2)_\Omega$, $\forall (v_1, v_2) \in \left(C(\overline{\Omega})\right)^2$

i.e. the left hand side of (14) $\longrightarrow$ the left hand side of (15) (17a)

Second, since $u_{1n} \longrightarrow u_1$ weakly in $L^2(\Omega)$ and $u_{2n} \longrightarrow u_2$ weakly in $L^2(\Gamma)$, then

$(f_2(x)(u_{1n} - u_1), v_1)_\Omega + (u_{2n} - u_2, v_2)_\Gamma \longrightarrow 0$ (17b)

From (17a) and (17b) give us that (14) converges to (15), and this convergence holds $\forall (v_1, v_2) \in \vec{V}$ (since $\left(C(\overline{\Omega})\right)^2$ is dense in $\vec{V}$) which gives the limit point $\vec{y} = \vec{y}_{\vec{u}}$ is a state " vector " solution of the (15).

Now, from the assumptions on $g_{01}(x, \vec{y}, u_1)$ and Lemma 4.2, the integral $\iint_\Omega g_{01}(x, \vec{y}, u_1) dx_1 dx_2$ is continuous w.r.t $(\vec{y}, u_1)$, and then $\iint_\Omega g_{01}(x, \vec{y}, u_1) dx_1 dx_2$ is weakly lower semicontinuous w.r.t u_1, (since $g_{01}(x, \vec{y}, u_1)$, is convex w.r.t u_1), i.e.

$$\iint_\Omega g_{01}(x, \vec{y}, u_1) dx_1 dx_2 \leq \varliminf_{n\to\infty} \iint_\Omega g_{01}(x, \vec{y}, u_{1n}) dx_1 dx_2$$
$$= \varliminf_{n\to\infty} \iint_\Omega [g_{01}(x, \vec{y}, u_{1n}) - g_{01}(x, \vec{y}_n, u_{1n})] dx_1 dx_2 + \varliminf_{n\to\infty} \iint_\Omega g_{01}(x, \vec{y}_n, u_{1n}) dx_1 dx_2$$
$$= \varliminf_{n\to\infty} \iint_\Omega g_{01}(x, \vec{y}_n, u_{1n}) dx_1 dx_2$$

The same way can be used to get

$$\int_\Gamma g_{02}(x, u_2) d\gamma \leq \varliminf_{n\to\infty} \int_\Gamma g_{02}(x, u_{2n}) d\gamma,$$

Hence, $G_0(\vec{u})$ is weakly lower semicontinuous w.r.t $(\vec{y}, \vec{u})$,

Therefore, the mixed control vector is obtained from

$$G_0(\vec{u}) \leq \varliminf_{n\to\infty} G_0(\vec{u}_n) = \lim_{n\to\infty} G_0(\vec{u}_n) = \inf_{\vec{w}\in\vec{W}} G_0(\vec{w}).$$

Conclusions

In this work, the continuity of the operator between the mixed control vector and the corresponding state vector solution is proved. Moreover, under suitable assumptions, the existence theorem of a continuous classical mixed-optimal control vector dominated by the considered PDEs is stated and proved. It is observed that, under suitable conditions, the Minty-Browder theorem is appropriate to prove the existence of a unique state vector solution of the coupled nonlinear elliptic PDEs, when the continuous classical mixed control vector is known.

References

[1] A. Diveev, E. Sofronova, and I. Zelinka, "Morphological and Optical Properties of CdS Quantum Dots Synthesized with Different pH values," *Mathematics* , vol. 8, pp. 1-18, 2020 .

[2] M. Sechilariu, "Intelligent Energy Management of Electrical Power Systems," *Appl., Sci*., vol 10, pp. 1-4, 2020.

[3] L. Mei and Q. Wang, " Structural Optimization in Civil Engineering: A Literature Review, " *Buildings,* vol 11, pp. 1-27, 2021.

[4] R. Bonalli, "Optimal Control of Aerospace Systems with Control-State Constraints and Delays," Ph. D. thesis Sorbonne University, Applied Mathematics Specialty, France, 2018 .

[5] N. L. Grigorenko, E. V. Grigorieva, P. K. Roi, and E. N. Khailov " Optimal Control Problems for a Mathematical Model of the Treatment of Psoriasis," *Computational mathematics and modeling*, vol. 30, No. 4, pp. 352-363, 2019.

[6] S. Federico, G. Ferrari and L. Regis, *Applications of Stochastic Optimal Control to Economics and Finance, Basel, Switzerland*, 2020.

[7] B. I. Eroğlu, D. Avci, and N. Özdemir, "Optimal Control Problem for a Conformable Fractional Heat Conduction Equation", *International Conference on Computational and Experimental Science and Engineering*, vol.132, No.3, 2017.

[8] J. Kahuru, L. S. Luboobi, and Y. Nkansah-Gyekye, "Optimal Control Techniques on a Mathematical Model for the Dynamics of tungiasis in a Community," *International Journal of Mathematics and Mathematical Sciences*, vol. 2017, No. 19, pp. 1–19, 2017 .

[9] M. Gerdts, *Optimal Control of ODEs and DAEs, De Gruyter, Germany*, 2012.

[10] I. Chryssoverghi and J. Al-Hawasy, "The Continuous Classical Optimal Control Problem of a semi linear Parabolic Equations (CCOCP)," *Journal of Karbala University*, Vol.8, No.3, pp. 57-70, 2010

[11] J. Al-Hawasy and E. Al-Rawdanee, "The Continuous Classical Optimal Control of a Coupled of non-linear Elliptic Equations," *Mathematical Theory and Modeling*, vol.4, No.14, pp. 211-219, 2014

[12] J. Al-Hawasy and G. Kadhem, "The Continuous Classical Optimal Control of a coupled of nonlinear parabolic Equations," *Al-Nahrain Journal of Sciences*, vol.19, No.1, pp. 173-186, 2016.

[13] J. Al-Hawasy, "The Continuous Classical Optimal Control of a Couple Nonlinear Hyperbolic Partial Differential Equations with equality and inequality Constraints," *Iraqi Journal of Science*, vol.57, No.2C, pp. 1528-1538, 2016.

[14] J. Al-Hawasy, and A. Naeif, "The Continuous Classical Boundary Optimal Control of a Couple Non Linear Parabolic Partial Differential Equations," *AL Nahrain Journal of Science*, Conference issue 0(1) pp.123-136, 2018.

[15] J. Al-Hawasy, and S. Al-Qaisi, "The solvability of the continuous classical boundary optimal control of couple nonlinear elliptic partial differential equations with state constraints"., *Al-Mustansiriyah Journal of Science*, Vol. 30, Issue 1,pp. 143-151, 2019.

[16] J. Al-Hawasy, "The Continuous Classical Boundary Optimal Control of Couple Nonlinear Hyperbolic Boundary Value Problem with Equality and Inequality Constraints," *Baghdad Science Journal*, Vol.16, No.4, pp. 1064-1074, 2019.

[17] J. Al-Hawasy, G. Kadhem, and A. Naeif, "Necessary conditions for optimal boundary-control of couple linear parabolic partial differential equations," *IOP Conference Series: Materials Science and Engineering*,Vol.871, pp. 1-12, 2020.

[18] R. Temam, *Navier-Stokes Equations, North-Holand Publishing Company*, 1977.