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On Fixed Point Theorems by Using Rational Expressions in Partially Ordered Metric Space

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Abstract
The main purpose of this paper is to introduce and prove some fixed point theorems for two maps that satisfy $(\phi-\psi)$-contractive conditions with rational expression in partially ordered metric spaces, our results improve and unify a multitude of fixed point theorems and generalize some recent results in ordered partially metric space.

Key words: fixed point, partially ordered set, rational expression, metric space

حول مبرهنات النقطة الصامدة بواسطة التعابير النسبية في الفضاءات المترية المرتبة جزئيا

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الخلاصه
يتناول هذا البحث استعراض ودراسة النقاط الصامدة للدوال التي تحقق الشرط $(\phi-\psi)$ مع التعابير النسبية في الفضاءات المترية المرتبة جزئيا. النتائج التي حصلنا عليها هي تحسين وتوحيد العديد من النتائج في مبرهنات النقطة الصامدة وتعميم بعض النتائج الحديثة في الفضاء المتري المرتب جزئيا.

1. Introduction

Banach contraction principle [1] is the one of most important tools in the study of nonlinear problems in analysis. Therefore, various generalization of Banach contraction principle either by weakening the contractive properties of the map or by extending the structure of the ambient space. For more details see [2] [3] [4] [5] [6].

Run and Reurings [7] developed the fixed point theory and obtained analogue of a Banach's theorem in partially ordered metric space. After this paper Nieto $et\ al.$ [8] [9] [10] proved the existence and uniqueness of solution for the first ordinary differential equation with the periodic boundary conditions, they present a new extension of Banach contraction mapping theorem to partially ordered metric space that allows to be discontinuous functions. A numerous papers have been published on partially ordered metric space, see for instance, ([11] [12] [13] [14] [15] [16] [17] [18]).

The purpose of this paper is to establish some common fixed point results satisfying a contractive condition of rational type endowed with partially ordered metric space.

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2. Preliminaries

We generally follow the definitions and notations used in [19], [20] and [21].

Definition 1 [7]**:** The triple $(M,d,\leq)$ is called a partially ordered metric space, if $(M,\leq)$ is a partially ordered set together with (M,d) is a metric space.

Definition 2 [7]**:** If (M,d) is a complete metric space, then the triple $(M,d,\leq)$ is called a partially ordered complete metric space.

Definition 3 [19]**:** A point $m\in B$, where B is a nonempty subset of a metric space (M,d) is called a common (Coincidence) fixed point of two self-mappings F and T if $Fm=Tm=m$ ($Fm=Tm$).

Definition 4 [19]**:** The two self-mappings F and T defined in a subset B of a metric space (M,d) are called as follows:

(i) Commuting, if $FTm=TFm$ for all $m\in B$.

(ii) Compatible, if for any sequence $\{q_n\}$ with $\lim_{n\to\infty} Fq_n=\lim_{n\to\infty} Tq_n=u$, for some $u\in B$ then $\lim_{n\to\infty} d(FTq_n,TFq_n)=0$.

(iii) Weakly compatible, if they commute at their coincidence points that means if $Fm=Tm$ then $FTm=TFm$.

Definition 5 [15]**:** Assume that F and T be two self-mappings defined in a partially ordered set $(M,\leq)$. A mapping T is called a monotone F non-decreasing if $Fm\leq Fn$ implies $Tm\leq Tn$, for all $m,n\in M$.

Definition 6 [15]**:** Assume that B be a nonempty subset of a partially ordered set $(M,\leq)$.If every two elements of B are comparable, then it is called well ordered set.

Definition 7 [15]**:** A partially ordered metric space $(M,d,\leq)$ is called an ordered complete, if for every convergent sequence $\{q_n\}_{n=0}^{+\infty}\subset M$, one of the following conditions holds

- If $\{q_n\}$ is an increasing sequence in M such that $q_n\to q$ implies $q_n\leq q$, for all $n\in N$ that is $q=\sup\{q_n\}$ or
- If $\{q_n\}$ is an decreasing sequence in M such that $q_n\to q$ implies $q_n\geq q$, for all $n\in N$ that is $q=\inf\{q_n\}$.

