Mixed Optimal Control Vector for a Boundary Value Problem of Couple Nonlinear Elliptic Equations

Safaa Juma Al-Qaisi 1, Ghufran M Kadhem 2, Jamil Amir Al-Hawasy 3*

1Diyala Directorate of Education, Diyala-Iraq
2Babylonian Directorate of Education, Babylon-Iraq
3Department of Mathematics, College of Science, Mustansiriyah University, Baghdad-Iraq

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Abstract
In this research, we study the classical continuous Mixed optimal control vector problem dominated by couple nonlinear elliptic PDEs. The existence theorem for the unique state vector solution of the considered couple nonlinear elliptic PDEs for a given continuous classical mixed control vector is stated and proved by applying the Minty-Browder theorem under suitable conditions. Under suitable conditions, the existence theorem of a classical continuous mixed optimal control vector associated with the considered couple nonlinear elliptic PDEs is stated and proved.

Keywords: State Vector Solution, Mixed Optimal Control Vector, Couple Nonlinear Elliptic Boundary Value Problem.

1. Introduction
The subject of control theory has wide applications for many real life problems, in particular in science and engineering, for example, robotics [1], electric power [2], civil
In the field of mathematical sciences, optimal control problems are usually formulated in general either as ODEs [9] or as PDEs [10]. In the last two decades, many authors are interesting to investigate the continuous classical optimal control problems expended to deal with more general types of PDEs as the studies of the coupled nonlinear of elliptic, parabolic and hyperbolic PDEs [11-13]. While, others interested in studying these three kinds of PDEs, which involve the boundary optimal control, see [14-16]. All these studies and the study of the classical continuous mixed optimal control vector problem dominated by parabolic PDEs [17] encourage us to interest about such problem.

This work is concerned at first, with the proof of the existence and uniqueness theorem of the state vector solution of the coupled nonlinear elliptic PDEs for a given control vector using the Minty- Browder theorem under suitable conditions. Second, the continuity of the Lipschitz operator between the state vector solutions and their corresponding control vector is proved. Finally, the existence theorem of a mixed optimal control vector dominating for the considered PDEs is developed and proved.

2. Description of the problem

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with Lipschitz boundary \( \Gamma = \partial \Omega \). Then the problem is considered as:

The state vector equations are assumed as follows:

\[
\begin{align*}
A_1 y_1 + a_0(x)y_1 - b(x)y_2 + f_1(x,\vec{y},u_1) &= f_2(x,u_1), \text{ in } \Omega \\
A_2 y_2 + b_0(x)y_2 + b(x)y_1 + h_1(x,\vec{y}) &= h_2(x), \text{ in } \Omega \\
\sum_{i,j=1}^n a_{ij} \frac{\partial y_1}{\partial x_i} &= 0, \text{ in } \Gamma \\
\sum_{i,j=1}^n b_{ij} \frac{\partial y_2}{\partial x_i} &= u_2, \text{ in } \Gamma
\end{align*}
\]

with

\[
A_1 y_1 = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial y_1}{\partial x_i} \right), \quad A_2 y_2 = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( b_{ij}(x) \frac{\partial y_2}{\partial x_i} \right),
\]

where \( a_0(x), b_0(x), b(x), a_{ij}(x), b_{ij}(x) \in C^\infty(\Omega) \) and \( \vec{u} = (u_1, u_2) = (u_1(x), u_2(x)) \in L^2(\Omega) \times L^2(\Gamma) \) is the mixed control vector, \( \vec{y} = (y_1, y_2) = (y_1(x), y_2(x)) \in (H^1(\Omega))^2 \) is its state vector solution and

\[
(f_1, h_1) = (f_1(x, y_1, u_1), h_1(x, y_1)) \in (L^2(\Omega))^2, \quad (f_2, h_2) = (f_2(x, u_1), h_2(x)) \in (L^2(\Omega))^2
\]

are a vector of given functions for all \( x \in \Omega \).

The set of admissible control \( \vec{W} \subset L^2(\Omega) \times L^2(\Gamma) \) is

\[
\vec{W} = \{ \vec{u} \in L^2(\Omega) \times L^2(\Gamma) | \vec{u} \in U_1 \times U_2 = \vec{U} \subset \mathbb{R}^2 \ \text{a.e. in } \Omega \times \Gamma \},
\]

where \( \vec{U} \) is a convex set.

