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On Controllability of Impulsive Fractional Integro-differential Nonlocal System with State-dependent Delay in a Banach Space

Naseif J. Al-Jawari, Abdul Khaleq O. Mazeel, Khudair O. Hussain*

Department of Mathematics, College of Science, AL-Mustansiriyah University, Baghdad, Iraq

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Abstract

This paper is concerned with the controllability of a nonlinear impulsive fractional integro-differential nonlocal control system with state-dependent delay in a Banach space. At first, we introduce a mild solution for the control system by using fractional calculus and probability density function. Under sufficient conditions, the results are obtained by means of semigroup theory and the Krasnoselskii fixed point theorem. Finally, an example is given to illustrate the main results.

Keywords: Controllability, State-dependent delay, Semigroup theory, Probability density function, Krasnoselskii fixed point theorem

حول إمكانية السيطرة لنظام تفاضلي – تكاملي كسري متسارع غير محلي مع تأخير يعتمد على الحالة في فضاء بناخ

نصيف جاسم الجواري ,عبد الخالق عويد مزعل , خضير عبيس حسين*

قسم الرياضات, كلية العلوم,الجامعة المستنصرية, بغداد ,العراق

الخلاصة

يهتم هذا البحث بإمكانية السيطرة لنظام تحكم تفاضلي – تكاملي كسري متسارع غير خطي غير محلي مع تأخير يعتمد على الحالة في فضاء باناخ. في البداية تم تقديم حلًا معتدلًا لنظام التحكم باستخدام حساب التفاضل والتكامل الكسري ودالة كثافة الاحتمال , في حالة وجود شروط كافية تم الحصول على النتائج عن طريق نظرية شبه الزمرة ونظرية النقطة الصامدة لكراسنوسيلسكي. أخيرًا ، تم إعطاء مثال لتوضيح النتائج الرئيسية.

1. Introduction

In 1695, fractional calculus was developed as an important mathematical field. The ideas of fractional calculus have recently been successfully applied to several fields, and researchers are increasingly finding that fractional calculus can accurately describe many events in the natural sciences and architecture. Rheology, fluid current, dispersion-like diffuse transport, and dynamic systems are among the most critical areas of fractional calculus today [1–5]. The fact that it provides an outstanding method for modelling diverse processes in many areas of physics, chemistry, economics, and other sciences. It is a clear impetus for further study into the fractional evolution equation. There are several kinds of fractional derivatives, like Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov, and Hilfer, for more details; see

*Email: khudair.hussain@qu.edu.iq

[5–7]. In several areas of science and engineering, such as fluid mechanics, biological simulations, and chemical kinetics, integro-differential equations are used. In [8], the authors give a comprehensive analysis of integro-differential equations and their solutions using the Laplace transform procedure. Recently fractional integro-differential equations have been used to model a wide variety of physical processes, including heat conduction in memory objects, mixed conduction, convection, and radiation problems [3,9]. The existence of mild solutions for fractional integro-differential equations was investigated by using the fixed point theorem in many publications[10–15].

Control theory is an important field of mathematics that deals with the structure and study of control systems. Controllability is one of the basic principles of mathematical control theory that has a profound influence on control systems, where it is important in many control problems for deterministic and stochastic control theories. The principle of controllability, on the other hand, is well developed for control problems expressed as abstract differential equations and solved using numerous methods of infinite and infinite-dimensional spaces. In general, controllability refers to the ability to steer a control system from an arbitrary initial state to an arbitrary final state by using a set of admissible controls. Recently, many authors investigated the controllability of fractional differential equations by using various techniques. In papers [16–20], the researchers discussed the controllability of various nonlinear fractional differential equations with the help of semigroup theory and fixed point theorems, as for the papers [21–27], the authors used Mittag-Leffler function and fixed point theorems to investigate controllability for fractional differential equations.

Delay differential equations with state-dependent delay arise normally from the modelling of infectious disease transmission, the modelling of immune response systems and the modelling of respiration, where the delay is due to the time required to accumulate an appropriate dosage of infection or antigen concentration[28]. On the other hand, delay differential equations are an abstract formulation of many mathematical models that can be used in various physical, chemical, and biological processes. It is mandatory to provide a delay in the simulation of many real-world problems, which is often based on the previous state of the unknown function. As a result, state-dependent delay differential equations emerge. The length of time to maturity, for example, is regarded as a continuous delay in a simple population dynamics model; see [29]. In [30], the author discovered that the time it takes for Antarctic whales and seals to reach adulthood changes depending on the population's health. Also, can be applied Differential equations with State-dependent delay on mathematical models to dynamics, control theory, and neural networks. The authors of [31] considered a population model with a state-dependent delay by

$$\mu(\delta) = \int_{\delta-\mathcal{G}(\mathcal{L}(\delta))}^{\delta} \alpha \mathcal{M}(\sigma) e^{-\omega(\delta-\sigma)}\, d\sigma.$$

Where $\mu(\delta)$ and $\mathcal{M}\ (\delta)$ represent immature individuals and mature individuals, respectively. $\mathcal{G}\big(\mathcal{L}(\delta)\big)$ describes a threshold age, which is the maturation time for an immature individual that matures at a time δ based on the overall population $\mathcal{L}(\delta) = \mu\delta) + \mathcal{M}\ (\delta)$. The constants ω, α refer to the rate of dies from individual $\mu(\delta)$ and the rate immature population produces, respectively. There are many papers on the controllability of the various fractional control systems with delay. For example, Ravichandran et al.[32] established controllability of nonlinear fractional integro-differential equations with state-dependent delay in Banach spaces by using Leray–Schauder alternative theorem and Krasnoselskii fixed point theorem with the resolvent operator. Aimene et al. [18] used the Drabo fixed point theorem and the properties of measures of noncompactness with semigroup theory to obtain the controllability of impulsive fractional differential equations with delay. Huang et al.[33] derived a set of sufficient conditions for the existence and controllability of mild solutions for a class of second-order neutral impulsive stochastic evolution integro-differential equations with state-dependent delay by using fixed point theorems. MA and LIU[17] discussed the exact controllability of fractional neutral integro-differential equations with state-dependent delay in Banach spaces with the help of fixed point theorems and fractional calculus. In this paper, our purpose is to study the controllability of a nonlinear impulsive fractional integro-differential nonlocal control system with state-dependent delay in a Banach space.

