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Harmonic Multivalent Functions Associated with Generalized Hypergeometric Functions

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Abstract
In this paper , certain subclass of harmonic multivalent function defined in the exterior of the unit disk by used generalize hypergeometric functions is introduced . In This study an attempting have been made to investigate several geometric properties such as coefficient property , growth bounds , extreme points , convolution property , and convex linear combination .

Keywords: Multivalent functions , Harmonic functions , Meromorphic function , Growth bonds , Hypergemetric functions .

الدوال متعددة التكافؤ التوافقية المرتبطة بالدوال فوق الهندسية المعممة

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الخلاصه
في هذا البحث ,فئة جزئية مؤكدة من الدوال متعددة التكافؤ التوافقية المعرفة في الجزء الخارجي من قرص الوحدة من خلال استخدام الدوال الفوق الهندسية المعممة هي المقدمة ,في هذه الدراسة , تمت محاولة لفحص العديد من الخصائص الهندسية مثل ملاءمة المعامل , حدود النمو , النقاط القصوى ,خاصية الالتفاف ,تركيب خطي محدب .

1. Introduction

A continuous function $\mathscr{F} = \upsilon + i\nu$ is complex valued harmonic function in $\mathcal{U} \subset \mathbb{C}$,if υ , ν are harmonic real in $\mathcal{U}$. The function $\mathscr{F}$ can be written in any simply connected domain $\mathcal{U}$ by $\mathscr{F} = \vartheta + \bar{\mathcal{G}}$, where ϑ and $\bar{\mathcal{G}}$ are analytic in $\mathcal{U}$.

Schober and Hengartner [1] considered harmonic sense preserving univalent mappings defined on $\bar{\mathbb{U}} = \{z : |z| > 1\}$ that map ∞ to ∞ and it is represented by

$\mathscr{F}(z) = \vartheta(z) + \overline{\mathcal{G}(z)} + B\log|z|$

where

$$\vartheta(z) = \alpha z + \sum_{s=0}^{\infty} a_n z^{-s} \ , \overline{\mathcal{G}(z)} = \beta z + \sum_{n=0}^{\infty} b_s z^{-s} \tag{1}$$

are analytic in $\bar{\mathbb{U}}$ with $|\alpha| > |\beta| \geq 0$, and $B \in \mathbb{C}$. Furthermore , let the family $\sum_p(\mathcal{M})$, which consists of each harmonic sense preserving meromorphic multivalent mapping

$$\mathscr{F}(z) = \vartheta(z) + \overline{\mathcal{G}(z)} \quad . \tag{2}$$

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where

$$\begin{cases} \vartheta(z) = z^{\wp} + \sum_{s=1}^{\infty} a_{s+\wp-1} z^{-(s+\wp-1)} \quad , \\ \mathcal{G}(z) = \sum_{s=1}^{\infty} b_{s+p-1} z^{-(s+\wp-1)}, |b_{\wp}| < 1 \,, |z| > 1 \end{cases} \quad . \tag{3}$$

In addition for real or complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_t$ *and* $\beta_1, \beta_2, \ldots, \beta_q$ $(\beta_j \neq 0, -1, -2, \ldots; j = 1,2 \ldots, q)$, we can define the generalized hypergeometric function ${}_tF_q(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z)$ by

$${}_tF_q(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z) = \sum_{s=1}^{\infty} \frac{\prod_{j=1}^{t} (\alpha_t) z^s}{\prod_{j=1}^{q} (\beta_q) s!}$$

$$(t \leq q + 1 \,; t, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \,; z \in \mathbb{C})$$

Wherever $(y)_s$, denotes the pochhammer symbol , which is defined by

$(y)_s = \frac{\Gamma(y+s)}{\Gamma(y)}$.

Which corresponds to a function

$\mathcal{K}_{\wp,\lambda}(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z) = z^p \; {}_tF_q(\alpha_1, \alpha_2, \ldots, \alpha_t; \beta_1, \beta_2, \ldots, \beta_q; z)$.