Corollary 2.1 [19]**:** Let $(M,d,\leq)$ be a partially ordered complete metric space and suppose that F and T are satisfied the following condition

$$d(Tm,Tn)\leq \alpha\frac{d(Fm,Tm)d(Fn,Tn)}{d(Fm,Fn)+d(Fm,Tn)+d(Fn,Tm)}+\beta d(Fm,Fn). \tag{1.1}$$

For all $m,n\in M$ with $Fm\leq Fn$ and for some $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$.

Suppose that

(i) $TM\subseteq FM$ such that (FM,d) is a complete subset of M

(ii) T is monotone F non-decreasing.

(iii) The pair (T,F) is compatible where F and T are continuous.

Then F and T have a coincidence point that is there exists $u\in M$ such that $Fu=Tu$.

The control function that alters distance between two points in a metric space is introduced by Khan et al. [20], which are called an altering distance function.

Definition 8 [20]**:** A function $\psi:[0,\infty]\to[0,\infty]$ is named an altering distance function if the following conditions are satisfied:

(i) ψ is monotone increasing and continuous function

(ii) $\psi(t)=0$ if and only if $t=0$.

In the following section, we will use the following class of functions.

We denote

$\Psi=\{\psi:[0,\infty]\to[0,\infty]:\psi$ an altering distance functiong $\}$ and

$\Phi=\{\phi:[0,\infty]\to[0,\infty]:\phi$ is lower semicontinuous $\phi(t)<\psi(t)$ and $\phi(0)=0\}$.

Theorem 2.2 [21]**:** Let $(M,d,\leq)$ be a partially ordered complete metric space. Let $T:M\to M$ be a non-decreasing mapping, which satisfies the inequality

$$\psi(d(Tm,Tn))\leq\psi(M_d(m,n))-\phi(M_d(m,n))+L\min\{d(m,Tn),d(n,Tm),d(m,Tm),d(n,Tn)\}$$

For all distinct points $m,n\in M$ with $m\leq n$ where $\phi\in\Phi$, $\psi\in\Psi$, $L\geq 0$ and

$$M_d(m,n)=\max\left\{\frac{d(m,Tm)d(n,Tn)}{d(m,n)},d(m,n)\right\}.$$

In addition, suppose that

(i) T is continuous or

(ii) If $\{q_n\}$ is a non-decreasing sequence in M such that $q_n\to q$, then $q=\sup\{q_n\}$.

If there exists $q_0\in M$ such that $q_0\leq Tq_0$, then T has a fixed point.

3. Main result

Theorem 3.1**:** Let $(M,d,\leq)$ be a partially ordered complete metric space and suppose that F and T are continuous self-mappings on M, $TM\subseteq FM$ and T is monotone F non-decreasing mapping and satisfying the following condition

$$\psi\big(d(Tm,Tn)\big)\leq\psi\big(M_d(m,n)\big)-\phi\big(M_d(m,n)\big)+LN_d(m,n) \tag{3.1}$$

For all $m,n\in M$ with $Fm\leq Fn$ where $\phi\in\Phi$, $\psi\in\Psi$, $L\geq 0$ and

$$M_d(m,n)=\max\left\{\frac{d(Fm,Tm)d(Fn,Tn)}{d(Fm,Fn)+d(Fm,Tn)+d(Fn,Tm)},d(Fm,Fn)\right\}$$

$$N_d(m,n)=\min\{d(Fm,Tn),d(Fn,Tm),d(Fm,Tm),d(Fn,Tn)\}$$

If there exists a point $q_0\in M$ such that $Fq_0\leq Tq_0$ and the mapping F and T are compatible, then F and T have a coincidence point in M.

Proof. Let $q_0\in M$ such that $Fq_0\leq Tq_0$. Since from hypotheses, we have $TM\subseteq FM$,then we can choose a point $q_1\in M$ such that $Fq_1=Tq_0$.But $Tq_1\in FM$, then there exists another point $q_2\in M$ such that $Fq_2=Tq_1$. By continuing the same way, we can define a sequence $\{q_n\}$ in M such that $Fq_{n+1}=Tq_n$, for all $n\in N$.