$$\begin{cases} {}^cD^{\alpha}[x(t)-g(t,x_t)] = Ex(t)+Bu(t)+\displaystyle\int_{t-\eta(x(t))}^{t} Gx(\zeta)\,d\zeta, t\in J:=[0,a]/\{t_1,t_2,t_3,\dots,t_m\} \\ \Delta x(t_r) = I_r\big(x(t_r^-)\big), \quad r=1,2,3,\dots,m \\ x_0(\theta)=\varphi(\theta)+\left(\displaystyle\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(\theta), \quad \theta\in[-\hat{a},0], \quad \hat{a}>0. \end{cases} \qquad (S_1)$$

Where $\alpha\in(0,1)$, the state x takes value in Banach space X, ${}^cD^{\alpha}$ is the Caputo fractional derivative of order α, $E: D(E)\subseteq X\longrightarrow X$ is a generator of a strongly continuous semigroup $\{T(t), t\geq 0\}$ on, $B: Y:=L^2([o,a],U)\to X$ is bounded linear operator and $u(.)$ is the control function takes its value in the space Y with U as a Banach space. $I_r: X\to X$ is the jump operator, $0=t_0<t_1<t_2<\cdots<t_r<t_{r+1}=a$, $x(t_r^+)$ and $x(t_r^-)$ represent respectively the right and left limits of $x(t)$ at $t=t_r$ with $x(t_r^-)=x(t_r)$, $\Delta x(t_r):=x(t_r^+)-x(t_r^-)$ represents the jump in the state x at time t_r. The functions $g, G, \hbar, \varphi, x_t$ and η will be defined below.

The outline of this paper is as follows. In the next section, we recall some necessary preliminaries related to fractional calculus, Krasnoselskii fixed point theorem, Hölder inequality and introduce the mild solution for system (S_1). In Section 3, we study controllability for system (S_1) under specific control u_x and sufficient conditions. In Section 4, we give an example to illustrate our main result.

2. Preliminaries

In this section, we will mention definitions and lemmas necessary to present our main results. Throughout this paper, we assume that E invertible and $E: D(E)\subseteq X\to X$ is the infinitesimal generator of a compact C_o- semigroup $\{T(t), t>0\}$ of uniformly bounded linear operators in X [34], i.e.

i- There exists $M>1$ such that $M=sup_{t\geq 0}\|T(t)\|$,

ii- for $x\in D(E), T(t)x\in D(E)$ and $\dot{T}(t)x=AT(t)x$. $\left(\cdot=\frac{d}{dt}\right)$

Let $D:=C([-\hat{a},0],X)$ be the space of all continuous functions from $[-\hat{a},0]$ into X, if D is endowed with the norm $\|\psi\|_D=sup\|\psi(\theta)\|_X$, for all $\psi\in D$, $\theta\in[-\hat{a},0]$, then, obviously $(D,\|.\|_D)$ is Banach space. Once again, let the set functions $PC([-\hat{a},a],X)=\{x:[-\hat{a},a]\to X,\ x\in C((t_r,t_{r+1}],X), r=0,1,2,\dots,m$ and there exist $x(t_r^+)$ and $x(t_r^-)$ with $x(t_r^-)=x(t_r)$, for $r=1,2,3,\ \dots,m\}$ be the space of piecewise continuous functions. Cleary, $(PC([-\hat{a},a],X),\|.\|_{PC})$ is a Banach space, where $\|x\|_{PC}=sup_{t\in[-\hat{a},a]}\|x(t)\|$. $g: J\times D\to D(A)$, $G: PC([-\hat{a},a],X)\to X$, and $\hbar: J\times D\to D$, are given functions to be specified later. An initial function φ belong to the space D and for any $x\in X$ defined on $[-\hat{a},a]$ and any time $t\in J$, the function $x_t\in D$ define by $x_t(\theta)=x(t+\theta)$, $\theta\in[-\hat{a},0]$ .The state-dependent delay $\eta: PC([0,a],X)\to J$ is an increasing differentiable function.

Next, we recall the following definitions related to fractional calculus.

Definition (2.1) [6]**.** The Riemann-Liouville fractional integral of order $q>0$ with the lower limit 0 for a function $\zeta:[0,\infty)\to R$ is defined by

$$I^q\zeta(\epsilon)=\frac{1}{\Gamma(q)}\int_0^{\epsilon}(\epsilon-s)^{q-1}\zeta(s)ds, \qquad \epsilon\in(0,\infty), q>0.$$

Where Γ is the Gamma function.

Definition (2.2) [6]. For a function $\zeta(\epsilon) \in C^1[0,\infty)$ the expression

$$^cD^q\zeta(\epsilon) = \frac{1}{\Gamma(1-q)}\int_0^{\epsilon}(\epsilon - s)^{-q}\,\dot{\zeta}(s)ds = I^{1-q}\dot{\zeta}(\epsilon) \qquad 0 < q < 1,$$

is called the Caputo derivative of order q.