Consider a linear operator $\mathcal{K}_{p,\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_t; \beta_1, \beta_2, \ldots, \beta_q; z)$ which is defined by the convolution

$$\mathcal{K}_{\wp}(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z) * \mathcal{K}_{p,\lambda}(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z) = \frac{z^{\wp}}{(1-z)^{\lambda+\wp}}$$

$(\lambda > -p)$. Let $\mathcal{K}_{\wp,t,q}(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z) : \sum_p(\mathcal{K}) \longrightarrow \sum_p(\mathcal{K})$ is defined by

$$\mathcal{K}_{p,t,q}(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z)\, \mathscr{F}(z) = \mathcal{K}_{\wp,\lambda}(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q; z) * \mathscr{F}(z) \tag{4}$$

$(\beta_j, \alpha_i \neq 0, -1, -2, \ldots; j = 1,2 \ldots, q, i = 1, 2 \ldots, t \;, \lambda > -p \,; \mathscr{F} \in \sum_{\wp}(\mathcal{K}) \;; z \in \overline{\mathbb{U}})$

We use the following shorter notation

$\mathcal{K}^{\lambda}_{,\wp,t,q}(\alpha_1) = \mathcal{K}_p(\alpha_1, \ldots, \alpha_t; \beta_1, \ldots, \beta_q)$.

Thus , from equation (4) after simple calculations we obtain

$$z(\mathcal{K}^{\lambda}_{p,t,q}(\alpha_1)\mathcal{F}(z)) = z^{\wp} + \sum_{s=1}^{\infty} \frac{(\lambda + \wp)_{s+\wp} (\beta_1)_{s+\wp} \ldots (\beta_q)_{s+\wp}}{(\alpha_1)_{s+\wp} \ldots (\alpha_t)_{s+\wp}} a_{s+\wp-1} z^{-(s+\wp-1)} \; . \tag{5}$$

Note that the linear operator $\mathcal{K}^{\lambda}_{,\wp,t,q}$ is closely related to the cho-saigo-srivastava operator [2]. In view of relationship (5) for harmonic function $\mathcal{F} = \vartheta + \bar{\mathcal{G}}$ that is given by (1) ,` we can define the operator

$$\mathcal{K}^{\lambda}_{,\wp,t,q}\mathcal{F}(z) = \mathcal{K}^{\lambda}_{p,t,q}\hbar(z) + \overline{\mathcal{K}^{\lambda}_{p,t,q}\mathcal{G}(z)} \quad . \tag{6}$$

2.Geometric outcomes

In this section , we have proceed to introduce a certain geometric subclass of $\sum_{\wp}(\mathcal{M})$, in [3],[4],[5] and [6] by invoking the operator $\mathcal{K}^{\lambda}_{,\wp,t,q}\mathcal{F}(z)$ that is given by (6) , then some properties are acquired by including coefficient bounds , growth formula , extreme points , convolution , and convex combinations as well as discuss a alas-preserving integral operator see [7] and [8] .

Definition1 A function $\mathfrak{f} \in \sum_{\wp}(\mathcal{M})$ is said to be in the subclass $\mathfrak{U}_{\wp,\mathfrak{b}}(t, q)$ if it satisfies the next inequality:

$$\mathscr{R}\left\{(1-\delta)\frac{z(\mathcal{K}^{\lambda}_{\wp,t,q}(\alpha_1)\mathcal{F}(z))}{z^{\wp}} + \delta\frac{[z(\mathcal{K}^{\lambda}_{\wp,t,q}(\alpha_1)\mathcal{F}(z))]'}{\wp z^{\wp-1}}\right\} \geq \frac{\mathfrak{b}}{\wp} \tag{7}$$

Where $\mathcal{K}^{\lambda}_{,\wp,t,q}(\alpha_1)\mathcal{F}(z)$ is given by (6) , $0 < \delta \leq 1$, $0 < \mathfrak{b} \leq \wp$.

and $\mathcal{K}\mathfrak{U}_{\wp,\mathfrak{b}}(t, q) = \mathfrak{U}_{\wp,\mathfrak{b}}(t, q) \cap \mathbb{N}\sum_{\wp}(\mathcal{M})$.

The next theorem provides a sufficient coefficient condition for function belong to the class$\mathfrak{U}_{p,\mathfrak{b}}(t\,,q\,)$.