Again, since we have $Fq_0\leq Tq_0=Fq_1$ and T is monotone F non-decreasing mapping then, we get $Tq_0\leq Tq_1$. Similarly, we obtain $Tq_1\leq Tq_2$, since $Fq_1\leq Fq_2$,then inductively, we get

$$Tq_0\leq Tq_1\leq Tq_2.........\leq Tq_n\leq Tq_{n+1}.........$$

If $d(Tq_{n_0+1},Tq_{n_0})=0$ for some $n_0\in N$ then $Tq_{n_0+1}=Tq_{n_0}=Fq_{n_0+1}$ then, q_{n_0+1} is coincidence point. Assume that $Tq_{n+1}\neq Tq_n$ for all $n\in N$. If $d(Tq_{n+1},Tq_n)>0$, then by condition (3.1), we have

$$\begin{aligned}\psi(d(Tq_{n+1},Tq_n)) &\le \psi(M_d(q_{n+1},q_n))-\phi(M_d(q_{n+1},q_n))+LN_d(q_{n+1},q_n)\\
&\le \psi(\max\left\{\frac{d(Fq_{n+1},Tq_{n+1})d(Fq_n,Tq_n)}{d(Fq_n,Fq_{n+1})+d(Fq_{n+1},Tq_n)+d(Fq_n,Tq_{n+1})},d(Fq_n,Fq_{n+1})\right\})\\
&-\phi(\max\left\{\frac{d(Fq_{n+1},Tq_{n+1})d(Fq_n,Tq_n)}{d(Fq_n,Fq_{n+1})+d(Fq_{n+1},Tq_n)+d(Fq_n,Tq_{n+1})},d(Fq_n,Fq_{n+1})\right\})\\
&+L\min\left\{d(Fq_n,Tq_{n+1}),d(Fq_{n+1},Tq_n),d(Fq_{n+1},Tq_{n+1}),d(Fq_n,Tq_n)\right\}\\
&=\psi(\max\left\{\frac{d(Tq_n,Tq_{n+1})d(Tq_{n-1},Tq_n)}{d(Tq_{n-1},Tq_n)+d(Tq_n,Tq_n)+d(Tq_{n-1},Tq_{n+1})},d(Tq_{n-1},Tq_n)\right\})\\
&-\phi(\max\left\{\frac{d(Tq_n,Tq_{n+1})d(Tq_{n-1},Tq_n)}{d(Tq_{n-1},Tq_n)+d(Tq_n,Tq_n)+d(Tq_{n-1},Tq_{n+1})},d(Tq_{n-1},Tq_n)\right\})\\
&+L\min\left\{d(Tq_{n-1},Tq_{n+1}),d(Tq_n,Tq_n),d(Tq_n,Tq_{n+1}),d(Tq_{n-1},Tq_n)\right\}\end{aligned}$$

$$\le\psi(\max\left\{d(Tq_{n-1},Tq_n),d(Tq_{n-1},Tq_n)\right\})-\phi(\max\left\{d(Tq_{n-1},Tq_n),d(Tq_{n-1},Tq_n)\right\})+L(0)$$

That is

$$\psi(d(Tq_n,Tq_{n+1}))\le\psi(d(Tq_{n-1},Tq_n))-\phi(d(Tq_{n-1},Tq_n)) \tag{3.2}$$

Therefore, the sequence $\left\{d(Tq_n,Tq_{n+1})\right\}$ is a decreasing sequence of positive real numbers, which is bounded below. So, there exists $r\ge 0$ such that $\lim_{n\to\infty}d(Tq_n,Tq_{n+1})=r$ .We claim that $r=0$. Suppose that $r>0$. By taking the limit of the supremum in the relation (3.2) as $n\to\infty$, we get

$\psi(r)\le\psi(r)-\phi(r)<\psi(r)$.

Which is a contradiction. Hence, we conclude that $r=0$ that is

$$\lim_{n\to\infty}d(Tq_n,Tq_{n+1})=0 \tag{3.3}$$

Now we prove that $\left\{Tq_{n+1}\right\}$ is Cauchy sequence in (M,d) . For that we suppose $\left\{Tq_{n+1}\right\}$ is not Cauchy sequence. Then there exists $\varepsilon>0$, such that for each positive integer k there exist $n(k)$ and $m(k)$ such that $n(k)>m(k)>k$ and

$$d(Tq_{m(k)},Tq_{n(k)})\ge\varepsilon \tag{3.4}$$

Further, we can choose $n(k)$ is the smallest integer with $n(k)>m(k)$ and satisfying

$$d(Tq_{m(k)},Tq_{n(k)-1})<\varepsilon \tag{3.5}$$

Now, from (3.4) and (3.5), we have

$$\varepsilon\le d(Tq_{m(k)},Tq_{n(k)})\le d(Tq_{m(k)},Tq_{n(k)-1})+d(Tq_{n(k)-1},Tq_{n(k)})$$