Remark (2.3) [6]**.**

1. For a constant c , $^cD^q(c) = 0$
2. $I^q\ {}^cD^q z(t) = z(t) - z(0), \qquad q \in (0,1)$
3. $I^q\big(z(t) + y(t)\big) = I^q z(t) + I^q y(t), \qquad q > 0$
4. For $\lambda > 0$, the Laplace transform of $I^q z(t)$ with parameter λ is $\lambda^{-q}z(\lambda)$, where $z(\lambda)$ is the Laplace transform of $z(t)$.

Lemma (2.4). If $x(t) \in PC([0,a],X)$, then the following equality

$$I^\alpha\ {}^cD^\alpha x(t) = \begin{cases} x(t) - x(0), & t \in [0,t_1] \\ x(t) - x(0) - \sum_{r=1}^{m}\Delta x(t_r), & t \in (t_r, t_{r+1}] \end{cases}$$

holds for any $\alpha \in (0,1)$.

Proof. If $t \in [0,t_1]$, then by Remark 2.3(2), we get

$$I^\alpha\ {}^cD^\alpha x(t) = x(t) - x(0).$$

If $t \in (t_r, t_{r+1}]$, then by Definitions (2.1) and (2.2) for $\alpha \in (0,1)$, we have

$$I^\alpha\ {}^cD^\alpha x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ {}^cD^\alpha \dot{x}(s)ds$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^\tau \frac{\dot{x}(\tau)}{(s-\tau)^\alpha}d\tau ds$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \dot{x}(\tau)d\tau\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{-\alpha}\,ds$$

Let $= \tau + \gamma(t-\tau)$. Then

$$\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{-\alpha}\,ds = \int_0^1 (t-\tau)^{\alpha-1}(1-\gamma)^{\alpha-1}(t-\tau)^{-\alpha}\gamma^{-\alpha}\,(t-\tau)\,d\gamma$$

$$= \int_0^1 (1-\gamma)^{\alpha-1}\gamma^{-\alpha}\ d\gamma = \mathcal{B}(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$$

Where $\mathcal{B}$ is the Beta function. Therefore

$$I^\alpha\ {}^cD^\alpha x(t) = \int_0^t \dot{x}(\tau)d\tau$$

By integration by parts for piecewise continuous functions, we have

$$I^\alpha\ {}^cD^\alpha x(t) = [x(\tau)]_{\tau=0}^{\tau=t} - \sum_{r=0}^{m}[x(\tau)]_{\tau=t_r^-}^{\tau=t_r^+} = x(t) - x(0) - \sum_{r=1}^{m}\Delta x(t_r),$$

for $t \in (t_r, t_{r+1}], r = 1,2,3,\dots,m.$ The proof is completed. □

Remark (2.5) [35] **.**Let

$$h_\delta(\mu) = \frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\,\mu^{-\delta i-1}\frac{\Gamma(i\delta+1)}{i!}\sin(i\pi\delta), \qquad \mu \in (0,\infty),$$

be the one-sided stable probability density. Then, the Laplace transform of $h_\delta(\mu)$ given by $\mathcal{L}\{h_\delta(\mu)\}(\lambda) = e^{-\lambda^\delta}$, $\delta \in (0,1), \lambda > 0$.In addition, let $\Psi_\delta(\mu) = \frac{1}{\delta}\mu^{-1-\frac{1}{\delta}}h_\delta\left(\mu^{\frac{-1}{\delta}}\right)$ be a probability density function defined on$(0,\infty)$.Then, for $\tau \in [0,1]$

$$\int_0^\infty \mu^\tau \Psi_\delta(\mu)d\mu = \int_0^\infty \frac{1}{\mu^{\delta\tau}} h_\delta(\mu)d\mu = \frac{\Gamma(1+\tau)}{\Gamma(1+\delta\tau)}, \qquad \delta \in (0,1), \mu \in (0,\infty).$$

Lemma (2.6). For $0 < \alpha < 1$, if $x(t) \in PC([-\hat{a}, a], X)$ and x is a solution of system (S_1), then x satisfies the following equation.

$$x(t) = \begin{cases} \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + g(t, x_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} \mathcal{H}_\alpha(t-s)\left[Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right] ds \\ \quad + \int_0^t (t-s)^{\alpha-1} E\, \mathcal{H}_\alpha(t-s) g(s, x_s) ds, \quad t \in [0, t_1] \\ \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + g(t, x_t) \\ + \sum_{r=1}^m \Delta x(t_r)\mathcal{K}_\alpha(t-t_r) + \int_0^t (t-s)^{\alpha-1} \mathcal{H}_\alpha(t-s)\left[Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right] ds \\ \quad + \int_0^t (t-s)^{\alpha-1} E\, \mathcal{H}_\alpha(t-s) g(s, x_s) ds, \quad t \in (t_r, t_{r+1}], r = 1,2,3,\dots,m \end{cases}$$

$$x_0(\theta) = \varphi(\theta) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(\theta), \quad \theta \in [-\hat{a}, 0],$$

where, $\mathcal{K}_\alpha(t) = \int_0^\infty \Psi_\alpha(\mu) T(\mu t^\alpha) d\mu$, $\mathcal{H}_\alpha(t) = \alpha \int_0^\infty \mu \Psi_\alpha(\mu) T(\mu t^\alpha) d\mu$, $\Psi_\alpha(\mu)$ it is previously mentioned in Remark (2.5) and $\mu \in (0,\infty), \alpha \in (0,1)$.

