Quasi-Radical Semiprime Submodules

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Abstract
In this paper, we introduce the concept of a quasi-radical semi prime submodule. Throughout this work, we assume that is a commutative ring with identity and is a left unitary module. A proper submodule of is called a quasi-radical semi prime submodule (for short Q-rad-semiprime), if , where is the intersection of all prime submodules of .

Keywords: Semi prime submodule, Quasi-semi prime submodule, Radical semi prime submodule, Quasi-radical semi prime submodule.

1. Introduction
A quasi-prime submodule was introduced and studied in 1999 by Abdul-Razak, M. H. in [1], which is generalization of a prime submodule. A proper submodule of an -module is called prime if whenever then either or . Several generalizations of prime submodules have been introduced such as Semi prime, nearly prime, and nearly quasi-prime submodules [3,4,5]. In this paper, we give another generalization of a prime submodule, where a proper submodule of is called a quasi-radical semi prime submodule (for short Q-rad-semiprime), if whenever , then . Where is the intersection of all prime submodules of .

2. Basic Properties of Quasi-Radical Semiprime Submodules
In this section, we introduce the concept of the quasi-radical semiprime submodule, we also give some examples, some basic properties, and characterizations of this concept.
Definition (2.1): A proper submodule $B$ of an $R$-module $D$ is said to be quasi-radical semi prime if whenever $a^2 by \in B + rad(D)$ for $a, b \in R$, $y \in D$ and $k \in Z^+$, then $aby \in B$. It is denoted by $Q$-rad-semiprime.

Theorem (2.2): A submodule $B$ of an $R$-module $D$ is $Q$-rad-semiprime if and only if $a^2 by \in B + rad(D)$ for $a, b \in R$, $y \in D$ implies that $aby \in B$.

Proof: Since $B$ is $Q$-rad-semiprime, then by definition (2.1) we have this direction. For the converse, we suppose that for $a, b \in R$, $y \in D, k \in Z^+$ such that $a^2 by \in B + rad(D)$. Now $a^2 b(a^{k-2} y) \in B + rad(D)$. This implies that $a^{k-1} by \in B$. After a finite number of steps, we get that $aby \in B$. This implies that $B$ is $Q$-rad-semi prime submodule of $D$.

Proposition (2.3): Let $B$ be a proper submodule of an $R$-module $D$, with $rad(D) \subseteq B$. Then $B$ is $Q$-rad-semiprime submodule if and only if $[B + rad(D): y]$ is a semiprime ideal of a ring $R$ for each $y \in D$.

Proof: Let $B$ be $Q$-rad-semiprime. To prove $[B + rad(D): (y)]$ is a semiprime ideal of $R$, it is enough to show

$$\sqrt{[B + rad(D): (y)]} \subseteq [B + rad(D): (y)].$$  

Now, let $a \in \sqrt{[B + rad(D): (y)]}$, then $a^k \in [B + rad(D): (y)]$ for some $k \in Z$, so that $a^k.1.y \in B + rad(D)$. But $B$ is $Q$-rad-semiprime, this implies $a^k.y \in B$. Since $rad(D) \subseteq B$, then $ay \in B + rad(D)$, which implies that $a \in [B + rad(D): (y)]$, so that $\sqrt{[B + rad(D): (y)]} \subseteq [B + rad(D): (y)]$ which means that $[B + rad(D): (y)]$ is a semiprime ideal of $R$.

Suppose that $[B + rad(D): (y)]$ is a semiprime ideal of $R$, let $a^kby \in B + rad(D)$ for $a, b \in R$, $y \in D$, and $k \in Z^+$, this implies that $a^k \in [B + rad(D): (y)]$. But $[B + rad(D): (y)]$ is a semiprime ideal, this implies that $ab \in [B + rad(D): (y)]$, hence $aby \in B + rad(D)$. Thus $aby \in B$, which implies that $B$ is $Q$-rad-semiprime submodule of $D$.