Theorem1. Let $\mathscr{F}=\hbar+\bar{\mathcal{G}}$ of form (1) . If

$$\sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+p\right]\left|a_{s+\wp-1}\right|+\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\left|b_{s+\wp-1}\right|\}\,\mathrm{H}_{\wp}^{\mu}(s) \leq \wp-\mathfrak{b} \tag{8}$$

Where $0<\delta\leq 1$, $0<\mathfrak{b}\leq\wp$, and $\mathrm{H}_{\wp}^{\lambda}(n)=\frac{(\lambda+\mathrm{p})_{s+\wp}(\beta_1)_{s+\wp}\cdots\cdots(\beta_q)_{s+\wp}}{(\alpha_1)_{s+\wp}\cdots\cdots(\alpha_t)_{s+\wp}}$, then $\mathcal{F}\in\mathfrak{U}_{\wp,\mathfrak{b}}(t\,,q\,)$

Proof :

Assume that $\mathcal{F}(\mathrm{z})=(1-\delta)\frac{z(\mathcal{K}_{\wp,t,q}^{\lambda}(\alpha_1)\mathcal{F}(\mathrm{z})\,)}{z^{\wp}}+\delta\frac{\left[z(\mathcal{K}_{\wp,t,q}^{\lambda}(\alpha_1)\mathcal{F}(\mathrm{z})\,)\right]'}{\wp z^{\wp-1}}$

Now to prove $\mathscr{R}\{\,\mathcal{F}(\mathrm{z})\}>\frac{\mathfrak{b}}{\wp}$ it suffices to prove that $|\wp-\mathfrak{b}+\wp\mathcal{F}(\mathrm{z})|\geq|\wp+\mathfrak{b}-\wp\mathcal{F}(\mathrm{z})|$.

We substitute for $\mathcal{F}(\mathrm{z})$ and use the equation (6) , we find that

$$|\wp-\mathfrak{b}+\wp\mathcal{F}(\mathrm{z})|\geq 2\,\wp-\mathfrak{b}-\sum_{s=1}^{\infty}\{[(-(s+\wp-1))\delta+\wp]\left|a_{s+\wp-1}\right| +[(-(s+\wp-1))\delta+\wp]\left|b_{s+\wp-1}\right|\}\,\mathrm{H}_{\wp}^{\mu}(s)\,|z|^{-(s+\wp-1)} \tag{9}$$

And

$$|\wp+\mathfrak{b}-\wp\mathcal{F}(\mathrm{z})|\leq\mathfrak{b}+\sum_{s=1}^{\infty}\{[(-(s+\wp-1))\delta+\wp]\left|a_{s+\wp-1}\right|. +[(-(s+\wp-1))\delta+\wp]\left|b_{s+\wp-1}\right|\}\,\mathrm{H}_{\wp}^{\mu}(s)\,|z|^{-(s+\wp-1)} \tag{10}$$

It is clear that the (9) and (10) in conjunction with (8) yield

$$|\wp-\mathfrak{b}+\wp\mathcal{F}(\mathrm{z})|\geq|\wp+\mathfrak{b}-\wp\mathcal{F}(\mathrm{z})|\geq 2[(\wp-\mathfrak{b})-\sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\left|a_{s+\wp-1}\right| +[(-(s+\wp-1))\delta+\wp]\left|b_{s+\wp-1}\right|\}\,\mathrm{H}_{\wp}^{\mu}(s)\geq 0\,. \tag{11}$$

The harmonic function

$$\mathcal{F}(\mathrm{z})=z^{\wp}+\sum_{s=1}^{\infty}\frac{x_{n+p-1}}{[(-(s+\wp-1))\delta+\wp]\,\mathrm{H}_{\wp}^{\lambda}(s)\left|a_{s+\wp-1}\right|}z^{-(s+\wp-1)}+\sum_{s=1}^{\infty}\frac{\overline{y_{s+\wp-1}}}{[(-(s+\wp-1))\delta+\wp]\mathrm{H}_{\wp}^{\lambda}(s)\left|a_{s+\wp-1}\right|}\overline{z}^{-(s+\wp-1)}\,,$$

Where $\sum_{s=1}^{\infty}\left|x_{s+\wp-1}\right|+\sum_{s=1}^{\infty}\left|y_{s+\wp-1}\right|=\wp-\mathfrak{b}$

It shows that coefficient bounds that is given by (8) is sharp .