The cost functional has to be minimized is given by

\[
G_0(\vec{u}) = \int_\Omega [g_{01}(x,\vec{y},u_1)] dx_1 dx_2 + \int_\Gamma [g_{02}(x,u_2)] dy
\]

(5)

The mixed optimal control problem is to minimize (5) subject to \( \vec{u} \in \vec{W} \).

Let \( \vec{V} = V \times V = H^1(\Omega) \times H^1(\Omega) \). We denote \((v, v)|_\Omega((v, v)|_\Gamma\) and \(|v|_{L^2(\Omega)}(|v|_{L^2(\Gamma)})\) the inner product and the norm in \( L^2(\Omega)(L^2(\Gamma)) \), respectively. The \((v, v)\) and \(|v|_{L^2(\Omega)}\) are the inner product and the norm in \( H^1(\Omega) \), respectively. While, the inner product and the norm in \( L^2(\Omega) \times L^2(\Omega) \) are denoted by \((\vec{v}, \vec{v})_0 = \sum_{i=1}^2 (v_i, v_i)\) and \(|\vec{v}|_{L^2(\Omega)}^2 = \sum_{i=1}^2 |v_i|_{L^2(\Omega)}^2\), respectively. We also denote the inner product and the norm in \( \vec{V} \) by \((\vec{v}, \vec{v}) = \sum_{i=1}^2 (v_i, v_i)\) and \(|\vec{v}|_{H^1(\Omega)}^2 = \sum_{i=1}^2 |v_i|_{H^1(\Omega)}^2\), respectively.

3. Existence of the unique state vector solution:

The weak form of (1- 4) is obtained by multiplying both sides of (1- 2) by \( v_1, v_2 \in V \), integrating both sides and then using Green's theorem in Hilbert space for the terms which have the second derivatives, once gets.

\[
a_1(y_1, v_1) + (a_0 y_1, v_1)_\Omega - (b y_2, v_1)_\Omega + (f_1(\vec{y}, u_1), v_1)_\Omega = (f_2(u_1), v_1)_\Omega, \quad \forall v_1 \in V_1
\]

and
We add these two equations to get the following
\[ a(\vec{y}, \vec{v}) + (f_1(x, \vec{y}, u_1), v_1)_{\Omega} + (h_1(\vec{y}), v_2)_{\Omega} = (h_2, v_2)_{\Omega} + (u_2, v_2)_{\Gamma}, \forall v_2 \in V_2 \]
(7)

Proof:
\[ a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + a_0(y_1, v_1) = a_1(y_1, v_1) + a_0(y_1, v_1) \]
where
\[ a_1(y_1, v_1) = \sum_{j=1}^{n} a_{ij} \frac{\partial y_1}{\partial x_i} \frac{\partial v_1}{\partial x_j}, \]
\[ a_0(y_1, v_1) = \sum_{j=1}^{n} b_{ij} \frac{\partial y_2}{\partial x_i} \frac{\partial v_2}{\partial x_j}, \]
with
\[ a_1(y_1, v_1) \geq c_i \| y_1 \|^2_{H^1(\Omega)}, \text{ where } c_i > 0, i = 1, 2 \]
\[ |a_i(y_i, v_i)| \leq \tilde{c}_i \| y_i \|^2_{H^1(\Omega)} \| v_i \|^2_{H^1(\Omega)}, \text{ where } \tilde{c}_i > 0, i = 1, 2. \]

The following assumptions are useful to prove the existence theorem of a unique solution of Assumptions (A):