Proof. It's easy to see that $x(.)$ can be decomposed $x_1(.) + x_2(.)$, where x_1 is a continuous mild solution for

$$\begin{cases} {}^cD^\alpha x_1(t) = {}^cD^\alpha g(t, x_t) + Ex_1(t) + Bu(t) + \int_{t-\eta(x(t))}^{t} Gx(\zeta)\, d\zeta, \quad t \in J \\ x_{10}(\theta) = \varphi(\theta) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(\theta), \quad \theta \in [-\hat{a}, 0] \end{cases} \tag{S_2}$$

and x_2 is the $PC-$mild solution for

$$\begin{cases} {}^cD^\alpha x_2(t) = Ex_2(t), & t \in J/\{t_1, t_2, \dots, t_m\}, \\ \Delta x(t_r) = I_r x(t_r^-), & r = 1,2,3,\dots,m \\ x_{20}(\theta) = 0, & \theta \in [-\hat{a}, 0]. \end{cases} \tag{S_3}$$

Indeed, by adding together system (S_2) with system (S_3), it follows by system (S_1). Since x_1 is continuous, then $x_1(t_r^+) = x_1(t_r^-)$, $r = 1,2,3,,\dots,m$.

First, we will calculate the mild solution of (S_2), by taking Riemann-Liouville integral on both sides of (S_2), then by Remark 2.3, we obtain

$$x_1(t) = \varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0) + g(t, x_t) + I^\alpha E x_1(t) + I^\alpha B u(t) + I^\alpha H\big(x(t)\big), \qquad (1)$$

where $H(x(t)) = \int_{t-\eta(x(t))}^{t} Gx(\zeta)d\zeta$.Apply Laplace transformation on both sides of Eq. (1); we get

$$\bar{x}_1(\lambda) = \lambda^{-1}\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + \bar{g}(\lambda) + \lambda^{-\alpha} E\bar{x}_1(\lambda) + \lambda^{-\alpha} B\bar{u}(\lambda) + \lambda^{-\alpha}\bar{H}(\lambda)$$

where,

$$\bar{x}_1(\lambda) = \int_0^\infty e^{-\lambda t}\, x_1(t)dt, \qquad \bar{g}(\lambda) = \int_0^\infty e^{-\lambda t}\, g(t, x_t)dt,$$

$$\bar{u}(\lambda) = \int_0^\infty e^{-\lambda t}\, u(t)dt, \qquad \bar{H}(\lambda) = \int_0^\infty e^{-\lambda t}\, H(x(t))dt, \quad \lambda > 0,$$

then, we get that

$$\bar{x}_1(u) = \lambda^{\alpha-1}(I\lambda^\alpha - E)^{-1}\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + \lambda^\alpha(I\lambda^\alpha - E)^{-1}\bar{g}(\lambda) + (I\lambda^\alpha - E)^{-1}B\bar{u}(\lambda) + (I\lambda^\alpha - E)^{-1}\bar{H}(\lambda)$$

$$= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}\, T(s)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right]ds + \lambda^\alpha \int_0^\infty e^{-\lambda^\alpha s}\, T(s)\bar{g}(\lambda)ds + \int_0^\infty e^{-\lambda^\alpha s}\, T(s)B\bar{u}(\lambda)ds + \int_0^\infty e^{-\lambda^\alpha s}\, T(s)\bar{H}(\lambda)ds \qquad (2)$$

Using Remark (2.5) and Eq. (2), we get that

$$\lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}\, T(s)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right]ds$$

$$= \lambda^{\alpha-1}\int_0^\infty e^{-(\lambda u)^\alpha}\, T(u^\alpha)\alpha u^{\alpha-1}\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right]du$$

$$= \int_0^\infty \frac{-1}{\lambda}\Big[\frac{d}{du} e^{-(\lambda u)^\alpha}\Big]\; T(u^\alpha)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right]du,$$

Since $$\frac{d}{du} e^{-(\lambda u)^\alpha} = \frac{d}{du}\int_0^\infty e^{-\lambda u \mu}\, h_\alpha(\mu)d\mu = \int_0^\infty -\lambda\mu e^{-\lambda u\mu}\, h_\alpha(\mu)d\mu$$

then, we have

$$\lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}\,T(s)\left[\varphi(0)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0)-g(0,x_0)\right]ds$$

$$=\int_0^\infty\int_0^\infty \mu e^{-\lambda u\mu}\,h_\alpha(\mu)\,T(u^\alpha)\left[\varphi(0)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0)\right.$$

$$\left.-\,g(0,x_0)\right]dud\mu$$

$$\int_0^\infty\int_0^\infty e^{-\lambda t}\,h_\alpha(\mu)\,T\left(\frac{t^\alpha}{\mu^\alpha}\right)\left[\varphi(0)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0)-g(0,x_0)\right]dtd\mu=$$

$$\int_0^\infty e^{-\lambda t}\left[\int_0^\infty \Psi_\alpha(\mu)\,T(\mu t^\alpha)\left[\varphi(0)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0)-g(0,x_0)\right]d\mu\right]dt$$

$$\int_0^\infty e^{-\lambda t}\,\mathcal{K}_\alpha\left[\varphi(0)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0)\right.$$

$$\left.-\,g(0,x_0)\right]dt. \qquad (3)$$

$$\lambda^\alpha\int_0^\infty e^{-\lambda^\alpha s}\,T(s)\bar{g}(\lambda)ds=\lambda^\alpha\int_0^\infty e^{-(\lambda u)^\alpha}\,T(u^\alpha)\bar{g}(\lambda)\alpha u^{\alpha-1}du$$

$$=\int_0^\infty \alpha(\lambda u)^{\alpha-1}\lambda e^{-(\lambda u)^\alpha}\,T(u^\alpha)\bar{g}(\lambda)du=\int_0^\infty[-T(u^\alpha)\bar{g}(\lambda)]\,de^{-(\lambda u)^\alpha}$$

$$=\left[e^{-(\lambda u)^\alpha}\left(-T(u^\alpha)\bar{g}(\lambda)\right)\right]_0^\infty+\int_0^\infty \alpha e^{-(\lambda u)^\alpha}\,ET(u^\alpha)u^{\alpha-1}\bar{g}(\lambda)du$$