From (8) the functions in $\mathfrak{U}_{\wp,\mathfrak{b}}(t\,,q\,)$ because

$$(\wp-\,\mathfrak{b})-\sum_{s=1}^{\infty}\{[(-(s+\wp-1))\delta+\wp]\left|a_{s+p-1}\right|+[(-(s+\wp-1))\delta +\wp]\left|b_{s+\wp-1}\right|\}\,H_{\wp}^{\mu}(s)$$

$=\sum_{s=1}^{\infty}\left|x_{s+\wp-1}\right|+\sum_{s=1}^{\infty}\left|y_{s+\wp-1}\right|=\wp-\mathfrak{b}$

This completes the proof .

The following theorem gives a sufficient coefficient condition for function to be in $\mathfrak{U}_{\wp,\mathfrak{b}}(t\,,q\,)$.

Theorem2 . Let $\mathscr{F}=\vartheta+\bar{\mathcal{G}}$ of the form (3) , then $\mathscr{F}\in\mathfrak{U}_{\wp,\mathfrak{b}}(t\,,q\,)$ if and only if the condition (8) is as follows:

$$\sum_{s=1}^{\infty}\{[(-(s+\wp-1))\delta+\wp]\left|a_{s+\wp-1}\right|+[(-(s+\wp-1))\delta+\wp]\left|b_{s+\wp-1}\right|\}\,\mathrm{H}_{\wp}^{\lambda}(s)\leq\mathrm{p}-\mathfrak{b}$$

Where , $0\leq\mathfrak{b}<\wp$.

Proof. As $\mathscr{F}\in\mathcal{K}\mathfrak{U}_{\wp,\mathfrak{b}}(t\,,q\,)\subset\mathcal{K}\mathfrak{U}_{\wp,\mathfrak{b}}(t\,,q\,)$, we have to prove only if part of these theorem .For $\mathscr{F}$ functions of the (3) condition (7) is following :

$$\mathscr{R}\left\{(1-\delta)\frac{z(\mathcal{K}_{\wp,t,q}^{\lambda}(\alpha_1)\mathcal{F}(\mathrm{z})\,)}{z^{\wp}}+\delta\frac{\left[z(\mathcal{K}_{\wp,t,q}^{\lambda}(\alpha_1)\mathcal{F}(\mathrm{z})\,)\right]'}{\wp z^{\wp-1}}\right\}\geq\frac{\mathfrak{b}}{\wp}$$

Which implies that

$\mathscr{R}\{[(\wp - \mathfrak{b}) - \sum_{s=1}^{\infty}\{[(-(s+\wp-1))\delta + p]\left|a_{s+\wp-1}\right|z^{-(s+\wp-11)} - [(-(s+\wp-1))\delta + \wp]\left|b_{s+\wp-1}\right|\overline{z}^{-(s+\wp-1)}\}\,\mathrm{H}_{\wp}^{\lambda}(s)]\} \geq 0$

The above-mentioned requires condition that should hold for each value of z in $\mathbb{C}$ when we choose the value of z on the real positive axis where $0 < |z| = r < 1$, most having

$$[(\wp - \mathfrak{b}) - \sum_{s=1}^{\infty}\{[(-(s+\wp-1))\delta + \wp]\left|a_{s+\wp-1}\right|r^{-(s+\wp-1)} - [(-(s+\wp-1))\delta + \wp]\left|b_{s+\wp-1}\right|\overline{r}^{-(s+\wp-1)}\}\,\mathrm{H}_{\wp}^{\lambda}(s)] \geq 0$$

Let $r \longrightarrow 1$ through the real value it gives that

$$[(\wp - \mathfrak{b}) - \sum_{s=1}^{\infty}\{[(-(s+\wp-1))\delta + \wp]\left|a_{s+\wp-1}\right| - [(-(s+\wp-1))\delta + \wp]\left|b_{s+\wp-1}\right|\}\,\mathrm{H}_{\wp}^{\lambda}(s)] \geq 0 \qquad (12)$$

The inequality (12) gives (8) ,and the result is obtained.

