DOI: 10.24996/ijs.2022.63.5.28

ISSN: 0067-2904

Quasi-Radical Semiprime Submodules

Omar M Al-Ragab, Nuhad S. Al-Mothafar
Department of Mathematics, College of Science, Baghdad University, Baghdad, Iraq

Received: 25/4/2021 Accepted: 27/7/2021 Published: 30/5/2022

Abstract
In this paper, we introduce the concept of a quasi-radical semi prime submodule. Throughout this work, we assume that R is a commutative ring with identity and D is a left unitary R- module. A proper submodule B of D is called a quasi-radical semi prime submodule (for short Q-rad-semiprime), if $a^k by \in B + rad(D)$ for a, $b \in R$, $y \in D$, and $k \in Z^+$ then $aby \in B$. Where $rad(D)$ is the intersection of all prime submodules of D.

Keywords: Semi prime submodule, Quasi-semi prime submodule, Radical semi prime submodule, Quasi- radical semi prime submodule.

المقاسات الجزئية شبه الأولية الجذرية الظاهرية

عمر محمد رجب، نهاد سالم المظفر
قسم الرياضيات, كلية العلوم ,جامعة بغداد, العراق

الخلاصه
في هذا البحث نقدم مفهوم المقاس شبه الاولي الجذري الظاهري. خلال هذا العمل تم فرض R حلقة ابدالية ذات عنصر محايد و D مقاسا احاديا ايسرا. ان المقاس الجزئي B من المقاس D يسمى مقاس جزئي شبه اولي جذري ظاهري (للاختصار مقاس شبه اولي من النوع Q–rad), كلما كان $a^k by$ $\in B + rad(D)$ حيث $a , b \in R$ $y \in D$ و $k \in Z^+$ و يعني ذلك $aby \in B$, حيث $rad(D)$ هو تقاطع كل المقاسات الجزئية الاالولية في D.

1. Introduction.

A quasi-prime submodule was introduced and studied in 1999 by Abdul-Razak, M. H. in [1], which is generalization of a prime submodule. A proper submodule B of an R – module D is called prime if whenever $ry \in B$, $for\ r \in R, and\ y \in D$, then either $y \in B$ or $r \in [B:_R D]$, where$[B:_R D] = \{ r \in R :, rD \subseteq B \}$[2]. Several generalizations of prime submodules have been introduced such as Semi prime, nearly prime, and nearly quasi-prime prime submodules [3,4,5]. In this paper, we give another generalization of a prime submodule, where a proper submodule B of D is called a quasi-radical semi prime submodule (for short Q-rad-semiprime), if whenever $a^k by \in B + rad(D)$ for a , $b \in R$, $y \in D$ and $k \in Z^+$, then $aby \in B$. Where $rad(D)$ is the intersection of all prime submodules of D.

2. Basic Properties of Quasi-Radical Semiprime Submodules

In this section, we introduce the concept of the quasi-radical semiprime submodule, we also give some examples, some basic properties, and characterizations of this concept.

*Email: omer.regab1203@sc.uobaghdad.edu.iq

Definition (2.1):
A proper submodule B of an R-module D is said to be quasi-radical semi prime if whenever $a^k by \in B + rad(D)$ for $a, b \in R, y \in D$ and $k \in Z^+$, then $aby \in B$. It is denoted by Q-rad-semiprime.

Theorem (2.2):
A submodule B of an R-module D is Q-rad-semiprime if and only if $a^2by \in B + rad(D)$ for $a, b \in R, y \in D$ implies that $aby \in B$.

Proof: Since B is Q-rad-semiprime, then by definition (2.1) we have this direction. For the converse, we suppose that for $a, b \in R, y \in D, k \in Z^+$ such that $a^2by \in B + rad(D)$. Now $a^2b(a^{k-2}y) \in B + rad(D)$. This implies that $a^{k-1}by \in B$. After a finite number of steps, we get that $aby \in B$. This implies that B is Q rad-semi prime submodule of D.