$$=\bar{g}(\lambda)+\int_0^\infty\int_0^\infty \alpha u^{\alpha-1}\,e^{-(\lambda u)^\alpha}\,ET(u^\alpha)e^{-\lambda s}g(s,x_s)dsdu$$

$$=\bar{g}(\lambda)+\int_0^\infty\int_0^\infty\int_0^\infty \alpha u^{\alpha-1}\,e^{-\lambda u\mu}\,h_\alpha(\mu)ET(u^\alpha)e^{-\lambda s}g(s,x_s)d\mu dsdu$$

$$=\bar{g}(\lambda)+\int_0^\infty\int_0^\infty\int_0^\infty \alpha\,e^{-\lambda(v+s)}\,h_\alpha(\mu)\frac{v^{\alpha-1}}{\mu^\alpha}ET\left(\frac{v^\alpha}{\mu^\alpha}\right)g(s,x_s)d\mu dsdu$$

$$=\bar{g}(\lambda)+\int_0^\infty\int_0^\infty\int_0^\infty \alpha\,e^{-\lambda t}\,h_\alpha(\mu)\frac{(t-s)^{\alpha-1}}{\mu^\alpha}ET\left(\frac{(t-s)^\alpha}{\mu^\alpha}\right)g(s,x_s)d\mu dsdu$$

$$=\bar{g}(\lambda)+\int_0^\infty e^{-\lambda t}\left[\int_0^t \alpha\int_0^\infty h_\alpha(\mu)\frac{(t-s)^{\alpha-1}}{\mu^\alpha}ET\left(\frac{(t-s)^\alpha}{\mu^\alpha}\right)g(s,x_s)d\mu ds\right]dt$$

$$=\bar{g}(\lambda)+\int_0^\infty e^{-\lambda t}\left[\int_0^t \alpha\int_0^\infty \mu\Psi_\alpha(\mu)\,(t-s)^{\alpha-1}ET(\mu(t-s)^\alpha)g(s,x_s)d\mu ds\right]dt$$

$$=\bar{g}(\lambda)+\int_0^\infty e^{-\lambda t}\left[\int_0^t (t-s)^{\alpha-1}E\mathcal{H}_\alpha(t-s)g(s,x_s)ds\right]dt \qquad (4)$$

$$\int_0^\infty e^{-\lambda^\alpha s}\,T(s)[B\bar{u}(\lambda)+\bar{H}(\lambda)]ds=\int_0^\infty \alpha u^{\alpha-1}e^{-(\lambda u)^\alpha}\,T(u^\alpha)[B\bar{u}(\lambda)+\bar{H}(\lambda)]du$$

$$= \int_0^\infty \int_0^\infty \alpha u^{\alpha-1} e^{-\lambda u\mu}\, h_\alpha(\mu) T(u^\alpha) \left[\int_0^\infty e^{-\lambda s} \left[Bu(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta \right] ds \right] d\mu du$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha \frac{v^{\alpha-1}}{\mu^\alpha} e^{-\lambda v}\, h_\alpha(\mu) T\left(\frac{v^\alpha}{\mu^\alpha}\right) e^{-\lambda s} \left[Bu(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta \right] ds d\mu dv$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha \frac{(t-s)^{\alpha-1}}{\mu^\alpha} e^{-\lambda t}\, h_\alpha(\mu) T\left(\frac{(t-s)^\alpha}{\mu^\alpha}\right) \left[Bu(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta \right] dt ds d\mu$$

$$= \int_0^\infty e^{-\lambda t} \left[\int_0^t \alpha \int_0^\infty \mu \Psi_\alpha(\mu)\, (t-s)^{\alpha-1} T(\mu (t-s)^\alpha) \left[Bu(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta \right] d\mu \mu ds \right] dt$$

$$= \int_0^\infty e^{-\lambda t} \left[\int_0^t (t-s)^{\alpha-1} \mathcal{H}_\alpha(t-s) \left[Bu(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta \right] ds \right] dt. \tag{5}$$

Using Eqs. $(3)-(5)$, we obtain

$$\bar{X}_1 = \int_0^\infty e^{-\lambda t} \left[\mathcal{K}_\alpha(t) \left[\varphi(0) + \left(\int\limits_0^a \hbar(\omega, x_\omega)\, d\omega \right)(0) - g(0, x_0) \right] dt \right.$$
$$+ \int_0^\infty e^{-\lambda t} \left[g(t, x_t) + \int_0^t (t-s)^{\alpha-1} E\mathcal{H}_\alpha(t-s) g(s, x_s) ds \right] dt,$$
$$+ \int_0^\infty e^{-\lambda t} \left[\int_0^t (t-s)^{\alpha-1} \mathcal{H}_\alpha(t-s) \left[Bu(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta \right] ds \right] dt. \tag{6}$$

Now, apply inverse Laplace transform on Eq.(6), we get

$$x_1(t) = \mathcal{K}_\alpha(t) \left[\varphi(0) + \left(\int\limits_0^a \hbar(\omega, x_\omega)\, d\omega \right)(0) - g(0, x_0) \right] + g(t, x_t)$$
$$+ \int_0^t (t-s)^{\alpha-1} E\mathcal{H}_\alpha(t-s) g(s, x_s) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} \mathcal{H}_\alpha(t-s) \left[Bu(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta \right] ds, \quad t \in J$$

Finally, we will calculate the $PC-$ mild solution of system (S_3).
By using Riemann-Liouville integral and Lemma (2.4), we have

$$x_2(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} E x_2(s) ds, & t \in [0,t_1] \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} E x_2(s) ds, + \sum_{r=1}^{m} \Delta x(t_r) & t \in (t_r, t_{r+1}], \quad r = 1,2,3,\dots,m \end{cases} \tag{7}$$