Corollary (2.4): A proper submodule $B$ is $Q$-rad-semiprime submodule of an $R$-module $D$ with $rad(D) \subseteq B$ if and only if $[B + rad(D): (y)]$ is a semiprime ideal of $R$ for each $y \in D$.

Remark and Examples (2.5):

1- Every prime submodule is $Q$-rad-semiprime submodule.

Proof: Let $B$ be a prime submodule of $D$, then $[B: D]$ is prime ideal of $R$ [6]. Let $a, b \in R$, $y \in D$ such that $a^2 by \in B + rad(D)$, since $B$ be a prime submodule of $D$ and $rad(D)$ is the intersection of prime submodule, then $a^2 by \in B$. Primness of $B$ implies that either $aby \in B$ or $a^2 \in [B + rad(D): D]$. Thus $a \in [B + rad(D): D]$ and $aby \in B$.

The next example shows that the converse of (1) is not true.

Example: Let $B = 10Z$ be a submodule of $Z$ as $Z$-module, and $B$ is $Q$-rad-semiprime, since $[10Z + rad(Z): y] = [10Z + 0: y] = [10Z: y] = 10Z$, then it is semiprime ideal of $Z$. However, it is not prime submodule.

2- Every $Q$-rad-semiprime submodule is semi prime submodule.

Proof: Let $a^2 y \in B$ such that $a \in R$, and $y \in D$, then $a^2 y \in B + rad(D)$ by definition (2.1), so that we have $ay \in B$. Hence $B$ is a semi prime submodule of $D$.

The following example proves that the converse of (2) is not true.

Example: Let $D = Z_2 \oplus Z_4$ as $Z$-module and $B = Z_2 \oplus 0$ is a submodule of $D$. $B + rad(D) = \{(0,0), (1,0), (0,2), (1,2)\}$, then $B$ is not $Q$-rad-semiprime submodule of, since $9 \times
1(1, 2) = 3^2 \times 1(1, 2) \subseteq B + \text{rad}(D)$, but $3 \times 1(1, 2) \not\subseteq B$, where $B = \{(0,0), (1,0)\}$, \text{rad}(D) = \text{rad}(Z_2) \oplus \text{rad}(Z_4) = \{(0,0), (0,2)\}[2]$. Therefore $B$ is a semi prime submodule.

3- Every maximal submodule of an R-module $D$ is a Q-rad-semiprime submodule. Since every maximal submodule of $D$ is prime, then by (1) it is Q-rad-semiprime.

4- Every quasi-prime submodule $B$ of an R-module $D$ with $\text{rad}(D) \subseteq B$ is Q-rad-semiprime submodule.

**Proof:** Let $B$ be a quasi-prime submodule of an R-module $D$, by [1]. $[B : \{y\}]$ is a prime ideal of $R$ for each $y \in D$, hence $[B : \{y\}]$ is a semi prime ideal of $R$ for each $y \in D$. But $\text{rad}(D) \subseteq B$ so $B + \text{rad}(D) = B$ implies $[B + \text{rad}(D) : \{y\}]$ is a semi prime ideal of $R$ for each $y \in D$. Thus $B$ is Q-rad-semiprime, by Corollary (2.4).

The converse is not true in general for example:

Let $D = Z_{12}$ as a Z-module and $= <\tilde{6}>$. $B$ is Q-rad-semiprime submodule of $D$ since $[<\tilde{6}> + \text{rad}(Z_{12}) : \{y\}] = [<\tilde{6}> + <\tilde{6}> : \{y\}] = 6Z$, which is a semi prime ideal of $R$. But $B = <\tilde{6}>$ is not quasi-prime submodule of $Z_{12}$, since $[<\tilde{6}> : <\tilde{1}>] = 6Z$ is not a prime ideal of a ring $Z$.