- a) \( a(\vec{y}, \vec{v}) \) is coercive, i.e. \( \frac{a(\vec{y}, \vec{v})}{\| \vec{y} \|^2_{(H^1(\Omega))^2}} \geq c \| \vec{v} \|^2_{(H^1(\Omega))^2}, \forall \vec{y} \in \vec{V} \).
- b) \| a(\vec{y}, \vec{v}) \| \leq \ell_1 \| \vec{y} \| \| \vec{v} \|_{(H^1(\Omega))^2}, \ell_1 > 0, \forall \vec{y}, \vec{v} \in \vec{V}.
- c) The functions \( f_1(x, \vec{y}, u_1) \) and \( h_1(x, \vec{y}) \) are of Carathéodory type on \( \Omega \times \mathbb{R}^2 \times U_1 \) and \( \Omega \times \mathbb{R}^2 \) respectively, and the following conditions for \( \phi_1, \phi_2 \in L^2(\Omega) \) and \( \tilde{c}_1, \tilde{c}_2 \geq 0 \) are satisfied:
\[ |f_1(x, \vec{y}, u_1)| \leq \phi_1(x) + \tilde{c}_1 \| y_1 \|^2, \| h_1(x, \vec{y})\| \leq \phi_2(x) + \tilde{c}_2 \| y_1 \|^2. \]
- d) \( f_1(x, \vec{y}, u_1) \) and \( h_1(x, \vec{y}) \) are monotone functions for all \( x \in \Omega \) w.r.t \( (\vec{y}, u_1) \) and \( \vec{y} \) respectively, with
\[ f_1(x, 0, u_1) = 0, \forall (x, u_1) \in \Omega \times U_1, h_1(x, 0) = 0, \forall x \in \Omega. \]
- e) The functions \( f_2(x, u_1) \) and \( h_2(x) \) are of “Carathéodory type “ on \( \Omega \times U_1 \) and satisfy for \( \phi_3, \phi_4 \in L^2(\Omega), \) and \( \tilde{c}_1 \geq 0 \)
\[ |f_2(x, u_1)| \leq \phi_3(x) + \tilde{c}_1 \| u_1 \|^2, (x, u_1) \in \Omega \times U_1 \text{ and } |h_2(x)| \leq \phi_4(x), \forall x \in \Omega. \]

Proposition 3.1[16]: Let \( f: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) is of “Carathéodory type “, let \( F \) be a functional such that \( F(y) = \int_\Omega f(x, y(x)) \, dx \), where \( \Omega \) is a measurable subset of \( \mathbb{R}^n \), and suppose that
\[ \| f(x, y) \| \leq \xi(x) + \eta(x) \| y \|^\alpha, \forall (x, y) \in \Omega \times \mathbb{R}^n \text{ and } \xi \in L^1(\Omega \times \mathbb{R}), \eta \in L^{\frac{n}{n-\alpha}}(\Omega \times \mathbb{R}) \text{ and } \alpha \in [1, P], \text{ if } P \in [1, \infty) \text{ and } \eta \equiv 0, \text{ if } P = \infty, \]
then \( F \) is continuous on \( L^p(\Omega \times \mathbb{R}^n) \).

Theorem 3.1 (Minty-Browder) [15]: Let \( V \) be a reflexive Banach space, and \( A: V \rightarrow V^* \) be a continuous nonlinear map such that
\[ \langle Av_1 - Av_2, v_1 - v_2 \rangle > 0, \forall v_1, v_2 \in V, v_1 \neq v_2 \text{ and } \lim_{\| v \| \rightarrow \infty} \frac{\langle Av, v \rangle}{\| v \|_{H^1(\Omega)}} = \infty. \]

Then for all \( f \in V^* \) there exists a unique solution \( y \in \vec{V} \) of the equation \( Ay = f \).

Theorem 3.2: In addition to assumptions A-(a and d). If one of the functions \( f_1 \) or \( h_1 \) in assumptions A(d) is strictly a monotone function, Then for each given \( \vec{u} \in \vec{W} \), the (8) has a unique solution \( \vec{y} \in \vec{V} \).

Proof: Let \( \vec{A}: \vec{V} \rightarrow \vec{V}^* \), then (8) can rewrite as follows:
\[ \langle \vec{A}(\vec{y}), \vec{v} \rangle = \langle \vec{F}(\vec{u}), \vec{v} \rangle \]
(9)
where \( \langle \vec{A}(\vec{y}), \vec{v} \rangle = a(\vec{y}, \vec{v}) + (f_1(x, \vec{y}, u_1), v_1)_{\Omega} + (h_1(x, \vec{y}), v_2)_{\Omega} \) and \( \langle \vec{F}(\vec{u}), \vec{v} \rangle = (f_2(x, u_1), v_1)_{\Omega} + (h_2(x), v_2)_{\Omega} + (u_2, v_2)_{\Gamma} \)

i) From assumptions A-(a and d), \( \vec{A} \) is coercive.
ii) From assumptions A-(b and c) with applying Proposition 3.1, the mapping \( \vec{y} \mapsto \langle \vec{A}(\vec{y}), \vec{v} \rangle \) is continuous w.r.t. \( \vec{y} \).
iii) From assumptions A-(a and d) with applying part (i), \( \vec{A} \) is strictly monotone w.r.t. \( \vec{y} \).
The uniqueness of the state vector solution \( \hat{y} \in \hat{V} \) of (9) is obtained from applying Theorem (3.1).