We can rewrite the previous equality as

$$x_2(t) = \sum_{r=1}^{m} \Delta x(t_r)\delta_r(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} E x_2(s) ds, \quad t \in [t_r, t_{r+1}]. \tag{8}$$

$$\delta_r(t) = \begin{cases} 0, & t \in [0,t_1] \\ 1, & t \in (t_r, t_{r+1}], \quad r = 1,2,3,\dots,m \end{cases}$$

By taking the Laplace transformation to Eq. (8), we get

$$\bar{X}_2(\lambda) = \sum_{r=1}^{m} \Delta x(t_r)\, \lambda^{-1} e^{-\lambda t_r} + \lambda^{-\alpha} E \bar{x}_2(\lambda).$$

That is,

$$\bar{X}_2(\lambda) = \sum_{r=1}^{m} \Delta x(t_r) \lambda^{\alpha-1} (I\lambda^{\alpha} - E)^{-1} e^{-\lambda t_r}.$$

In the same way as before, we can show that the $PC -$ mild solution of (S_3), as follows

$$x_2(t) = \sum_{r=1}^{m} \Delta x(t_r)\, \mathcal{K}_\alpha(t - t_r).$$

From the above, the $PC -$ mild solution of (S_1) is as follows

$$x(t) = \begin{cases} \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0,x_0)\right] + g(t,x_t) \\ \quad + \int_0^t (t-s)^{\alpha-1} \mathcal{H}_\alpha(t-s)\left[Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right] ds \\ \quad + \int_0^t (t-s)^{\alpha-1} E\, \mathcal{H}_\alpha(t-s) g(s,x_s) ds, \qquad t \in [0,t_1] \\ \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0,x_0)\right] + g(t,x_t) \\ \quad + \sum_{r=1}^{m} \Delta x(t_r)\mathcal{K}_\alpha(t-t_r) + \int_0^t (t-s)^{\alpha-1} \mathcal{H}_\alpha(t-s)\left[Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right] ds \\ \quad + \int_0^t (t-s)^{\alpha-1} E\, \mathcal{H}_\alpha(t-s) g(s,x_s) ds, \qquad t \in (t_r, t_{r+1}], \qquad r = 1,2,3,\dots,m \end{cases}$$

$$x_0(\theta) = \varphi(\theta) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(\theta), \qquad \theta \in [-\hat{a}, 0],$$

where $r = 1,2,3, \dots, m$, $\quad \mathcal{K}_\alpha(t) = \int_0^\infty \psi_\alpha(\mu)\, T(\mu t^\alpha) d\mu$,

$$\mathcal{H}_\alpha(t) = \alpha \int_0^\infty \mu \psi_\alpha(\mu)\, T(\mu t^\alpha) d\mu$$

$$\psi_\alpha(\mu) = \frac{1}{\alpha} \mu^{-\frac{1}{\alpha}-1} h_\alpha\left(\mu^{-\frac{1}{\alpha}}\right),$$

$$h_\alpha(\mu) = \frac{1}{\pi} \sum_{i=1}^{\infty} (-1)^{i-1} \mu^{-i\alpha-1} \frac{\Gamma(i\alpha+1)}{i!} \sin(i\pi\alpha), \quad 0 < \mu < \infty.$$

The proof is complete.

In the following Lemma, we will show some properties of the operators $\mathcal{K}_\alpha$ and $\mathcal{H}_\alpha$.

Lemma (2.7). The operators $\mathcal{K}_\alpha$ and $\mathcal{H}_\alpha$ are linear, bounded and strongly continuous for any fixed $t \geq 0$, additionally, if $\{T(t), t > 0\}$ is compact operator, then $\mathcal{K}_\alpha(t)$ and $\mathcal{H}_\alpha(t)$ are also compact operators.

Proof: Since $T(t)$ is a linear operator for any fixed ≥ 0 , it is clear to check that $\mathcal{K}_\alpha(t)$ and $\mathcal{H}_\alpha(t)$ are also linear operators. With the same technique that was used in[13], by Remark (2.5), if $\tau = 1$, we have

$$\int_0^\infty \mu \psi_\alpha(\mu)\, d\mu = \int_0^\infty \mu^{-\alpha}\, h_\alpha(\mu) d\mu = \frac{1}{\Gamma(1+\alpha)}$$

then, for any $x \in D(E)$, we get

$$\|\mathcal{H}_\alpha(t)x\| = \left\|\alpha \int_0^\infty \mu \psi_\alpha(\mu)\, T(\mu t^\alpha) x d\mu\right\| \leq \frac{M}{\Gamma(\alpha)} \|x\|,$$

if $\tau = 0$, we obtain

$$\int_0^\infty \psi_\alpha(\mu)\, d\mu = 1 \text{ and } \|\mathcal{K}_\alpha(t)x\| = \left\|\int_0^\infty \psi_\alpha(\mu)\, T(\mu t^\alpha) x d\mu\right\| \leq M\|x\|.$$

That is, $\{\mathcal{K}_\alpha(t)\}_{t\geq 0}$ and $\{\mathcal{H}_\alpha(t)\}_{t\geq 0}$ are bounded operators. Now, we show that the operators $\mathcal{K}_\alpha(t)$ and $\mathcal{H}_\alpha(t)$ are strongly continuous, for each $x \in D(E)$ and $0 \leq t_1 < t_2 \leq a$, we get that

$$\|\mathcal{K}_\alpha(t_2)x - \mathcal{K}_\alpha(t_1)x\| = \left\|\int_0^\infty \psi_\alpha(\mu)\, [T(t_2^\alpha \mu) - T(t_1^\alpha \mu)] x d\mu\right\|$$

$$= \left\|\int_0^\infty \psi_\alpha(\mu)\, [T(t_2^\alpha \mu - t_1^\alpha \mu + t_1^\alpha \mu) - T(t_1^\alpha \mu)] x d\mu\right\|$$

$$\leq M \int_0^\infty \psi_\alpha(\mu)\, \|[T(t_2^\alpha \mu - t_1^\alpha \mu) - I]x\| d\mu.$$