5- A submodule of Q-rad-semiprime needs not to be Q-rad semi prime. Example: $<\tilde{2}>$ in $Z_{12}$ as a Z-module is Q-rad-semiprime submodule, since $<\tilde{2}>$ is a prime. But $<\tilde{4}>$ is a submodule of $<\tilde{2}>$, which is not Q-rad-semiprime of $Z_{12}$, since $2^2 \times 1 \times \tilde{1} \not\subseteq <\tilde{4}> + \text{rad}(Z_{12})$ but $2 \times 1 \times \tilde{1} \not\subseteq <\tilde{4}>$.

6- In the Z-module $D = Z$ the submodule $B = nZ$ is Q-rad-semiprime if $n$ is a prime number.

7- It is clear that every semi-prime submodule is Q-rad-semiprime submodule.

**Proposition (2.6):** Let $D$ be an R-module and $B$ be a proper submodule of $D$. Then $B$ is Q-rad-semiprime submodule of $D$ if and only if $I^kJC \subseteq B + \text{rad}(D)$ for some ideals $I, J \in R, k \in Z^+$ and $C$ some submodule of $D$ implies $IJC \subseteq B$.

**Proof:**

$\rightarrow$) Assume that $B$ is a Q-rad-semiprime submodule of $D$, and $I^kJC \subseteq B + \text{rad}(D)$ for some ideal $I, J$ of $R$ and some submodule $C$ of $D$ and $k \in Z^+$, we have to show that $IJC \subseteq B$. Let $x \in IJC$, then $x = r_1s_1x_1 + r_2s_2x_2 + \cdots + r_ns_nx_n$ where $r_i \in I, s_i \in J, x_i \in C, i = 1, 2, \ldots, n$, thus $r_is_ix_i \in IJC$ for each $i = 1, 2, \ldots, n$, then $r_is_ix_i \in I^kJC \subseteq B + \text{rad}(D)$, but $B$ is Q-rad-semiprime submodule of $D$, therefore $r_is_ix_i \in B$ for each $i = 1, 2, \ldots, n$, thus $x \in B$, which implies that $IJC \subseteq B$.

$\leftarrow$) Suppose $r^ksy \in B + \text{rad}(D), where r, s \in R, y \in D, and k \in Z^+$ implies $r^kys > < r^ks > < y > \subseteq B + \text{rad}(D)$, by hypothesis $< r^ks > < y > \subseteq B$, hence $rsy \in B$. Thus $B$ is rad-semiprime submodule in $D$.

**Corollary (2.7):**

$B$ is Q-rad-semiprime submodule of $D$ if and only if $< a >^k < b > C \subseteq B + \text{rad}(D)$, where $k \in Z^+$ and $C$ some submodule of $D$ implies $< a > < b > C \subseteq B$.

**Corollary (2.8):**

$B$ is Q-rad-semiprime submodule of $D$ if and only if $a^kbc \subseteq B + \text{rad}(D)$, where $k \in Z^+$ and $C$ some submodule of $D$ implies $abc \subseteq B$.

Recall that a proper submodule $B$ of an R-module $D$ is said to be quasi-semi prime for short (Q-semi prime) if whenever $a^kby \in B$ where $a, b \in R, y \in D$ and $k \in Z^+$ implies that $aby \in B$ [8].

**Remark (2.9):**

If $B$ is Q-rad-semiprime submodule of an R-module $D$. Then $[B : D]$ is a Q-semi prime ideal of $R$.

**Proof:** Let $a^2bt \in [B : D], where a, b, and t \in R$, which implies $a^2b(tD) \subseteq B \subseteq B + \text{rad}(D)$, since $B$ is Q-rad-semiprime submodule of $D$, then by Corollary
(2.8), we have \( ab(tD) \subseteq B \). Hence \( abt \in [B:D] \). Therefore \([B:D]\) is a Q-semi prime ideal of \( R \).

Remark (2.10):
Let \( B \) be a submodule of an \( R \)-module \( D \), if \([B:D]\) is Q-rad-semiprime ideal of \( R \), then \( B \) cannot be Q rad-semi prime in general. For example: Let \( D = \mathbb{Z} \oplus \mathbb{Z} \) be a \( \mathbb{Z} \)-module and \( B = \langle 18 \rangle + \langle 0 \rangle \) then \([B:D] = \langle 0 \rangle \) is Q-rad-semiprime ideal of \( \mathbb{Z} \) but \( B \) is not Q-rad-semiprime submodule of \( D \), since \( 3^2 \times 2 \times (1,0) \in B + rad(D) \), but \( 3 \times 2 \times (1,0) \notin B \).