Proposition (2.3):
Let B be a proper submodule of an R-module D, with $rad(D) \subseteq B$. Then B is Q-rad-semiprime submodule if and only if $[B + rad(D): y]$ is a semiprime ideal of a ring R for each $y \in D$.

Proof:
Let B be Q-rad-semiprime. To prove $[B + rad(D): \langle y \rangle]$ is a semiprime ideal of R, it is enough to show $\sqrt{[B + rad(D): \langle y \rangle]} \subseteq [B + rad(D): \langle y \rangle]$. Now, let $a \in \sqrt{[B + rad(D): \langle y \rangle]}$, then $a^k \in [B + rad(D): \langle y \rangle]$ for some $k \in Z$, so that $a^k.1.y \in B + rad(D)$. But B is Q-rad-semiprime, this implies $ay \in B$. Since $rad(D) \subseteq B$, then $ay \in B + rad(D)$, which implies that $a \in [B + rad(D): \langle y \rangle]$, so that $\sqrt{[B + rad(D): \langle y \rangle]} \subseteq [B + rad(D): \langle y \rangle]$ which means that $[B + rad(D): \langle y \rangle]$ is a semiprime ideal of R.

Suppose that $[B + rad(D): \langle y \rangle]$ is a semiprime ideal of R, let $a^k by \in B + rad(D)$ for $a, b \in R, y \in D$, and $k \in Z^+$, this implies that $a^k b \in [B + rad(D): \langle y \rangle]$. But $[B + rad(D): \langle y \rangle]$ is a semiprime ideal, this implies that $ab \in [B + rad(D): \langle y \rangle]$, hence $aby \in B + rad(D)$. Since $rad(D) \subseteq B$, then $[B + rad(D) = B$. Thus $aby \in B$, which implies that B is Q-rad-semiprime submodule of D.

Corollary (2.4):
A proper submodule B is Q-rad-semiprime submodule of an R-module D with $rad(D) \subseteq B$ if and only if $[B + rad(D): (y)]$ is a semiprime ideal of R for each $y \in D$.

Remark and Examples (2.5):
1- Every prime submodule is Q-rad-semiprime submodule.

Proof:
Let B be a prime submodule of D, then $[B: D]$ is prime ideal of R **[6].** Let $a, b \in R, y \in D$ such that $a^2by \in B + rad(D)$, since B be a prime submodule of D and $rad(D)$ is the intersection of prime submodule, then $a^2by \in B$. Primness of B implies that either $aby \in B$ or $a^2 \in [B + rad(D): D]$. Thus $a \in [B + rad(D): D]$ and $aby \in B$.

The next example shows that the converse of (1) is not true.

Example: Let $B = 10Z$ be a submodule of Z as Z-module, and B is Q-rad-semiprime, since $[10Z + rad(Z): y] = [10Z + 0: y] = [10Z: y] = 10Z$, then it is semiprime ideal of Z. However, it is not prime submodule.

2- Every Q-rad-semiprime submodule is semi prime submodule.

Proof:
Let $a^2y \in B$ such that $a \in R, and\ y \in D$, then $a^2y \in B + rad(D)$ by definition (2.1),so that we have $ay \in B$. Hence B is a semi prime submodule of D.

The following example proves that the converse of (2) is not true.

Example: Let $D = Z_2 \oplus Z_4$ as Z-module and $B = Z_2 \oplus 0$ is a submodule of D. $B + rad(D) = \{(0,0), (1,0), (0,2), (1,2)\}$, then B is not Q-rad-semiprime submodule of, since $9 \times$

$1(\bar{1},\bar{2}) = 3^2 \times 1(\bar{1},\bar{2}) \in B + rad(D)$, but $3 \times 1(\bar{1},\bar{2}) \notin B$, where $B = \{(0,0),(1,0)\}$, $rad(D) = rad(Z_2) \oplus rad(Z_4) = \{(0,0),(0,2)\}$[2]. Therefore B is a semi prime submodule

3- Every maximal submodule of an R-module D is a Q-rad-semiprime submodule. Since every maximal submodule of D is prime, then by (1) is Q-rad-semiprime

4- Every quasi-prime submodule B of an R-module D with $rad(D) \subseteq B$ is Q-rad-semiprime submodule.