Based on the strongly continuity of $T(t), (t \geq 0)$ $\|\mathcal{K}_\alpha(t_2)x - \mathcal{K}_\alpha(t_1)x\|$ approaches to zero as $t_2 \to t_1$, consequently the operator $\{\mathcal{K}_\alpha(t)\}_{t\geq 0}$ is strongly continuous. In the same manner, we can also obtain that $\{\mathcal{H}_\alpha(t)\}_{t\geq 0}$ is strongly continuous.

Now, assume that $T(t)$ $(t > 0)$ is compact operator and $\{x_n\}$ is a bounded sequence in X, then the sequence $\{T(t)x_n\}_{t>0}$ has convergent subsequence $\{T(t)x_m\}_{t>0}$ for x_p, x_q, $p > q$, and $t > 0$

$$\|\mathcal{K}_\alpha(t)x_p - \mathcal{K}_\alpha(t)x_q\| = \left\|\int_0^\infty \psi_\alpha(\mu)\, [T(\mu t^\alpha)x_p - T(\mu t^\alpha)x_q] d\mu\right\|$$

$$\leq \sup_{\mu,t>0} \|T(\mu t^\alpha)x_p - T(\mu t^\alpha)x_q\| \to 0 \quad \text{as } p \to q,$$

then, $\lim_{p\to q} \|\mathcal{K}_\alpha(t)x_p - \mathcal{K}_\alpha(t)x_q\| = 0$, which means that the sequence $\{\mathcal{K}_\alpha(t)x_m\}$ is Cauchy sequence, hence by the completeness of X, $\{\mathcal{K}_\alpha(t)x_m\}$ is convergent. Similarly, we can also obtain that $\{\mathcal{H}_\alpha(t)\}_{t>0}$ is a compact operator. The proof is completed.

Add to the above we can see that,

$$E\mathcal{H}_\alpha(t)x = \alpha\int_0^\infty \mu\psi_\alpha(\mu)E\,T(\mu t^\alpha)xd\mu = \alpha\int_0^\infty \mu\psi_\alpha(\mu)\,T(\mu t^\alpha)Exd\mu = \mathcal{H}_\alpha(t)Ex,$$

for any $x \in D(E)$ and fixed $t \geq 0$.

Based on Lemma (2.6), we give the following definition of a mild solution of system (S_1).

Definition (2.8). A function $x \in PC([-\hat{a}, a], X)$ is called a PC −mild solution of the control problem (S_1) if $x_0(\theta) = \varphi(\theta) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(\theta)(\theta \in [-\hat{a}, 0])$ and satisfies

$$x(t) = \begin{cases} \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + g(t, x_t) \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{H}_\alpha(t-s)\left[Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right]ds, \qquad t \in [0, t_1], \\ \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + g(t, x_t) + \sum_{r=1}^{m} \Delta x(t_r)\mathcal{K}_\alpha(t - t_r) \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{H}_\alpha(t-s)\left[Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right]ds, t \in (t_r, t_{r+1}], \\ r = 1,2,3, \dots, m \end{cases}$$

where $r = 1,2,3, \dots, m$, $\mathcal{K}_\alpha(t) = \int_0^\infty \psi_\alpha(\mu)\,T(\mu t^\alpha)d\mu$, $\mathcal{H}_\alpha(t) = \alpha\int_0^\infty \mu\psi_\alpha(\mu)\,T(\mu t^\alpha)d\mu$ and $\psi_\alpha(\mu)$ we have already mentioned it in Remark (2.5).

Lemma (2.9) Krasnoselskii fixed point theorem [36].

Let D be a convex closed nonempty subset of a Banach space . Suppose that $\Phi, \overline{\Phi}$ be maps D into X such that

i- $\Phi x_1 + \overline{\Phi}x_2 \in D$ for every pair $x_1, x_2 \in D$,

ii- Φ is a contraction,

iii- $\overline{\Phi}$ is completely continuous.

Then, $\Phi z + \overline{\Phi}z = z \in D$, for some $z \in D$

Lemma (2.10) Hölder inequality.

Let $\gamma, \bar{\gamma} \in [1, \infty)$ be such that $\frac{1}{\gamma} + \frac{1}{\bar{\gamma}} = 1$. If $v \in L^\gamma(J, R)$, $\overline{v} \in L^{\bar{\gamma}}(J, R)$, then for $1 \leq \gamma \leq \infty$, $v\bar{v} \in L^1(J, R)$ and $\|v\bar{v}\|_{L^1(J)} \leq \|v\|_{L^\gamma(J)}\|\bar{v}\|_{L^{\bar{\gamma}}(J)}$.

Theorem (2.11) Arzela-Ascoli's theorem [37].

A set $D \subset C[J, X]$ is relatively compact if and only if it is bounded and equicontinuous.

3. Controllability result.

In this section, we provide the main results on completely controllable of system (S_1). To do this, we need to list the following assumptions

$\mathcal{A}_1$. $g: J \times D \to D(E)$ is a continuous function, and there exists two constants $L_g, L_{\bar{g}} > 0$ such that $\|Eg(t, x_t) - Eg(t, x_t)\| \leq L_g\|x - y\|$ and $L_{\bar{g}} = \sup_{t\in[0,a]}\|Eg(t, 0)\|$.