Next theorem considers the growth of the function $\mathscr{F} \in \mathcal{KU}_{p,\mathfrak{b}}(t, q)$

Theorem3. Let $\mathscr{F} \in \mathcal{KU}_{\wp,\mathfrak{b}}(t, q)$, then for $r = |z| < 1$,

$|\mathcal{F}(z)| \leq r^{\wp}(1 + |b_{\wp}|) + \frac{[\wp(1-|b_{\wp}|)-\mathfrak{b}]}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)}r^{-\wp}$

And

$|\mathcal{F}(z)| \geq r^{\wp}(1 + |b_{\wp}|) + \frac{[\wp(1-|b_{\wp}|)-\mathfrak{b}]}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)}r^{-\wp}$

Proof . Let $\mathscr{F} \in \mathcal{KU}_{p,\mathfrak{b}}(t, q)$. If we take the modulus value of $\mathscr{F}$, and use Theorem2
Then we have :

$|\mathcal{F}(z)| \leq r^{\wp}(1 + |b_{\wp}|) + \sum_{s=1}^{\infty}(|a_{s+\wp-1}| + |b_{s+\wp-1}|)r^{-(s+\wp-1)}$

$\leq r^{\wp}(1 + |b_{\wp}|) + \sum_{s=1}^{\infty}(|a_{s+\wp-1}| + |b_{s+\wp-1}|)r^{-\wp}$

$\leq r^{\wp}(1 + |b_{\wp}|) + \frac{r^{-\wp}}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)} \times (\sum_{s=1}^{\infty}\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)(|a_{s+\wp-1}| + |b_{s+\wp-1}|))$

$\leq r^{\wp}(1 + |b_{\wp}|) + \frac{r^{-\wp}}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)} \times (\sum_{s=1}^{\infty}\mathrm{H}_{\wp}^{\lambda}(s)\,[(-(s+\wp-1))\delta + \wp](|b_{s+\wp-1}| + |a_{s+\wp-1}|)]$

$\leq r^{\wp}(1 + |b_{\wp}|) + \frac{r^{-\wp}\,(\wp(1-|b_{\wp}|-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)}$

In addition ,

$|\mathcal{F}(z)| \geq r^{\wp}(1 + |b_{\wp}|) - \sum_{s=1}^{\infty}(|a_{s+\wp-1}| + |b_{s+\wp-1}|)r^{-(s+\wp-1)}$

$\geq r^{\wp}(1 + |b_{\wp}|) - r^{-\wp}\sum_{n=1}^{\infty}(|a_{n+p-1}| + |b_{n+p-1}|)$

$\geq r^{\wp}(1 + |b_{\wp}|) - \frac{r^{-\wp}}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)} \times (\sum_{s=1}^{\infty}\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)(|a_{s+\wp-1}| + |b_{s+\wp-1}|))$

$\geq r^{\wp}(1 + |b_{\wp}|) - \frac{r^{-\wp}}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)} \times \sum_{s=1}^{\infty}\mathrm{H}_{\wp}^{\mu}(s)\,[(-(s+\wp-1))\delta + \wp](|b_{s+\wp-1}| + |a_{s+\wp-1}|)]$

$\geq r^{\wp}(1 + |b_{\wp}|) - \frac{r^{-\wp}\,(\wp(1-|b_{\wp}|-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\,(-\delta\wp+\wp)}$

This completes the proof .

Next theorem determines the extreme points of convex closed hulls of $\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$ which is denoted by $\overline{co}\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$.