Proof: Let B be a quasi-prime submodule of an R-module D, by [1]. $[B:(y)]$ is a prime ideal of R for each $y \in D$, hence $[B:(y)]$ is a semi prime ideal of R for each $y \in D$. But $rad(D) \subseteq B$ so $B + rad(D) = B$ implies $[B + rad(D):(y)]$ is a semi prime ideal of for each $y \in D$. Thus B is Q-rad-semiprime, by Corollary (2.4).

The converse is not true in general for example:

Let $D = Z_{12}$ as a Z-module and $= < \bar{6} >$. B is Q-rad-semiprime submodule of D since $[< \bar{6} > + rad(Z_{12}):y] = [< \bar{6} > + < \bar{6} >:y] = [< \bar{6} >:y] = 6Z$,which is a semi prime ideal of Z. But $B = < \bar{6} >$ is not quasi-prime submodule of Z_{12}, since$[< \bar{6} >:< \bar{1} >] = 6Z$ is not a prime ideal of a ring Z.

5- A submodule of Q-rad-semiprime needs not to be Q rad-semi prime.

Example: $< \bar{2} >$ in Z_{12} as a Z-module is Q-rad-semiprime submodule, since $< \bar{2} >$ is a prime. But $< \bar{4} >$ is a submodule of $< \bar{2} >$,which is not Q-rad-semiprime of Z_{12}, since $2^2 \times 1 \times \bar{1} \in < \bar{4} > + rad(Z_{12})$ but $2 \times 1 \times \bar{1} \notin < \bar{4} >$.

6- In the Z-module $D = Z$ the submodule $B = nZ$ is Q-rad-semiprime if n is a prime number.

7- It is clear that every rad-semi prime submodule is Q-rad-semiprime submodule.

Proposition (2.6):

Let D be an R-module and B be a proper submodule of D.Then B is Q-rad-semiprime submodule of D if and only if $I^kJC \subseteq B + rad(D)$ for some ideals I,J in R , $k \in Z^+$ and C some submodule of D implies $IJC \subseteq B$.

Proof:

→) Assume that B be a Q-rad-semiprime submodule of D, and $I^kJC \subseteq B + rad(D)$ for some ideal I,J of R and some submodule C of D and $k \in Z^+$, we have to show that $IJC \subseteq B$. Let $x \in IJC$, then $x = r_1s_1x_1 + r_2s_2x_2 + \cdots + r_ns_nx_n$ where $r_i \in I$ $s_i \in J$, $x_i \in C$, $i = 1,2,\ldots\ldots,n$, thus $r_is_ix_i \in IJC$ for each $i = 1,2,\ldots\ldots,n$, then ${r_i}^k s_ix_i \in I^kJC \subseteq B + rad(D)$, but B is Q-rad-semiprime submodule of D, therefore $r_is_ix_i \in B$ for each $i = 1,2,\ldots\ldots,n$, thus $x \in B$, which implies that $IJC \subseteq B$.

←) Suppose $r^ksy \in B + rad(D)$,where $r,s \in R$, $y \in D$, and $k \in Z^+$ implies $< r^k >< s >< y > \subseteq B + rad(D)$, by hypothesis $< r >< s >< y > \subseteq B$, hence $rsy \in B$. Thus B is rad-semiprime submodule in D.

Corollary (2.7):

B is Q-rad-semiprime submodule of D if and only if $< a >^k < b > C \subseteq B + rad(D)$, where $k \in Z^+$ and C some submodule of D implies $< a >< b > C \subseteq B$.