That is

$$\varepsilon\le d(Tq_{m(k)},Tq_{n(k)})\le\varepsilon+d(Tq_{n(k)-1},Tq_{n(k)})$$

Letting $k\to\infty$ in the above inequality and using (3.3), we have

$$\lim_{k\to\infty}d(Tq_{m(k)},Tq_{n(k)})=\varepsilon . \tag{3.6}$$

Hence,

$$d(Tq_{m(k)-1},Tq_{n(k)-1})\le d(Tq_{m(k)-1},Tq_{m(k)})+d(Tq_{m(k)},Tq_{n(k)})+d(Tq_{n(k)},Tq_{n(k)-1})$$

and

$d(Tq_{m(k)},Tq_{n(k)}) \le d(Tq_{m(k)},Tq_{m(k)-1}) + d(Tq_{m(k)-1},Tq_{n(k)-1}) + d(Tq_{n(k)-1},Tq_{n(k)})$

Letting $k \to \infty$ in the above inequality and using (3.3) and (3.6), we have

$$\lim_{k\to\infty} d(Tq_{m(k)-1},Tq_{n(k)-1}) = \varepsilon . \qquad (3.7)$$

Thus,

$d(Tq_{m(k)-1},Tq_{n(k)}) \le d(Tq_{m(k)-1},Tq_{n(k)-1}) + d(Tq_{n(k)-1},Tq_{n(k)})$

and

$d(Tq_{m(k)-1},Tq_{n(k)-1}) \le d(Tq_{m(k)-1},Tq_{n(k)}) + d(Tq_{n(k)},Tq_{n(k)-1})$

Letting $k \to \infty$ in the above inequality and using (3.3) and (3.7), we have

$$\lim_{k\to\infty} d(Tq_{m(k)-1},Tq_{n(k)}) = \varepsilon . \qquad (3.8)$$

Hence,

$d(Tq_{m(k)},Tq_{n(k)-1}) \le d(Tq_{m(k)},Tq_{n(k)}) + d(Tq_{n(k)},Tq_{n(k)-1})$

and

$d(Tq_{m(k)},Tq_{n(k)}) \le d(Tq_{m(k)},Tq_{n(k)-1}) + d(Tq_{n(k)-1},Tq_{n(k)})$

Letting $k \to \infty$ in the above inequality and using (3.3) and (3.6), we have

$$\lim_{k\to\infty} d(Tq_{m(k)},Tq_{n(k)-1}) = \varepsilon \qquad (3.9)$$

We apply the condition (3.1) to $m = Tq_{m(k)}$ and $n = Tq_{n(k)}$

$$M_d(Tq_{m(k)},Tq_{n(k)}) = \max\left\{ \frac{d(Tq_{m(k)-1},Tq_{m(k)})d(Tq_{n(k)-1},Tq_{n(k)})}{d(Tq_{m(k)-1},Tq_{n(k)-1}) + d(Tq_{m(k)-1},Tq_{n(k)}) + d(Tq_{n(k)-1},Tq_{m(k)})}, d(Tq_{m(k)-1},Tq_{n(k)-1}) \right\} (3.10)$$

and

$$N_d(Tq_{m(k)},Tq_{n(k)}) = \min\left\{ d(Tq_{m(k)-1},Tq_{n(k)}), d(Tq_{n(k)-1},Tq_{m(k)}), d(Tq_{m(k)-1},Tq_{m(k)}), d(Tq_{n(k)-1},Tq_{n(k)}) \right\}$$

(3.11)

Letting $k \to \infty$ in (3.10) and (3.11) using (3.3) (3.6) (3.8) and (3.9), we have

$$\lim_{k\to\infty} M_d(Tq_{m(k)},Tq_{n(k)}) = \max\{0,\varepsilon\} = \varepsilon \qquad (3.12)$$

$$\lim_{k\to\infty} N_d(Tq_{m(k)},Tq_{n(k)}) = \min\{\varepsilon,\varepsilon,0,0\} = 0 \qquad (3.13)$$

Since $n(k) > m(k)$,we have $\{q_{m(k)}\} < \{q_{n(k)}\}$. Now by applying (3.1) and using (3.10) (3.11), we have

$\psi(d(Tq_{m(k)},Tq_{n(k)})) \le \psi(M_d(Tq_{m(k)},Tq_{n(k)})) - \phi(M_d(Tq_{m(k)},Tq_{n(k)})) + LN_d(Tq_{m(k)},Tq_{n(k)})$

If we take the limit supremum in both sides of the above inequality, and use (3.7) (3.12) (3.13) to the property of ψ and the continuity of ϕ , we obtain

$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon)$.