Theorem4 . The function $\mathcal{F}\in \overline{co}\mathcal{KU}_{p,\mathfrak{b}}(t,q)$ if and only if

$$\mathcal{F}(z) = \sum_{s=1}^{\infty}(\chi_{s+\wp-1}\,\xi_{s+\wp-1}(z) + \psi_{s+\wp-1}\,\zeta_{s+\wp-1}(z)), \tag{13}$$

where

$$\xi_{\wp}(z) = z^{\wp} \quad ,$$

$$\xi_{s+\wp-1}(z) = z^{\wp} - \frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]} z^{-(s+\wp-1)} \quad ,$$

And

$$\zeta_{s+\wp-1}(z) = z^{\wp} - \frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]}\overline{z}^{-(s+\wp-1)},$$

$\sum_{s=1}^{\infty}(\chi_{s+\wp-1} + \psi_{s+\wp-1}) = 1$, $\chi_{s+\wp-1} \geq 0$, $\psi_{s+\wp-1} \geq 0$.

Proof. The function $\mathcal{F}$ in (13) has

$$\mathcal{F}(z) = \sum_{s=1}^{\infty}(\chi_{s+\wp-1}\,\xi_{s+\wp-1}(z) + \psi_{s+\wp-1}\,\zeta_{s+\wp-1}(z))$$

$$=\sum_{s=1}^{\infty}(\chi_{s+\wp-1} + \psi_{s+\wp-1})\ z^{p}$$

$$-\frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]}\chi_{n+\wp-1}z^{-(s+\wp-1)} - \frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]}\psi_{s+\wp-1}\overline{z}^{-(s+\wp-1)}$$

$$=z^{\wp} - \frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]}\chi_{s+\wp-1}\,z^{-(s+\wp-1)}$$

$$- \frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]}\psi_{s+\wp-1}\overline{z}^{-(s+\wp-1)}$$

By Theorem2 we obtain

$$\sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\mathrm{H}_{\wp}^{\lambda}(s)\ \times\left[\frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]}\chi_{s+\wp-1}\right]$$

$$+ \sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\mathrm{H}_{\wp}^{\lambda}(s)\ \times\left[\frac{(\wp-\mathfrak{b})}{\mathrm{H}_{\wp}^{\lambda}(s)\left[\left(-(s+\wp-1)\right)\delta+\wp\right]}\psi_{s+\wp-1}\right]$$

$$\leq (\wp-\mathfrak{b})\left(\sum_{n=1}^{\infty}(\chi_{s+\wp-1} + \psi_{s+\wp-1}) - \chi_{\wp}\right)$$

$$= (\wp-\mathfrak{b})(1-\chi_{\wp}) \leq \wp - \mathfrak{b} .$$

$\mathcal{F}\in \overline{co}\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$. Conversely , suppose that $\mathcal{F}\in \overline{co}\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$,and set

$$\chi_{s+\wp-1} = \left[\left(-(s+\wp-1)\right)\delta+\wp\right]\frac{\mathrm{H}_{\wp}^{\lambda}(s)}{(\wp-\mathfrak{b})}\left|a_{s+\wp-1}\right| ,$$

$$\psi_{s+\wp-1} = \left[\left(-(s+\wp-1)\right)\delta+\wp\right]\frac{\mathrm{H}_{\wp}^{\lambda}(s)}{(\wp-\mathfrak{b})}\left|b_{s+\wp-1}\right|,$$

In the Theorem2 ,we note that $0 \leq \chi_{s+\wp-1} \leq 1$, $0 \leq \psi_{s+\wp-1} \leq 1$, *let* $\chi_{p} = 1 - \sum_{s=1}^{\infty}\chi_{s+\wp-1} + \sum_{s=1}^{\infty}\psi_{s+\wp-1}$ by Theorem2 ,we have $\chi_{\wp} \geq 0$. consequently , the function $\mathcal{F}(z) = \sum_{s=1}^{\infty}(\chi_{s+\wp-1}\,\xi_{s+\wp-1}(z) + \psi_{s+\wp-1}\,\zeta_{s+\wp-1}(z))$ is obtained.

The following theorem shows that the subclass $\mathcal{KU}_{p,\mathfrak{b}}(t,q)$ is close under convolution.

Theorem5. Assume that $\mathcal{F}\in\mathcal{KU}_{\wp,e}(t,q)$ and $\mathcal{M}\in \mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$ then $(\mathcal{F}*\mathcal{M})\in\mathcal{KU}_{\wp,e}(t,q) \subset \mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$. $z^{-(s+\wp-1)}\overline{z}^{-(s+\wp-1)}$, for $0 \leq e \leq 1$.