4. Existence of the continuous classical mixed control vector

This section deals with the state and proof of the existence theorem will be done under suitable assumptions. The following lemmas and assumptions are necessary for the proof of the existence theorem.

**Lemma 4.1:** In addition to the assumptions (A), if the functions \( f_1, f_2 \) are Lipschitz w.r.t. \( u_1 \) and \( h_1 \) is Lipschitz w.r.t. \( \hat{y} \), the function \( h_2 \) is bounded. Then the operator \( \hat{u} \mapsto \hat{y} \) from \( \hat{W} \) to \( L^2(\Omega) \times L^2(\Gamma) \), is Lipschitz continuous, i.e.

\[
\| \hat{\Delta y} \|_{L^2(\Omega) \times L^2(\Gamma)} \leq L \| \Delta \hat{u} \|_{L^2(\Omega) \times L^2(\Gamma)}, \quad \text{with } L > 0.
\]

**Proof:** Let \( \hat{u}, \hat{u}' \in \hat{W} \) are two given mixed control vectors, then by Theorem 3.1, \( \hat{y} \) and \( \hat{y}' \) are the state vector solutions of (8). Subtracting these two weak forms one from the other, setting

\[
\Delta \hat{y} = \hat{y}' - \hat{y} \quad \text{and} \quad \Delta \hat{u} = \hat{u}' - \hat{u}, \quad \text{with } \hat{v} = \Delta \hat{y},
\]

then adding the obtained equations to get

\[
\begin{align*}
\alpha_1 (\Delta y_1, \Delta y_1) + (a_0 \Delta y_1, \Delta y_1)_{\Omega} + \alpha_2 (\Delta y_2, \Delta y_2)_{\Omega} + (b_0 \Delta y_2, \Delta y_2)_{\Omega} \\
+ (f_1(x, \hat{y} + \Delta \hat{y}, u_1) - f_1(x, \hat{y}, u_1)) + h_1(x, \hat{y} + \Delta \hat{y}) - h_1(x, \hat{y}), \Delta \hat{y})_{\Omega} \\
= (f_2(x, u_1 + \Delta u_1) - f_2(x, u_1), \Delta y_1)_{\Omega} + (\Delta u_2, \Delta y_2)_{\Gamma}.
\end{align*}
\]

Applying assumptions A-(a) in (10), then taking the absolute value for both sides to get

\[
c \| \Delta \hat{y} \|_{(H^1(\Omega))^2}^2 \leq |(f_2(x, u_1 + \Delta u_1) - f_2(x, u_1), \Delta y_1)_{\Omega}| + |(\Delta u_2, \Delta y_2)_{\Gamma}|
\]

Using the Lipschitz property, the Cauchy-Schwarz inequality and then the trace operator to get

\[
c \| \Delta \hat{y} \|_{(H^1(\Omega))^2}^2 \leq 2 \| \Delta \hat{u} \|_{L^2(\Omega) \times L^2(\Gamma)} \| \Delta \hat{y} \|_{(H^1(\Omega))^2}^2 \\
\Rightarrow \| \Delta \hat{y} \|_{(H^1(\Omega))^2}^2 \leq c_2 \| \Delta \hat{u} \|_{L^2(\Omega) \times L^2(\Gamma)}, \quad \text{where } c_2 = \frac{2c_1}{c}
\]

which gives the requirement result

\[
\| \Delta \hat{y} \|_{L^2(\Omega) \times L^2(\Gamma)} \leq L \| \Delta \hat{u} \|_{L^2(\Omega) \times L^2(\Gamma)}, \quad \text{where } L = cc_2
\]

**Assumptions (B):**

Assume that \( g_{01} \) and \( g_{02} \) are of CAT on \( \Omega \times \mathbb{R}^2 \times U_1 \) and \( \Omega \times U_2 \), respectively. And the following conditions are satisfied for \( \hat{y} \in \mathbb{R}^2 \), \( \hat{u} \in \hat{U} \) with \( y_{01}, y_{02} \in L^1(\Gamma) \) and \( c_{01}, \bar{c}_{01}, c_{02} \geq 0; \)

\[
|g_{01}(x, \hat{y}, u_1)| \leq y_{01}(x) + c_{01} \hat{y}^2 + \bar{c}_{01} u_1^2 \quad \text{and} \quad |g_{02}(x, u_2)| \leq y_{02}(x) + c_{02} u_2^2
\]

**Lemma 4.2:** With assumptions (B), the functional \( \hat{u} \mapsto G_0(\hat{u}) \) which is defined on \( L^2(\Omega) \times L^2(\Gamma) \) is continuous.