Which is a contradiction. Therefore, the sequence $\{Tq_n\}$ is a Cauchy sequence in M . So by the completeness of M , there exists a point $u \in M$ such that $Tq_n \to u$ as $n \to \infty$. By continuity of T , we have.

$\lim_{n\to\infty} T(Tq_n) = T(\lim_{n\to\infty} Tq_n) = Tu$.

But $Fq_{n+1} = Tq_n$, then $Fq_{n+1} \to u$ as $n \to \infty$ and from the compatibility for T and F , we have

$$\lim_{n\to\infty} (T(Fq_n), F(Tq_n)) = 0 .$$

Further, by triangular inequality, we have

$d(Tu,Fu) = d(Tu,T(Fq_n)) + d(T(Fq_n),F(Tq_n)) + d(F(Tq_n),Fu)$

By taking the limit as $n \to \infty$ in both sides of the previous equation, then we get $d(Tu, Fu) = 0$. due to the continuity of T and F. Thus, $Tu = Fu$ and u is a coincidence point of T and F in M .This completes the proof.

Corollary 3.1**:** Let $(M, d, \leq)$ be a partially ordered complete metric space and suppose that F and T are continuous self-mappings on M, $TM \subseteq FM$ and T is monotone F non-decreasing mapping and satisfying the following condition

$$\psi\left(d\left(Tm,Tn\right)\right) \leq \psi\left(M_d\left(m,n\right)\right) - \phi\left(M_d\left(m,n\right)\right)$$

(3.14)

For all $m, n \in M$ with $Fm \leq Fn$ where $\phi \in \Phi$, $\psi \in \Psi$ and

$$M_d\left(m,n\right) = \max\left\{\frac{d(Fm,Tm)d(Fn,Tn)}{d(Fm,Fn)+d(Fm,Tn)+d(Fn,Tm)}, d(Fm,Fn)\right\}.$$

If there exist a point $q_0 \in M$ such that $Fq_0 \leq Tq_0$ and the mapping F and T are compatible, then F and T have a coincidence point in M.

Proof. Take $L = 0$ in the Theorem (3.1).

Corollary 3.2**:** Let $(M, d, \leq)$ be a partially ordered complete metric space and suppose that F and T are continuous self-mappings on M, $TM \subseteq FM$ and T is monotone F non-decreasing mapping and satisfying the following condition

$$d\left(Tm,Tn\right) \leq M_d\left(m,n\right) - \phi\left(M_d\left(m,n\right)\right) + LN_d\left(m,n\right) \tag{3.15}$$

For all $m, n \in M$ with $Fm \leq Fn$ where $\phi \in \Phi$, $L \geq 0$ and

$$M_d\left(m,n\right) = \max\left\{\frac{d(Fm,Tm)d(Fn,Tn)}{d(Fm,Fn)+d(Fm,Tn)+d(Fn,Tm)}, d(Fm,Fn)\right\}$$

$$N_d(m,n) = \min\left\{d(Fm,Tn), d(Fn,Tm), d(Fm,Tm), d(Fn,Tn)\right\}$$

If there exist a point $q_0 \in M$ such that $Fq_0 \leq Tq_0$ and the mapping F and T are compatible, then F and T have a coincidence point in M.

Proof. Take $\psi(t) = t$ in the Theorem (3.1).

Corollary 3.3**:** Let $(M, d, \leq)$ be a partially ordered complete metric space and suppose that F and T are continuous self-mappings on M, $TM \subseteq FM$ and T is monotone F non-decreasing mapping and satisfying the following condition

$$d\left(Tm,Tn\right) \leq kM_d\left(m,n\right) + LN_d\left(m,n\right) \tag{3.16}$$

For all $m, n \in M$ with $Fm \leq Fn$ where $k < 1$, $L \geq 0$ and

$$M_d\left(m,n\right) = \max\left\{\frac{d(Fm,Tm)d(Fn,Tn)}{d(Fm,Fn)+d(Fm,Tn)+d(Fn,Tm)}, d(Fm,Fn)\right\}$$

$$N_d(m,n) = \min\left\{d(Fm,Tn), d(Fn,Tm), d(Fm,Tm), d(Fn,Tn)\right\}$$

If there exist a point $q_0 \in M$ such that $Fq_0 \leq Tq_0$ and the mapping F and T are compatible, then F and T have a coincidence point in M.