Proof. Use convolution concept , the harmonic function

$$\mathcal{F}(z) = z^{\wp} - \sum_{s=1}^{\infty}\left|a_{s+\wp-1}\right|z^{-(s+\wp-1)} - \sum_{s=1}^{\infty}\left|b_{s+\wp-1}\right|\overline{z}^{-(s+\wp-1)} .$$

And $\mathcal{M}(\mathrm{z}) = z^{\wp} - \sum_{s=1}^{\infty}|L_{s+\wp-1}|z^{-(s+\wp-1)} - \sum_{s=1}^{\infty}|J_{s+\wp-1}|\overline{z}^{-(s+\wp-1)}$.
Then the convolution of $\mathcal{F}$ and $\mathcal{M}$ is
$(\mathcal{F} * \mathcal{M})(\mathrm{z}) = z^{\wp} - \sum_{s=1}^{\infty}|a_{s+\wp-1}L_{s+\wp-1}|z^{-(s+\wp-1)} - \sum_{s=1}^{\infty}|b_{s+\wp-1}J_{s+\wp-1}|\overline{z}^{-(s+\wp-1)}$.
In Theorem2 , $\mathcal{M}(\mathrm{z}) \in \mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$ deduces that $|L_{s+\wp-1}| \leq 1$, $|J_{s+\wp-1}| \leq 1$. However , $\mathcal{F} \in \mathcal{KU}_{\wp,e}(t,q)$, then we have

$$\sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\frac{\mathrm{H}_{\wp}^{\lambda}(s)}{(\wp-e)}|a_{s+\wp-1}|$$
$$+\sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\frac{\mathrm{H}_{\wp}^{\lambda}(s)}{(\wp-e)}|b_{s+\wp-1}|$$
$$\leq \sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\frac{\mathrm{H}_{\wp}^{\lambda}(s)}{(\wp-\mathfrak{b})}|a_{s+\wp-1}|$$
$$+\sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\frac{\mathrm{H}_{\wp}^{\lambda}(s)}{(\wp-\mathfrak{b})}|b_{s+\wp-1}|$$
$$\leq 1.$$

Thus$(\mathcal{F} * \mathcal{M}) \in \mathcal{KU}_{\wp,e}(t,q) \subset \mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$.
The next result shows that the convex combination of subclass $\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$.
Let $\mathcal{F}_i(\mathrm{z})$ defined by , $\mathcal{F}_i(\mathrm{z}) = z^{\wp} + \sum_{s=1}^{\infty}|a_{(s+\wp-1),i}|z^{-(s+\wp-1)} - \sum_{s=1}^{\infty}|b_{(s+\wp-1),i}|\overline{z}^{-(s+\wp-1)}$. for $i = 1, 2, \ldots,$ (14)

Theorem6. Let the function $\mathcal{F}_i(\mathrm{z})$ that defined by (14) in $\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$ for $i = 1, 2 \ldots,$ then the function $\phi(\mathrm{z})$ is defined as follows

$$\phi(\mathrm{z}) = \sum_{i=1}^{\infty} e_i\mathcal{F}_i(z) \qquad (0 \leq e_i \leq 1). \qquad (15)$$

is also in $\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$, where $\sum_{i=1}^{\infty} e_i = 1$.
proof. Depending to the definition of ϕ , we can write

$$\phi(\mathrm{z}) = z^{\wp} + \sum_{s=1}^{\infty}\left(\sum_{i=1}^{\infty} e_i|a_{(s+\wp-1),i}|\right)z^{-(s+\wp-1)} - \sum_{s=1}^{\infty}\left(\sum_{i=1}^{\infty} e_i|b_{(s+\wp-1),i}|\right)\overline{z}^{-(s+\wp-1)}.$$

then , by Theorem2 , we have

$$\sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\mathrm{H}_{\wp}^{\lambda}(s)\left(\sum_{i=1}^{\infty} e_i|a_{(s+\wp-1),i}|\right)$$
$$+\sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\mathrm{H}_{\wp}^{\lambda}(s)\left(\sum_{i=1}^{\infty} e_i|b_{(s+\wp-1),i}|\right)$$
$$= \sum_{i=1}^{\infty} e_i\left(\sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\mathrm{H}_{\wp}^{\lambda}(s)\,|a_{(s+\wp-1),i}|\right.$$
$$\left.+\sum_{i=1}^{\infty} e_i\left(\sum_{s=1}^{\infty}[(-(s+\wp-1))\delta+\wp]\mathrm{H}_{\wp}^{\lambda}(s)\,|b_{(s+\wp-1),i}|\right.\right. \leq \sum_{i=1}^{\infty} e_i = 1.$$