Corollary (2.8):

B is Q-rad-semiprime submodule of D if and only if $a^kbC \subseteq B + rad(D)$, where $k \in Z^+$ and C some submodule of D implies $abC \subseteq B$.

Recall that a proper submodule B of an R-module D is said to be quasi- semi prime for short (Q -semi prime) if whenever $a^kby \in B$where $a , b \in R$, $y \in D$ and $k \in Z^+$ implies that $aby \in B$ [8].

Remark (2.9):

If B is Q-rad-semiprime submodule of an R-module D. Then $[B:D]$is a Q-semi prime ideal of R.

Proof: Let $a^2bt \in [B:D]$,where $a,b, and\ t \in R$, which implies $a^2b\langle tD\rangle \subseteq B \subseteq B + rad(D)$, since B is Q-rad-semiprime submodule of D, then by Corollary

(2.8), $we\ have\ ab\langle tD\rangle \subseteq B$. Hence $abt \in [B: D]$. Therefore $[B: D]$is a Q-semi prime ideal of R.

Remark (2.10):

Let B be a submodule of an R-module D, If $[B: D]$ is Q-rad-semiprime ideal of R, then B cannot be Q rad-semi prime in general. For example: Let $D = Z \oplus Z$ be a Z-module and $B = \langle 18\rangle + \langle 0\rangle$ then $[B: D] = (0)$ is Qrad-semiprime ideal of Z but B is not Q-rad-semiprime submodule of D, since $3^2 \times 2 \times (1,0) \in B + rad(D)$, but $3 \times 2 \times (1,0) \notin B$.

Proposition (2.11):

Let D be an R-module. $B, and\ K$ are submodules of D. If B is Q-rad-semiprime submodule of D and K is semiprime submodule with $rad(D) \subseteq K$. Then $B \cap K$ is Qrad-semiprime of D.

Proof: Let $a^n by \in (B \cap K) + rad(D)$ for $a, b \in R$, $y \in D$ and $n \in Z^+$, from the modular law, we have $a^n by \in \big(B + rad(D)\big) \cap K$, then $a^n by \in B + rad(D)$ and $a^n by \in K$, B is Q-rad-semiprime submodule of D and K is semiprime submodule, this implies that $aby \in B$ and $aby \in K$. Hence $aby \in B \cap K$, which implies that $B \cap K$ is Q-rad-semiprime in D.

Corollary (2.12):

If B , K are Q-rad-semiprime submodule of an R-module D with $rad(D) \subseteq K$, then $B \cap K$ is a Q-rad-semiprime submodule of D.

Proposition (2.13):

Let B , K be submodules of an R- module D with $rad(K) = rad(D) \cap K$ such that B is a Q-rad-semiprime submodule of D and K is not contained in B. Then $B \cap K$ is a Qrad-semiprime submodule of K.

Proof: Since $K \not\subset B$, we get $B \cap K$ is a proper submodule of K. Let a , $s \in R$, and $y \in K$, such that $a^2 s\, y \in (B \cap K) + rad(K)$. Since $rad(K) = rad(D) \cap K$, then $a^2 s\, y \in (B \cap K) + rad(D) \cap K$,this implies $a^2 s\, y \in \big(B + rad(D)\big) \cap K$ so that $a^2 s\, y \in B + rad(D)$ and $a^2 s\, y \in K$. Because of B is a Q-rad-semiprime submodule of D and by definition (2.1), we have $a\, s\, y \in B$. Since $y \in K$ implies that $as\, y \in K \cap B$. Hence $K \cap B$ is a Q-rad-semi prime submodule of K.

Proposition (2.14):

Let D and D' be R-modules and let $\Phi: D \to D'$ be R- homomorphism. If B is a Qrad-semiprime submodule of D', then $\Phi^{-1}(B)$ is Q-rad-semi prime submodule of D .