Proof. Take $\phi(t) = (1-k)t$ in Corollary (3.2).

Theorem 3.2**:** Let $(M, d, \leq)$ be a partially ordered complete metric space and suppose that F and T are self-mappings on M, $TM \subseteq FM$ and T is monotone F non-decreasing mapping and satisfying condition (3.1). If there exist a point $q_0 \in M$ such that $Fq_0 \le Tq_0$ and if $\{q_n\}$ is a non-decreasing sequence in M such that $q_n \to q$ then $q_n \le q$, for all $n \in N$, that is $q = \sup\{q_n\}$.

Then F and T have a coincidence point M , whenever FM is a complete subset of M . Further, if F and T are weakly compatible, then, F and T have a common fixed point in M .

Proof. From proof of Theorem (3.1), we have that $\{Tq_n\}$ is a Cauchy sequence. As $Fq_{n+1} = Tq_n$, so $\{Fq_n\}$ is a Cauchy sequence in (FM,d) .Since FM is complete, there is $Fu \in FM$ such that $\lim_{n\to\infty} Tq_n = \lim_{n\to\infty} Fq_n = Fu$. Notice that the sequences $\{Tq_n\}$ and $\{Fq_n\}$ are non-decreasing. Then from our assumptions, we have $Tq_n \le Fu$ and $Fq_n \le Fu$ for all $n \in N$. Since T is monotone F non-decreasing, we get $Tq_n \le Tu$ for all $n \in N$. Making $n \to \infty$ so we obtain $Fu \le Tu$. Now if $Fu = Tu$,then u is coincidence fixed point of F and T . Suppose that $Fu < Tu$, Construct a sequence $\{u_n\}$ as $u_0 = u$ and $Fu_{n+1} = Tu_n$ for all $n \in N$. An argument is similar to that in the proof of Theorem (3.1) yields that $\{Fu_n\}$ is a non-decreasing sequence and $\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Fu_n = Fv$ for some $v \in M$. From our assumptions, it follows that $\sup Fu_n \le Fv$ and $\sup Tu_n \le Fv$. Notice that

$$Fq_n \le Fu < Fu_1 \le \cdots\cdots \le Fu_n \le \cdots \le Fv .$$

Now if $Fq_{n_0} = Fu_{n_0}$ for some $n_0 \in N$ then, we have

$Fq_{n_0} = Fu = Fu_{n_0} = Tu_{n_1} = Tu$.

Hence, u is coincidence point of T and F in M .

Assume that $Fq_{n_0} \ne Fu_{n_0}$ for all $n \in N$,then from condition (3.17), we have

$$\psi\left(d\left(Fq_{n+1}, Fu_{n+1}\right)\right) = \psi\left(d\left(Tq_n, Tu_n\right)\right)$$

$$\le \psi\left(\max\left\{ \frac{d(Fq_n,Tq_n)d(Fu_n,Tu_n)}{d(Fq_n,Fu_n)+d(Fq_n,Tu_n)+d(Fu_n,Tq_n)}, d(Fq_n,Fu_n) \right\} \right)$$

$$-\phi\left(\max\left\{ \frac{d(Fq_n,Tq_n)d(Fu_n,Tu_n)}{d(Fq_n,Fu_n)+d(Fq_n,Tu_n)+d(Fu_n,Tq_n)}, d(Fq_n,Fu_n) \right\} \right)$$

$$+L\min\left\{d(Fq_n,Tu_n), d(Fu_n,Tq_n), d(Fq_n,Tq_n), d(Fu_n,Tu_n)\right\}$$

Taking limit supremum as $n \to \infty$ on both sides of the above inequality, we get

$$\psi\left(d\left(Fu,Fv\right)\right) \le \psi\left(d\left(Fu,Fv\right)\right) - \phi\left(d\left(Fu,Fv\right)\right) < \psi\left(d\left(Fu,Fv\right)\right)$$

Which is contradictions. Hence, $Fu = Fv = Fu_1 = Tu$ and u is a coincidence point of T and F .