Proposition (2.11):
Let \( D \) be an \( R \)-module. \( B \), \( K \) are submodules of \( D \) and \( K \) is semi prime submodule with \( rad(D) \subseteq K \). Then \( B \cap K \) is Q-rad-semiprime of \( D \).

Proof: Let \( a^nby \in (B \cap K) + rad(D) \) for \( a, b \in R \), \( y \in D \) and \( n \in \mathbb{Z}^+ \), from the modular law, we have \( a^nby \in (B + rad(D)) \cap K \), then \( a^nby \in B + rad(D) \) and \( a^nby \in K \) is Q-rad-semiprime submodule of \( D \) and \( K \) is semi prime submodule, this implies that \( aby \in B \) and \( aby \in K \). Hence \( aby \in B \cap K \), which implies that \( B \cap K \) is Q-rad-semiprime in \( D \).

Corollary (2.12):
If \( B, K \) are Q-rad-semiprime submodule of an \( R \)-module \( D \) with \( rad(D) \subseteq K \), then \( B \cap K \) is a Q-rad-semiprime submodule of \( D \).

Proposition (2.13):
Let \( B, K \) be the submodules of an \( R \)-module \( D \) with \( rad(K) = rad(D) \cap K \) such that \( B \) is a Q-rad-semiprime submodule of \( D \) and \( K \) is not contained in \( B \). Then \( B \cap K \) is a Q-rad-semiprime submodule of \( K \).

Proof: Since \( K \not\subseteq B \), we get \( B \cap K \) is a proper submodule of \( K \). Let \( a, s \in R \), and \( y \in K \), such that \( a^2s y \in (B \cap K) + rad(K) \). Since \( rad(K) = rad(D) \cap K \), then \( a^2s y \in (B + rad(D)) \cap K \), this implies \( a^2s y \in (B + rad(D)) \cap K \) so that \( a^2s y \in B + rad(D) \) and \( a^2s y \in K \). Because of \( B \) is a Q-rad-semiprime submodule of \( D \) and by definition (2.1), we have \( a s y \in B \). Since \( y \in K \) implies that \( a s y \in K \cap B \). Hence \( K \cap B \) is a Q-rad-semiprime submodule of \( K \).

Proposition (2.14):
Let \( D \) and \( D' \) be \( R \)-modules and let \( \Phi : D \rightarrow D' \) be \( R \)-homomorphism. If \( B \) is a Q-rad-semiprime submodule of \( D' \), then \( \Phi^{-1}(B) \) is Q-rad-semi prime submodule of \( D \).

Proof: Let \( a, b \in R \) and \( y \in D \), such that \( a^2b \Phi(y) \in B + \Phi(rad(D)) \), then \( a^2b \Phi(y) \in B + \Phi(rad(D')) \), since \( \Phi(rad(D)) = \Phi(rad(D')) \) [2]. But \( B \) is a Q-rad-semiprime submodule of \( D' \), hence \( ab \Phi(y) \in B \). Thus \( aby \in \Phi^{-1}(B) \). Therefore \( \Phi^{-1}(B) \) is Q-rad-semi prime submodule of \( D' \).

Proposition (2.15):
Let \( D \) and \( D' \) be \( R \)-modules and let \( \Phi : D \rightarrow D' \) be an epimorphism. If \( B \) is a Q-rad-semiprime submodule of \( D \) with \( Ker\Phi \subseteq D \), then \( \Phi(B) \) is Q-rad-semiprime submodule of \( D' \).