**Proof:** The functional \( \iint g_{01}(x, \hat{y}, u_1) \, dx_1 \, dx_2 \) and \( \int g_{02}(x, u_2) \, dy \) are continuous on \( (L^2(\Omega))^2 \) and \( L^2(\Gamma) \), respectively. (from Assumptions (B) and by using Proposition 3.1).

Hence,

\[
G_0(\hat{u}) = \iint g_{01}(x, \hat{y}, u_1) \, dx_1 \, dx_2 + \int g_{02}(x, u_2) \, dy \text{ is continuous on } L^2(\Omega) \times L^2(\Gamma).
\]

**Theorem 4.1:** In addition to assumptions (A), assume that \( \hat{W} \neq \emptyset \), \( f_1 \) and \( h_4 \) are independent functions of \( u_1 \) and \( u_2 \), respectively. The functions \( f_2 \) and \( h_2 \) are bounded such that

\[
|f_1(x, \hat{y}, u_1)| \leq \bar{f}_1(x, \hat{y}, u_1), \quad |f_2(x, \hat{y}, u_1)| \leq \bar{f}_2(x, \hat{y}, u_1)
\]

\[
|h_1(x, \hat{y})| \leq \bar{h}_1(x, \hat{y}), \quad |h_2(x)| \leq \bar{h}_2(x)
\]

If \( G_0 \) is coercive, then there exists a mixed optimal control vector.

**Proof:** Since \( \hat{U} \) is convex and bounded, then \( \hat{W} \) is convex and bounded, and it is closed (from Egorov’s theorem), hence \( \hat{W} \) is weakly compact.
Since \( \bar{W} \neq \emptyset \), then there exists \( \bar{\omega} \in \bar{W} \), and a minimum sequence \( \{\bar{u}_n\} = \{(u_{1n}, u_{2n})\} \in \bar{W} \) for each \( n \), s.t.:
\[
\lim_{n \to \infty} G_0(\bar{u}_n) = \inf_{\bar{w} \in \bar{W}} G_0(\bar{w}).
\]
But \( \bar{W} \) is weakly compact, then there exists a subsequence of \( \{\bar{u}_n\} \), say again \( \{\bar{u}_n\} \) which converges weakly to some point \( \bar{u} \) in \( \bar{W} \), i.e. \( \bar{u}_n \to \bar{u} \) weakly in \((L^2(\Omega) \times L^2(\Gamma))\).

Then from the proof of Theorem 3.2, corresponding to this sequence \( \{\bar{u}_n\} \) there is a sequence of solution \( \{\bar{y}_n\} \) (with \( \|\bar{y}_n\|_{H^\alpha(\Omega)} \)) is bounded for all \( n \), of the sequence:
\[
\begin{align*}
& a_1(y_{1n}, v_1) + (a_0 y_{1n}, v_1) + (b_{0} y_{2n}, v_1) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) \\
& + (f_1(x, \bar{y}_n), v_1) + (h_1(x, \bar{y}_n), v_2) = (f_2(x, u_{1n}, v_1) + (h_2(x, u_{2n}, v_2) + (u_{2n}, v_2),)
\end{align*}
\]
(14)

Then by Alaoglu theorem, there exists a subsequence of \( \{\bar{y}_n\} \), say again \( \{\bar{y}_n\} \) such that \( \bar{y}_n \to \bar{y} \) weakly in \( \bar{V} \).