Proof: Let a , $b \in R$ and $y \in D$, such that $a^2\, by \in \Phi^{-1}(B) + rad(D)$, so that $a^2\, b\, \Phi(y) \in B + \Phi(rad(D))$, then $a^2\, b\, \Phi(y) \in B + rad(D')$, since $\Phi\big(rad(D)\big) = rad(D')$**[2]**. But B is a Q-rad-semiprime submodule in D', hence $a\, b\, \Phi\,(y) \in B$. Thus $aby \in \Phi^{-1}(B)$. Therfore $\Phi^{-1}(B)$ is Q-rad-semi prime submodule of D'.

Proposition (2.15):

Let D and D' be R-modules and let $\Phi: D \to D'$ be an epimorphism. If B is a Qrad-semiprime submodule of D with $Ker\Phi \subseteq B$, then $\Phi(B)$ is Qrad-semiprime submodule of D' .

Proof: Assume that $\Phi(B)$ is not proper submodule of D',so that $\Phi(B) = D'$, let $y \in D$, then $\Phi(y) \in D' = \Phi(B)$, which implies that $\Phi(y) = \Phi(b)$ for some $b \in B$, it follows that $\Phi(y - b) = 0$, so $y - b \in Ker\Phi \subseteq B + rad(D)$, hence $y \in B$, that means $B = D$, we get a contradiction. Because of $Ker\Phi \subseteq B$ this implies $Ker\Phi \subseteq B + rad(D)$, therefore $\Phi(B)$ is a proper submodule of D'. Now let a , $b \in R$ and $y' \in D'$, such that $a^2 sy' \in \Phi(B) + rad(D')$, since Φ is epimorphism, then $\Phi(y) = y'$ for some $y \in D$, so $a^2\, s\, \Phi(y) \in \Phi(B) + rad(D')$ this implies that $a^2\, s\, \Phi(y) \in \Phi(B) + \Phi(rad(D))$ **[2]** , it follows that $a^2\, s\, \Phi(y) = \Phi(x) + \Phi(s)$ for some $x \in B, s \in rad(D)$ that is $\Phi(a^2 sy - x - s) = 0$, so $a^2 sy - x - s \in Ker\Phi \subseteq B \subseteq B + rad(D)$, which implies that $a^2 sy \in B + rad(D)$. But B is a Q-rad-semiprime submodule of D, so $asy \in B$, it follows that $as\Phi(y) \in \Phi(B)$. Hence $asy' \in \Phi(B)$. Thus $\Phi(B)$ is Q rad-semi prime submodule of D'.

Proposition (2.16):
Let $D = D_1 \oplus D_2$ where D_1 and D_2 be R-modules. If $B + rad(D) = (B_1 + rad(D_1)) \oplus (B_2 + (rad(D_2))$ is Q-rad-semiprime submodule of D with $B \subseteq rad(D)$then B_1 and B_2 are Q-rad-semiprime of D_1 and D_2, respectively.

Proof: To prove that B_1 is Q-rad-semiprime of D_1. Let $a, s \in R$, and $y_1 \in D_1$ such that $a^2 s\ y_1 \in B_1 + rad(D_1)$, and $(a^2 s\ y_1, 0) \in (B_1 + rad(D_1)) \oplus (B_2 + rad(D_2))$, so that $a^2 s\ (y_1, 0) \in \big(B_1 + rad(D_1)\big) \oplus \big(B_2 + rad(D_2)\big) = B + rad(D_1)$,and we have $a^2 s\ (y_1, 0) \in (B_1 \oplus B_2) + \big(rad(D_1) \oplus rad(D_2)\big) = B + rad(D)$. But B is Q-rad-semiprime submodule of D, so $a\, s\, (y_1, 0) \in B_1 \oplus B_2 = B$, thus $a\, s\, y_1 \in B_1$. Therefore B_1 is Q-rad-semiprime of D_1. By similar way we can prove that B_2 is Q-rad-semiprime of D_2.

3. Quasi Radical- Semiprime Submodules in Multiplication modules

Remark (3.1):
If B is a Q-rad-semiprime submodule of an R-module D, then $[B{:}D]$ needs not to be Q-rad-semiprime ideal of R.