Proof: Assume that \( \Phi(B) \) is not proper submodule of \( D' \), so that \( \Phi(B) = D' \), let \( y \in D \), then \( \Phi(y) \in D' = \Phi(B) \), which implies that \( \Phi(y) = \Phi(b) \) for some \( b \in B \), so that \( \Phi(y - b) = 0 \), so \( y - b \in Ker\Phi \subseteq B + rad(D) \), hence \( y \in B \), that means \( B = D \), we get a contradiction. Because of \( Ker\Phi \subseteq B \) this implies \( Ker\Phi \subseteq B + rad(D) \), therefore \( \Phi(B) \) is a proper submodule of \( D' \). Now let \( a, b \in R \) and \( y' \in D' \), such that \( a^2s y' \in \Phi(B) + \Phi(rad(D')) \), since \( \Phi \) is epimorphism, then \( \Phi(y) = y' \) for some \( y \in D \), so \( a^2s \Phi(y) \in \Phi(B) + \Phi(rad(D')) \) this implies that \( a^2s \Phi(y) \in \Phi(B) + \Phi(rad(D)) \) [2], it follows that \( a^2s \Phi(y) = \Phi(x) + \Phi(s) \) for some \( x \in B, s \in rad(D) \) that is \( \Phi(a^2s - x - s) = 0 \), so \( a^2s - x - s \in Ker\Phi \subseteq B + rad(D) \), which implies that \( a^2s y \in B + rad(D) \). But \( B \) is a Q-rad-semiprime submodule of \( D \), so \( asy \in B \), it follows that \( asy' \in \Phi(B) \). Hence \( asy' \in \Phi(B) \). Thus \( \Phi(B) \) is Q rad-semi prime submodule of \( D' \).

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Proposition (2.16):
Let \( D = D_1 \oplus D_2 \) where \( D_1 \) and \( D_2 \) be \( R \)-modules. If \( B + \text{rad}(D) = (B_1 + \text{rad}(D_1)) \oplus (B_2 + \text{rad}(D_2)) \) is \( Q \)-rad-semiprime submodule of \( D \) with \( B \subseteq \text{rad}(D) \) then \( B_1 \) and \( B_2 \) are \( Q \)-rad-semiprime of \( D_1 \) and \( D_2 \), respectively.

Proof: To prove that \( B_1 \) is \( Q \)-rad-semiprime of \( D_1 \). Let \( a, s \in R \), and \( \gamma_1 \in D_1 \) such that \( a^2s \gamma_1 \in B_1 + \text{rad}(D_1) \), and \( (a^2s \gamma_1, 0) \in (B_1 + \text{rad}(D_1)) \oplus (B_2 + \text{rad}(D_2)) \), so that \( a^2s (\gamma_1, 0) \in (B_1 + \text{rad}(D_1)) \oplus (B_2 + \text{rad}(D_2)) = B + \text{rad}(D) \), and we have \( a^2s (\gamma_1, 0) \in (B_1 \oplus B_2) + (\text{rad}(D_1) \oplus \text{rad}(D_2)) = B + \text{rad}(D) \). But \( B \) is \( Q \)-rad-semiprime submodule of \( D \), so \( a \in B_1 \) for some ideal \( \gamma_1 \in B_1 \). Therefore \( B_1 \) is \( Q \)-rad-semiprime submodule of \( D_1 \). By similar way we can prove that \( B_2 \) is \( Q \)-rad-semiprime of \( D_2 \).

3. Quasi Radical-Semiprime Submodules in Multiplication modules

Remark (3.1):
If \( B \) is a \( Q \)-rad-semiprime submodule of an \( R \)-module \( D \), then \( B \) need not to be \( Q \)-rad-semiprime ideal of \( R \).

Proposition (3.2):
Let \( a \) submodule of an \( R \)-module \( D \). If \( B \) is a \( Q \)-rad-semiprime submodule of \( D \) with \( \text{rad}(R) \subseteq [B: D] \), then \( [B: D] \) is \( Q \)-rad-semiprime ideal of \( R \).