Now suppose that T and F are weakly compatible. Let $w = Fz = Tz$ .Then $Tw = TFz = FTz = Fw$ Consider,

$$\psi\left(d\left(Tz,Tw\right)\right) \le \psi\left(\max\left\{ \frac{d(Fz,Tz)d(Fm,Tw)}{d(Fz,Fw)+d(Fz,Tw)+d(Fw,Tz)}, d(Fz,Fw) \right\} \right)$$

$$-\phi\left(\max\left\{ \frac{d(Fz,Tz)d(Fm,Tw)}{d(Fz,Fw)+d(Fz,Tw)+d(Fw,Tz)}, d(Fz,Fw) \right\} \right)$$

$$+LN = \min\left\{d(Fz,Tw), d(Fw,Tz), d(Fz,Tz), d(Fw,Tw)\right\}$$

$$\le \psi\left(d\left(Tz,Tw\right)\right) - \phi\left(d\left(Tz,Tw\right)\right) < \psi\left(d\left(Tz,Tw\right)\right)$$

Then, $\psi\left(d\left(Tz,Tw\right)\right)=0$. Therefore, $Tz=Tw$ since $Fw=Tw$ and $w=Tz$ that is w is a common fixed point of T and F. This completes the proof.

***Corollary 3.4*:** Let $(M,d,\leq)$ be a partially ordered complete metric space and suppose that F and T are self-mappings on M, $TM\subseteq FM$ and T is monotone F non-decreasing mapping and satisfying condition (3.14). If there exists a point $q_0\in M$ such that $Fq_0\leq Tq_0$ and if $\{q_n\}$ is a non-decreasing sequence in M such that $q_n\to q$ implies $q_n\leq q$, for all $n\in N$.

Then, F and T have a coincidence point M whenever FM is a complete subset of M. Further, if F and T are weakly compatible, then F and T have a common fixed point in M.

Proof. Take $L=0$ in Theorem (3.2)

***Corollary 3.5*:** Let $(M,d,\leq)$ be a partially ordered complete metric space and suppose that F and T are self-mappings on M such that $TM\subseteq FM$ and T is monotone F non-decreasing mapping and satisfying condition (3.15). If there exists a point $q_0\in M$ such that $Fq_0\leq Tq_0$ and if $\{q_n\}$ is a non-decreasing sequence in M such that $q_n\to q$ implies that $q_n\leq q$ for all $n\in N$.

Then, F and T have a coincidence point M whenever FM is a complete subset of M. Further, if F and T are weakly compatible, then F and T have a common fixed point in M.

Proof. Take $\psi(t)=t$ in Theorem (3.2).

***Corollary 3.6*:** Let $(M,d,\leq)$ be a partially ordered complete metric space and suppose that F and T are self-mappings on M, $TM\subseteq FM$ and T is monotone F non-decreasing mapping and satisfying condition (3.16). If there exists a point $q_0\in M$ such that $Fq_0\leq Tq_0$ and if $\{q_n\}$ is a non-decreasing sequence in M such that $q_n\to q$ implies that $q_n\leq q$ for all $n\in N$.

Then, F and T have a coincidence point M whenever FM is a complete subset of M. Further, if F and T are weakly compatible, then F and T have a common fixed point in M.

Proof. Take $\phi(t)=(1-k)t$ in Corollary (3.5).

***Theorem 3.3*:** Adding definition (6) to the hypotheses of Theorem (3.2), we obtain that u is a unique common fixed point of T and F.

Proof. Suppose that the set of common fixed points of T and F is well ordered. We claim that the common fixed point of T and F is unique. Assume on the contrary that $Tu=Fu=u$ and $Tv=Fv=v$ but $u\neq v$. Consider

$$
\begin{aligned}
&\psi\left(d\left(u,v\right)\right)=\psi\left(d\left(Tu,Tv\right)\right)\\
&\leq\psi\left(\max\left\{\frac{d(Fu,Tu)d(Fv,Tv)}{d(Fu,Fv)+d(Fu,Tv)+d(Fv,Tu)},d(Fu,Fv)\right\}\right)\\
&-\phi\left(\max\left\{\frac{d(Fu,Tu)d(Fv,Tv)}{d(Fu,Fv)+d(Fu,Tv)+d(Fv,Tu)},d(Fu,Fv)\right\}\right)\\
&\qquad+LN=\min\left\{d(Fu,Tv),d(Fv,Tu),d(Fu,Tu),d(Fv,Tv)\right\}\\
&\qquad\leq\psi\left(d\left(u,v\right)\right)-\phi\left(d\left(u,v\right)\right)<\psi\left(d\left(u,v\right)\right)
\end{aligned}
$$

This implies that $\psi\left(d\left(u,v\right)\right)=0$. Hence, $u=v$. Conversely, if T and F have only one common fixed point then the set of common fixed points of T and F being a singleton is well ordered.