A closure property of $\mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$ is examined under the generalized Livingston-Bernardi-Libera integral operator $\mathcal{M}(\mathrm{z})$ which is defined as follows : (see [1])

$$\mathcal{M}(\mathrm{z}) = \frac{(\lambda+\wp)}{z^{\lambda}}\int_0^z \mathcal{T}^{\lambda-1}\,\mathcal{F}(\mathcal{T})d\mathcal{T}, \qquad (\lambda > -\wp).$$

Theorem7. Let $\mathcal{F} \in \mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$. then $\mathcal{M} \in \mathcal{KU}_{\wp,\mathfrak{b}}(t,q)$.
Proof. Let

$$\mathcal{F}(\mathrm{z}) = z^{\wp} - \sum_{s=1}^{\infty}|a_{s+\wp-1}|z^{-(s+\wp-1)} - \sum_{s=1}^{\infty}|b_{s+\wp-1}|\overline{z}^{-(s+\wp-1)}.$$

From the representation of $\mathcal{M}(\mathrm{z})$, we have

$$\mathcal{M}(\mathrm{z}) = \frac{(\lambda+\wp)}{z^{\lambda}}\int_0^z \mathcal{T}^{\lambda-1}\{\vartheta(\mathrm{z}) + \overline{\mathcal{G}(z)}\}d\mathcal{T}$$
$$= \frac{(\lambda+\mathrm{p})}{z^{\lambda}}\left\{\int_0^z \mathcal{T}^{\lambda-1}\left(\mathcal{T}^{\mathrm{p}} - \sum_{s=1}^{\infty}|a_{s+\wp-1}|\mathcal{T}^{-(s+\wp-1)}\right)d\mathcal{T}\right.$$
$$\left. - \overline{\int_0^z \mathcal{T}^{\lambda-1}}\left(\sum_{s=1}^{\infty}|b_{s+\wp-1}|\mathcal{T}^{-(s+\wp-1)}\right)d\mathcal{T}\right\}$$

$= z^{\wp} - \sum_{s=1}^{\infty} L_{s+\wp-1} z^{-(s+\wp-1)} - \sum_{s=1}^{\infty} J_{s+\wp-1}\, \overline{z}^{-(s+\wp-1)}$,

where

$L_{s+\wp-1} = (\frac{(\lambda+\wp)}{(\lambda+s+\wp-1)})\,|a_{s+\wp-1}|$, $J_{s+\wp-1} = (\frac{(\lambda+\wp)}{(\lambda+s+\wp-1)})\,|b_{s+\wp-1}|$.

Because of $\mathcal{F} \in \mathcal{KU}_{p,\mathfrak{b}}(t\,,q)$,

$\sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\mathrm{H}_{\wp}^{\lambda}(s)\,(\frac{(\lambda+\wp)}{(\lambda+s+\wp-1)})\,|a_{s+\wp-1}|$

$+ \sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\mathrm{H}_{\wp}^{\lambda}(s)(\frac{(\lambda+\wp)}{(\lambda+s+\wp-1)})\,|b_{s+\wp-1}|$

$\leq \sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\mathrm{H}_{\wp}^{\lambda}(s)\;|a_{s+\wp-1}|$

$+ \sum_{s=1}^{\infty}\left[\left(-(s+\wp-1)\right)\delta+\wp\right]\mathrm{H}_{\wp}^{\lambda}(s)\,|b_{s+\wp-1}|$

$\leq \wp - \mathfrak{b}$

Then from Theorem2 , we have $\mathcal{M}(z) \in \mathcal{KU}_{\wp,\mathfrak{b}}(t\,,q)$.

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