Proof: Let \( a^2bt \in [B: D] + \text{rad}(R) \) where \( a, b, t \in R \), we have to show that \( aby \in [B: D] \).
Since \( \text{rad}(R) \subseteq [B: D] \), which implies \( [B: D] + \text{rad}(R) = [B: D] \). Hence \( a^2bt \in [B: D] \), this implies that \( a^2b(tD) \subseteq B \subseteq B + \text{rad}(D) \), but \( B \) is \( Q \)-rad-semi prime submodule of \( D \), then by Corollary (2.7) \( ab(tD) \subseteq B \). Hence \( aby \in [B: D] \). Therefore \( [B: D] \) is \( Q \)-rad-semiprime ideal of \( R \).

Recall an \( R \)-module \( D \) is called a multiplication submodule if for each submodule \( B \) of \( D \) there exists an ideal \( I \) of \( R \) such that \( B = ID \). In fact \( D \) is called a multiplication module if \( [B: D]D = B \) for each submodule \( B \) of \( D \).

Proposition (3.3):
Let \( D \) be a finitely generated multiplication faithful \( R \)-module with \( \text{rad}(D) = \text{rad}(R) \). If \( [B: D] \) is a \( Q \)-rad-semiprime ideal of \( R \), then \( B \) is a \( Q \)-rad-semiprime submodule of \( D \).

Proof: Let \( a^2by \in B + \text{rad}(D) \) for \( a, b \in R \), and \( \gamma \in D \), then \( a^2b(y) \subseteq B + \text{rad}(D) \). Since \( D \) is multiplication \( R \)-module, then \( (y) = ID \) for some ideal \( I \) of \( R \) and \( B \subseteq [B: D]D \), hence \( a^2bID \subseteq [B: D]D + \text{rad}(D)D \) where \( \text{rad}(D) = \text{rad}(R)D \).
But \( D \) is finitely generated multiplication faithful \( R \)-module, so \( a^2bI \subseteq [B: D] + \text{rad}(R) \), since \( [B: D] \) is a \( Q \)-rad-semiprime ideal of \( R \), then \( abI \subseteq [B: D] \), hence \( abID \subseteq [B: D]D \), then \( abID \subseteq B \), so \( ab(y) \subseteq B \), implies \( aby \in B \). Thus \( B \) is a \( Q \)-rad-semi prime submodule of \( D \).

Proposition (3.4):
Let \( D \) be a finitely generated multiplication faithful \( R \)-module with \( \text{rad}(D) = \text{rad}(R) \). If \( I \) is a \( Q \)-rad-semiprime ideal of \( R \), then \( ID \) is a \( Q \)-rad-semi prime submodule of \( D \).

Proof: Let \( a^2by \in ID + \text{rad}(D) \) for \( a, b \in R \), and \( \gamma \in D \), then \( a^2b(y) \subseteq ID + \text{rad}(D) \). Since \( D \) is multiplication \( R \)-module, then \( (y) = JD \) for some ideal \( J \) of \( R \) and since \( \text{rad}(D) = \text{rad}(R)D \), then \( a^2bJD \subseteq ID + \text{rad}(R)D \), then \( a^2bJD \subseteq (I + \text{rad}(R))D \). Since \( D \) is a finitely generated multiplication faithful \( R \)-module, then \( a^2bJ \subseteq I + \text{rad}(R) \). But \( I \) is \( Q \)-rad-semi prime ideal of \( R \), then \( ab \subseteq I \), so \( abJD \subseteq ID \). Hence \( ab < y > \subseteq ID \), which implies that \( aby \in ID \). Thus \( ID \) is \( Q \)-rad-semi prime submodule of \( D \).

Proposition (3.5):
Let \( D \) be a faithful finitely generated multiplication \( R \)-module, and \( B \) be a proper submodule of \( D \) with \( \text{rad}(D) = \text{rad}(R)D \) and \( \text{rad}(R) \subseteq [B: D] \). then the following statements are equivalent:
1- \( B \) is \( Q \)-rad-semiprime submodule of \( D \).
2- \([B:D]\) is Q-rad-semiprime ideal of \(R\).