Proposition (3.2):
Let B asubmodule of an R-module D. If B is a Q-rad-semiprime submodule of D with $rad(R) \subseteq [B{:}D]$, then $[B{:}D]$ is Q-rad-semi prime ideal of R.

Proof: Let $a^2 bt \in [B{:}D] + rad(R)$ where $a, b, t \in R$, we have to show that $aby \in [B{:}D]$. Since $rad(R) \subseteq [B{:}D]$, which implies $[B{:}D] + rad(R) = [B{:}D]$. Hence $a^2 bt \in [B{:}D]$, this implies that $a^2 b\langle tD\rangle \subseteq B \subseteq B + rad(D)$, but B is Q-rad-semi prime submodule of D, then by Corollary (2.7) $ab\langle tD\rangle \subseteq B$. Hence $aby \in [B{:}D]$. Therefore $[B{:}D]$ is Q-rad-semiprime ideal of R.

Recall an R-module D is called a multiplication module if for each submodule B of D there exists an ideal I of R such that $B = ID$. In fact D is called a multiplication module if $[B{:}D]D = B$ for each submodule B of D[8].

Proposition (3.3):
Let D be a finitely generated multiplication faithful R-module with $rad(D) = rad(R) . D$. If $[B{:}D]$ is a Q-rad-semiprime ideal of R, then B is a Q-rad-semiprime submodule of D.

Proof: Let $a^2 by \in B + rad(D)$ for $a, b \in R$,and $y \in D$,then $a^2 b(y) \subseteq B + rad(D)$. Since D is multiplication R-module, then $(y) = ID$ for some ideal I of R and $B = [B{:}D]D$, hence $a^2 bID \subseteq [B{:}D]D + rad(R)$D where $rad(D) = rad(R).D$.
But D is finitely generated multiplication faithful R-module, so $a^2 bI \subseteq [B{:}D] + rad(R)$, since $[B{:}D]$ is a Q-rad-semiprime ideal of R, then $abI \subseteq [B{:}D]$, hence $abID \subseteq [B{:}D]D$,then $abID \subseteq B$, so $ab(y) \subseteq B$, implies $aby \in B$.Thus B is a Q-rad-semi prime submodule of D.

Proposition (3.4):
Let D be a finitely generated multiplication faithful R-module with $rad(D) = rad(R).D$. If I is a Q-rad-semiprime ideal of R, then ID is a Q rad-semi prime submodule of D.

Proof: Let $a^2 by \in ID + rad(D)$ for $a, b \in R$,and $y \in D$,then $a^2 b(y) \subseteq ID + rad(D)$. Since D is multiplication R-module, then $(y) = JD$ for some ideal J of R and since $rad(D) = rad(R).D$, then $a^2 bJD \subseteq ID + rad(R).D$, then $a^2 JD \subseteq (I + rad(R))D$. Since D is a finitely generated multiplication faithful , then $a^2 bJ \subseteq I + rad(R)$. But I is Q-rad-semiprime ideal of R, then $abJ \subseteq I$, so $abJD \subseteq ID$. Hence $ab < y > \subseteq ID$, which implies that $aby \in ID$. Thus ID is Q-rad-semi prime submodule of D.

Proposition (3.5):
Let D be a faithful finitely generated multiplication R-module, and B be a proper submodule of D with $rad(D) = rad(R)D$ and $rad(R) \subseteq [B{:}D]$. then the following statements are equivelant:
1- B is Q-rad-semiprime submodule of D.

2-$[B\colon D]$ is Q-rad-semiprime ideal of R .
3-$B = ID$ for some Q rad-semi prime ideal I of R.