To prove that (14) converges to
\[
\begin{align*}
& a_1(y_{1n}, v_1) + (a_0 y_{1n}, v_1) + (b_{0} y_{2n}, v_1) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) \\
& + (f_1(x, \bar{y}), v_1) + (h_1(x, \bar{y}), v_2) = (f_2(x, u_1, v_1) + (h_2(x, u_2, v_2) + (u_2, v_2),)
\end{align*}
\]
(15)

Let \( (y_1, y_2) \in (C(\bar{\Omega}))^2 \), and first for the left hand sides, we have \( y_{1n} \to y_1 \) weakly in \( V_1 \), i.e.
\[
y_{1n} \to y_1 \text{ weakly in } L^2(\Omega), \quad V_1 = 1,2.
\]

Then from the left hand sides of (14), (15), by using Cauchy- Schwarz inequality, one has
\[
\begin{align*}
& |a_1(y_{1n}, v_1) + (a_0 y_{1n}, v_1) + (b_{0} y_{2n}, v_1) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) + (b_{0} y_{2n}, v_2) \\
& + (f_1(x, \bar{y}_n), v_1) + (h_1(x, \bar{y}_n), v_2) = (f_2(x, u_{1n}, v_1) + (h_2(x, u_{2n}, v_2) + (u_{2n}, v_2),)
\end{align*}
\]
(14)

From assumptions (B), and Proposition 3.1, the functional \( \int_{\Omega} a_1(x, \bar{y}_n)v_1dx_1dx_2 \) and
\[
\int_{\Omega} a_1(x, \bar{y}_n)v_2dx_1dx_2 \text{ are continuous w.r.t. } \bar{y}_n. \text{ But } \bar{y}_n \to \bar{y} \text{ weakly in } (L^2(\Omega))^2, \text{ then by}
\]
using the compactness theorem in [16], to get \( \bar{y}_n \to \bar{y} \) strongly in \( (L^2(\Omega))^2 \), then
\[
\begin{align*}
& (f_1(x, \bar{y}_n), v_1) + (h_1(x, \bar{y}_n), v_2) = (f_1(x, \bar{y}), v_1) + (h_1(x, \bar{y}, v_2), V(v_1, v_2) \in (C(\bar{\Omega}))^2
\end{align*}
\]
(15)

Second, since \( u_{1n} \to u_1 \) weakly in \( L^2(\Omega) \) and \( u_{2n} \to u_2 \) weakly in \( L^2(\Gamma) \), then
\[
\begin{align*}
& (f_2(x)(u_{1n} - u_1), v_1) + (u_{2n} - u_2, v_2) \to 0
\end{align*}
\]
(17b)

From (17a) and (17b) give us that (14) converges to (15), and this convergence holds
\[
V(v_1, v_2) \in \bar{V} \text{ (since } (C(\bar{\Omega}))^2 \text{ is dense in } \bar{V}) \text{ which gives the limit point } \bar{y} = \bar{y}_{\text{lt}} \text{ is a state “vector” solution of the (15).}
\]

Now, from the assumptions on \( g_{01}(x, \bar{y}, u_1) \) and Lemma 4.2, the integral
\[
\int_{\Omega} g_{01}(x, \bar{y}, u_1)dx_1dx_2 \text{ is continuous w.r.t. } (\bar{y}, u_1), \text{ and then } \int_{\Omega} g_{01}(x, \bar{y}, u_1)dx_1dx_2 \text{ is}
\]
weakly lower semicontinuous w.r.t \( u_1 \), (since \( g_{01}(x, \bar{y}, u_1) \), is convex w.r.t \( u_1 \), i.e.
\[
\begin{align*}
& \int_{\Omega} g_{01}(x, \bar{y}, u_1)dx_1dx_2 \leq \lim_{n \to \infty} \int_{\Omega} g_{01}(x, \bar{y}, u_{1n})dx_1dx_2
\end{align*}
\]
(17b)

The same way can be used to get
\[
\int_{\Gamma} g_{02}(x, u_2)dy \leq \lim_{n \to \infty} \int_{\Gamma} g_{02}(x, u_{2n})dy.
\]
Hence, \( G_0(\bar{u}) \) is weakly lower semicontinuous w.r.t \( (\bar{y}, \bar{u}) \),

Therefore, the mixed control vector is obtained from
\[
G_0(\bar{u}) = \lim_{n \to \infty} G_0(\bar{u}_n) = \inf_{\bar{w} \in \bar{W}} G_0(\bar{w}).
\]
Conclusions

In this work, the continuity of the operator between the mixed control vector and the corresponding state vector solution is proved. Moreover, under suitable assumptions, the existence theorem of a continuous classical mixed-optimal control vector dominated by the considered PDEs is stated and proved. It is observed that, under suitable conditions, the Minty-Browder theorem is appropriate to prove the existence of a unique state vector solution of the coupled nonlinear elliptic PDEs, when the continuous classical mixed control vector is known.

References