3- \(B = ID\) for some Q rad-semi prime ideal \(I\) of \(R\).

**Proof:**

1→2) It is clear by Proposition (3.2).

2→1) It is clear by Proposition (2.11).

2→3) Since \([B:D]\) is Q-rad-semiprime ideal of \(R\) and \(B = [B:D]D\). It follows that \(B = ID\) and \(I = [B:D]\) an Q-rad-semiprime ideal of \(R\).

3→2) Assume that \(B = ID\) and \(I = [B:D]\) an Q-rad-semiprime ideal of \(R\). But \(D\) be multiplication we have \(B = [B:D]D = ID\), since \(D\) be a faithful finitely generated multiplication, then \(I = [B:D]\), implies that \([B:D]\) is Q-rad-semiprime ideal of \(R\).

**Proposition (3.4):**

If \(B\) is a proper submodule of an \(R\)-module \(D\). Then \(B\) is a Q-rad-semiprime submodule of \(D\) if and only if \([B:D]\) is a Q-rad-semiprime submodule of \(D\) for every ideal \(J\) of \(R\) where \(\text{rad}(D).J \subseteq B\).

**Proof:**

→) Let \(a^2by \in [B:D] + \text{rad}(D)\) for \(a,b \in R\), and \(y \in D\), since \(\text{rad}(D).J \subseteq B\), then \(\text{rad}(D) \subseteq [B:D]J\), which implies that \([B:D] + \text{rad}(D) = [B:D]\), that is \(a^2by \in [B:D]\), so \(a^2by \subseteq B\) and \(a^2by \subseteq B\) for each \(t \in J\), since \(B\) is a Q-rad-semiprime submodule of \(D\), hence \(aby \in B\) for each \(t \in J\), so \(aby \subseteq B\), thus \(aby \in [B:D]\), which implies that \([B:D]\) is a Q-rad-semiprime submodule of \(D\).

←) Suppose that \([B:D]\) is a Q-rad-semiprime for each ideal \(J\) of \(R\). If \(J = R\), this implies that \([B:R] = B\) is a \(Q\)-rad-semi prime submodule of \(D\).

**Proposition (3.5):**

Let \(D\) be a multiplication \(R\)-module and \(B\) be proper submodule of \(D\), then \(B\) is a Q-rad-semiprime submodule of \(D\) if and only if \(A^2Cy \subseteq B + \text{rad}(D)\) implies that \(ACy \subseteq B\) for each submodules \(A\) and \(C\) of \(D\) and \(y \in D\).

**Proof:**

→) Assume that \(A^2Cy \subseteq B + \text{rad}(D)\) where \(A\), \(B\), and \(C\) are submodule of \(D\) with \(y \in D\). Since \(D\) is a multiplication \(R\)-module. Then \(A = ID\) and \(C = JD\) for som ideal \(I, J\) of \(R\). Hence \(A^2Cy = (ID)^2(JD)y = I^2Jy \subseteq B + \text{rad}(D)\). But \(B\) is a Q-rad-semiprime submodule of \(D\), which implies that by corollary(2.7) \(I^2Jy \subseteq B\). Hence \(ACy \subseteq B\).

←) Suppose that \(I^2Jy \subseteq B + \text{rad}(D)\) where \(I\) and \(J\) are ideal in \(R\) and \(y \in D\). But \(D\) is a multiplication \(R\)-module, so that \(A^2Cy \subseteq B + \text{rad}(D)\) where \(A^2 = (ID)^2 = I^2D\) and \(C = JD\).

By assumption, we have \(ACy \subseteq B\), so that \(I^2Jy \subseteq B\). Hence by corollary (2.7) \(B\) is a Q-rad-semi prime submodule of \(D\).

**Proposition (3.6):**

Let \(B\) be a proper submodule of a multiplication \(R\)-module \(D\). Then \(B\) is a Q-rad-semiprime submodule of \(D\) if and only if \(A^2CK \subseteq B + \text{rad}(D)\) where \(A,C,\) and \(K\) are submodules of \(D\), then \(ACK \subseteq B\).