Proof:

1→2) It is clear by Proposition (3.2).
2→1) It is clear by Proposition (2.11).
2→3) Since $[B\colon D]$ is Q-rad-semiprime ideal of R and $B = [B\colon D]D$. It follows that $B = ID$ and $I = [B\colon D]$ an Q-rad-semiprime ideal of R.
3→2) Assume that $B = ID$ and $I = [B\colon D]$ an Q-rad-semiprime ideal of R. But D be multiplication we have $B = [B\colon D]D = ID$, since D be a faithful finitely generated multiplication, then $I = [B\colon D]$, implies that $[B\colon D]$ is Q-rad-semiprime ideal of R .

Proposition (3.4):

If B is a proper submodule of an R-module D. Then B is a Q-rad-semiprime submodule of D if and only if $[B:_D J]$ is a Q-rad-semiprime submodule of D for every ideal J of R where $rad(D).J \subseteq B$.

Proof:

→) Let $a^2by \in [B:_D J] + rad(D)$ for $a, b \in R,\ and\ y \in D,$ since $rad(D).J \subseteq B$, then $rad(D) \subseteq [B:_D J]$, which implies that$[B:_D J] + rad(D) = [B:_D J]$, that is $a^2by \in [B:_D J]$, so $a^2byJ \subseteq B$ and $a^2byt \subseteq B$ for each $t \in J$, since B is a Q-rad-semiprime submodule of D, hence $abyt \in B$ for each $t \in J$, so $abyJ \subseteq B$, thus $aby \in [B:_D J]$, which implies that $[B:_D J]$ is a Qrad-semiprime submodule of D.
←) Suppose that $[B:_D J]$ is a Q-rad-semiprime for each ideal J of R. If $J = R$, this implies that $[B\colon R] = B$ is a Q rad-semi prime submodule of D.

Proposition (3.5):

Let D be a multiplication R-module and B be proper submodule of D, then B is a Q-rad-semiprime submodule of D if and only if $A^2Cy \subseteq B + rad(D)$ implies that $ACy \subseteq B$ for each submodules A and C of D and $y \in D$.

Proof:

→) Assume that $A^2Cy \subseteq B + rad(D)$ where $A, and\ C$ are submodule of D with $y \in D$. Since D is a multiplication R-module. Then $A = ID$ and $C = JD$ for som ideal I,J of R. Hence $A^2Cy = (ID)^2(JD)y = I^2Jy \subseteq B + rad(D)$. But B is a Q-rad-semiprime submodule of D, which implies that by corollary(2.7) $IJy \subseteq B$. Hence $ACy \subseteq B$.
←) Suppose that $I^2Jy \subseteq B + rad(D)$ where $I, and\ J$ are ideal in R and $y \in D$. But D is a multiplication R-module, so that $A^2Cy \subseteq B + rad(D)$ where $A^2 = (ID)^2 = I^2D$ and $C = JD$. By assumption, we have $ACy \subseteq B$, so that $IJy \subseteq B$. Hence by corollary (2.7) B is a Q-rad-semi prime submodule of D.

Proposition (3.6):

Let B be a proper submodule of a multiplication R-module D. Then B is a Q-rad-semiprime submodule of D if and only if $A^2CK \subseteq B + rad(D)$ where A,C, and K are submodules of D , then $ACK \subseteq B$.

Proof:

→) Assume that $A^2CK \subseteq B + rad(D)$ where $A, C,$and K are submodule of D. Since D is a multiplication R-module. Then $A = ID$ and $C = JD$ for som ideal I,J of R. Hence $A^2CK = I^2JK \subseteq B + rad(D)$. But B is a Q-rad-semiprime submodule of D. From Corollary(2.7), we have $IJK \subseteq B$. Hence $ACK \subseteq B$.
←) Suppose that $I^2JK \subseteq B + rad(D)$ where I, and J are ideal in R. Since D is a multiplication R-module, then $I^2JK = A^2CK \subseteq B + rad(D)$ where $A^2 = (ID)^2 = I^2D$ and $C = JD$. By assumption, we have $ACK \subseteq B$, which implies $IJK \subseteq B$. By Corollary (2.7), we get B is a Q rad-semi prime submodule of D.