Remark:

1. If $k=\alpha+\beta$ where $\alpha,\beta\in[0,1)$ and $L=0$, in Corollary (3.3) we obtain Corollary (3.6) in [21]

2. If M is metric space, $k=\alpha+\beta$ where $\alpha,\beta\in[0,1)$, $L=0$ and $F=I$ in Corollary (3.3) we obtain Theorem (1) in [5]

***Example 1*:** The following example supports our Theorem (3.1). Let $M=\{(1,0),(0,1),(1,1)\}\subset R^2$ with the Euclidean distance d_2 and $\leq:=\{(m,m):m\in M\}\cup\{(0,1),(1,1)\}$. We also consider $T,F:M\to M$ which is given by $T(1,0)=(1,0),T(0,1)=(0,1),T(1,1)=(1,0),F(1,0)=(1,1),\quad F(0,1)=(0,1),F(1,1)=(1,1)$ and take $\phi,\psi:[0,\infty]\to[0,\infty]$ such that $\psi(t)=4t$ and $\phi(t)=3t$. First, T and F are trivial continuous and T is monotone F non-decreasing. Let $m,n\in M$ with $Fm\leq Fn$ and then we have $m=(1,0),n=(1,1)$. Then,

$$d_2(T(1,0),T(1,1))=0$$

(3.17)

$$M_{d_2}((1,0),(1,1))=\max\left\{\frac{d(F(1,0),T(1,0))d(F(1,1),T(1,1))}{d(F(1,0),F(1,1))+d(F(1,0),T(1,1))+d(F(1,1),T(1,0))},d(F(1,0),F(1,1))\right\}$$

$$=\max\left\{\frac{1}{2},0\right\}=\frac{1}{2} \tag{3.18}$$

$$N_{d_2}((1,0),(1,1))=\min\{d(F(1,0),T(1,0)),d(F(1,1),T(1,1)),d(F(1,0),T(1,1)),d(F(1,1),T(1,0))\}$$

$$=\min\{1,1,1,1\}=1. \tag{3.19}$$

From (3.17), (3.18) and (3.19), we have

$$\psi\left(d_2(T(1,0),T(1,1))\right)=0$$

$$\leq\psi\left(M_{d_2}((1,0),(1,1))\right)-\phi\left(M_{d_2}((1,0),(1,1))\right)+LN_{d_2}((1,0),(1,1))$$

Thus condition (3.1) holds , since T and F are weakly compatible then T and F have one common fixed point such that $T(0,1)=F(0,1)=(0,1)$.

If (M,d) is a metric space the condition (3.1) is not applicable, take $m=(1,0)$ and $n=(0,1)$.

$$d_2(T(1,0),T(0,1))=\sqrt{2} \tag{3.20}$$

$$M_{d_2}((1,0),(0,1))=\max\left\{\frac{d(F(1,0),T(1,0))d(F(0,1),T(0,1))}{d(F(1,0),F(0,1))+d(F(1,0),T(0,1))+d(F(0,1),T(1,0))},d(F(1,0),F(0,1))\right\}$$

$$=\max\{0,1\}=1 \tag{3.21}$$

$$N_{d_2}((1,0),(0,1))=\min\{d(F(1,0),T(1,0)),d(F(0,1),T(0,1)),d(F(1,0),T(0,1)),d(F(0,1),T(1,0))\}$$

$$=\min\{0,1,0,1\}=0 \tag{3.22}$$

From (3.20), (3.21) and (3.22), we have,

$$\psi\left(d_2(T(1,0),T(0,1))\right)=4\sqrt{2}$$

$$>\psi\left(M_{d_2}((1,0),(0,1))\right)-\phi\left(M_{d_2}((1,0),(0,1))\right)+LN_{d_2}((1,0),(0,1))$$

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