**Proof:**

→) Assume that \(A^2CK \subseteq B + \text{rad}(D)\) where \(A, C,\) and \(K\) are submodule of \(D\). Since \(D\) is a multiplication \(R\)-module. Then \(A = ID\) and \(C = JD\) for som ideal \(I, J\) of \(R\). Hence \(A^2CK = I^2JK \subseteq B + \text{rad}(D)\). But \(B\) is a Q-rad-semiprime submodule of \(D\). From Corollary(2.7), we have \(IJK \subseteq B\). Hence \(ACK \subseteq B\).

←) Suppose that \(I^2JK \subseteq B + \text{rad}(D)\) where \(I, J\) are ideal in \(R\). Since \(D\) is a multiplication \(R\)-module, then \(I^2JK = A^2CK \subseteq B + \text{rad}(D)\) where \(A^2 = (ID)^2 = I^2D\) and \(C = JD\). By assumption, we have \(ACK \subseteq B\), which implies \(IJK \subseteq B\). By Corollary (2.7), we get \(B\) is a Q rad-semi prime submodule of \(D\).

Recall that a proper submodule \(B\) of an \(R\)-module \(D\) is called a primary submodule if for
each $a \in R$, $y \in D$ such that $ay \in B$, then either $a \in B$ or $a^k \in [B : D]$ for some $k \in \mathbb{Z}^+[9]$. 

**Proposition (3.7):**

Let $B$ be a primary submodule of an $R$-module $D$ with $\text{rad}(D) \subseteq B$. Then the following statements are equivalent:

1. $B$ is a quasi-prime submodule of $D$.
2. $B$ is a $Q$-rad-semi prime of $D$.
3. $B$ is semi prime submodule of $D$.

**Proof:**

(1) $\implies$ (2) by Remark and Example (2.5,4) every quasi-prime submodule is $Q$ rad-semi prime submodule.

(2) $\implies$ (3) is a semiprime submodule of $D$ by Remark and Example (2.5, 2).

(3) $\implies$ (1) Let $a, b \in R$ and $y \in D$, such that $a \cdot b \cdot y \in B$, we have to show that $a \cdot y \in B$ or $b \cdot y \in B$, let $b \notin B$, since $B$ is a primary submodules of $D$, then $a^k \in [B : D]$ and we get $a \in \sqrt{[B : D]}$. But $B$ is a semiprime of $D$, then $[B : D]$ is semiprime ideal by [6], so $a \in [B : D]$. Therefore $a \cdot y \in B$ for all $y \in D$. Which implies that $B$ is a quasi-prime submodule of $D$.

**Proposition (3.8):**

If $B$ is a proper submodule of multiplication $R$-module $D$ with $\text{rad}(D) \subseteq B$ and $[B : D]$ is a primary ideal of $R$. Then the following statements are equivalent:

1. $B$ is a quasi-prime submodule of $D$.
2. $B$ is a $Q$-rad-semi prime submodule of $D$.
3. $B$ is $Q$-semi prime submodule of $D$.

**Proof:**

1$\implies$2) Since $a \cdot d(D) \subseteq B$ then by Remark and Example (2.5,4) $B$ is $Q$-rad-semi prime submodule.

2$\implies$3) It follows from Remark (2.9) $B$ is $Q$-semi prime submodule of $D$.

3$\implies$1) Since $B$ is a $Q$-semi prime submodule of then $[B : D]$ is semi prime ideal of $R$ by [7]. From assumption $[B : D]$ is a primary ideal of $R$, we get $[B : D]$ is a prime ideal. Since $D$ is multiplication $R$-module then $B$ is a quasi-prime submodule of $D$ [1, proposition (2.1.9)].

3$\implies$1) Since $\text{rad}(D) \subseteq B$ implies that $B$ is rad-semi prime. Then $B$ is a $Q$-rad-semiprime submodule of $D$ by Remark and Example (2.5, 7).

**References:**