Recall that a proper submodule B of an R −module D is called a primary submodule if for each $a \in R, y \in D$ such that $ay \in B$, then either $a \in B$ or $a^k \in [B: D]$ for some $k \in Z^+$[9].

Proposition (3.7):

Let B be a primary submodule of an R-module D with $rad(D) \subseteq B$. Then the following statements are equivalent:

1. B is a quasi-prime submodule of D.

2. B is a Q-rad-semi prime of D.

3. B is semi prime submodule of D.

Proof:

(1) → (2) by Remark and Example (2.5,4) every quasi-prime submodule is Q rad-semi prime submodule.

(2) → (3) B is a semiprime submodule of D by Remark and Example (2.5, 2).

(3) → (1) Let $a, b \in R$ and $y \in D$, such that $a\, b\, y \in B$, we have to show that $a\, y \in B$ or $by \in B$, let $by \notin B$, since B is a primary submodules of D, then $a^k \in [B : D]$ and we get $a \in \sqrt{[B : D]}$. But B is a semiprime of D, then [B : D] is semiprime ideal by [6], so $a \in [B : D]$. Therefore $ay \in B$ for all $y \in D$. Which implies that B is a quasi-prime submodule of D.

Proposition (3.8):

If B is a proper submodule of multiplication R-module D with $rad(D) \subseteq B$ and $[B: D]$ is a primary ideal of R. Then the following statements are equivalent**:**

1- B is a quasi-prime submodule of D.

2- B is a Q-rad-semi prime submodule of D.

3- B is Q-semi prime submodule of D.

Proof:

1→2) Since $ad(D) \subseteq B$ then by Remark and Example (2.5,4) B is Q-rad-semi prime submodule.

2→3) It follows from Remark (2.9) B is Q-semi prime submodule of D.

3→1) Since B is a Q-semi prime submodule of then $[B: D]$ is semi prime ideal of R by [7]. From assumption $[B: D]$ is a primary ideal of R, we get $[B: D]$ is a prime ideal. Since D is multiplication R -module then B is a quasi-prime submodule of D [1, proposition (2.1.9)].

3→1) Since $rad(D) \subseteq B$ implies that B is rad-semi prime.Then B is a Q-rad-semiprime submodule of D by Remark and Example (2.5, 7).

References:

[1] Abdul-Razak, M. H. "Quasi-Prime Modules and Quasi- Prime Submodules", M. Sc. Thesis, University of Baghdad, 1999.

[2] James Jenkins, Patrick F. Smith “The Prime Radical of a Modules Over Commutative Rings”, *Communications in Algebra*, vol. 20, no. 1, pp. 3593 – 3602, 1992.

[3] Bülent Saraç. “On Semi prime Submodules”, *Communications in Algebra*, vol. 37, no. 7, pp. 2485 – 2495, 2009.

[4] Al-Mothafar S. Nuhad and Abdul-Alkalik, A. J. “Nearly Prime Submodules”, *International J. of Advanced Scientific and Technical Research*, vol. 5, no. 6, pp. 166-173, 2015.

[5] Al-Mothafar S. Nuhad and Abdul-Alkalik, A. J. “Nearly Quasi-Prime Submodules”, *International J. of Advanced and Research*, vol. 5, no. 1, pp. 170-180, 2017.

[6] Athab, E. A. "Prime Submodules and Semi prime Submodules”, M. Sc.Thesis, University of Baghdad, 1996.

[7] Omar M Al-Ragab and Al-Mothafar N.S. “Some Result on Quasi-Semi prime Submodules”, *Journal of physics*: Conference Series, vol. 1818, 012104, 2021.

[8] Smith P.F. "Some Remarks on Multiplication Modules", *Arch. Math.*, vol. 50, (1988), pp. 223 - 225, 1988.

[9] R. L. McCasland and P. F. Smith. "Prime Submodules of Noetherian Modules", *Rocky Mt. Math*. Vol. 23, pp. 1041-1062, 1993.