$\mathcal{A}_2$. $\hbar: J \times D \to D$ is continuous function such that $\hbar(\omega, x_\omega)(\theta)$ is continuous for $(\omega, x_\omega, \theta) \in J \times D \times [-\hat{a}, 0]$ and there exists a constant $L_\hbar > 0$ such that $\|\hbar(\omega, x_\omega) - \hbar(\omega, y_\omega)\| \leq L_\hbar\|x - y\|$, whenever $x, y \in PC([-\hat{a}, a], X)$, $\omega \in J$.

$\mathcal{A}_3$. $G: PC([-\hat{a}, a], X) \to X$ is a continuous function and there exists $\alpha_1 \in (0, \alpha)$ and $\Im \in L^{\frac{1}{\alpha_1}}(J, R^+)$ such that $\left\| \int_0^t Gx(\zeta)\, d\zeta \right\| \le \Im(t)$ for all $x \in PC([-\hat{a}, a], X), t \in J$.
$\mathcal{A}_4$. $I_r: X \to X$, $(r = 1,2,3, \dots, m)$ are continuous functions, and there exists constants $L_r > 0$, with $\sum_{r=1}^{m} L_r = L$ such that $\|I_r(x) - I_r(y)\| \le L_r \|x - y\|$.
$\mathcal{A}_5$.The state-dependent delay $\eta: PC([0, a], X) \to J$ is an increasing differentiable function and satisfies the inequality $\eta(x(t)) \le t$ for each $t \in J$.
$\mathcal{A}_6$. The linear operator $W: Y \to X$ is defined by

$$Wu = \int_0^a (a - s)^{\alpha - 1} \mathcal{H}_\alpha(a - s) Bu(s) ds,$$

has an inverse operator W^{-1}: Rang$(W) \to \frac{Y}{kerW}$ and there exits a constant k > 0 such that
$\|W^{-1}\| \le k$.

For simplicity, we denote $2\|\Im\|_{L^{\frac{1}{\alpha_1}}[0,t]} \left(\frac{1-\alpha_1}{\alpha - \alpha_1}\right)^{(1-\alpha_1)} a^{\alpha - \alpha_1}$ by Σ

Definition (3.1). The system (S_1) is said to be controllability on $[0, a]$ if for every initial function $\varphi \in C([-\hat{a}, 0], X)$ and final state $x_1 \in X$, there exists an admissible control $u \in Y$ such that the PC −mild solution $x(t)$ of system (S_1) satisfies $x(a) = x_1$.

Theorem (3.2). Let $\varphi \in C([-\hat{a}, 0], X)$ be an initial function. If the hypotheses $\mathcal{A}_1 - \mathcal{A}_6$ are holds, then the control system (S_1) is controllability on $[0, a]$ provided that

$$C\left[\left(1 + \frac{Ma^\alpha}{\Gamma(\alpha + 1)} \|B\| K\right)\right] < 1, \qquad (9)$$

where, $$C = M\left(L + aL_{\hbar} + \frac{a^\alpha L_g}{\Gamma(\alpha + 1)}\right) + (M + 1)\|E^{-1}\| L_g$$

Proof. By using condition $\mathcal{A}_6$, for any function $x(.) \in PC([-\hat{a}, a], X)$ choose the control u_x associated with the control problem (S_1) as follows:

$$u_x(t) = \begin{cases} W^{-1}\left[\begin{array}{c} x_1 - \mathcal{K}_\alpha(a)\left[\varphi(0) + \left(\int\limits_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] - g(a, x_a) \\ -\int_0^a (a - s)^{\alpha - 1} \mathcal{H}_\alpha(a - s)\left[Eg(s, x_s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta\right] ds, \\ t \in [0, t_1], \end{array}\right](t) \\ W^{-1}\left[\begin{array}{c} x_1 - \mathcal{K}_\alpha(a)\left[\varphi(0) + \left(\int\limits_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] - g(a, x_a) \\ -\int_0^a (a - s)^{\alpha - 1} \mathcal{H}_\alpha(a - s)\left[Eg(s, x_s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta) d\zeta\right] ds \\ -\sum_{r=1}^{m} \Delta x(t_r) \mathcal{K}_\alpha(a - t_r), t \in (t_r, t_{r+1}], \quad \text{r} = 1,2,3, \dots, \text{m}. \end{array}\right](t) \end{cases}$$

By using this control, we will show that the operator $\Pi: PC([-\hat{a}, a], X) \to PC([-\hat{a}, a], X)$ defined as

$$(\Pi x)(t) = \begin{cases} \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int\limits_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + g(t, x_t) \\ + \int_0^t (t-s)^{\alpha-1}\, \mathcal{H}_\alpha(t-s)\left[Eg(s, x_s) + Bu_x(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right] ds, \\ t \in [0, t_1], \\ \mathcal{K}_\alpha(t)\left[\varphi(0) + \left(\int\limits_0^a \hbar(\omega, x_\omega)\, d\omega\right)(0) - g(0, x_0)\right] + g(t, x_t) \\ + \int_0^t (t-s)^{\alpha-1}\, \mathcal{H}_\alpha(t-s)\left[Eg(s, x_s) + Bu_x(s) + \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right] ds \\ + \sum_{r=1}^{m} \Delta x(t_r)\mathcal{K}_\alpha(t-t_r), t \in (t_r, t_{r+1}], r = 1,2,3, \dots, m \end{cases}$$

$$(\Pi x)(\theta) = \varphi(\theta) + \left(\int\limits_0^a \hbar(\omega, x_\omega)\, d\omega\right)(\theta), \qquad \theta \in [-\hat{a}, 0],$$

has a fixed point. This fixed point is a PC −mild solution of control problem (S_1), obviously that $(\Pi x)(a) = x_1$ .For any positive constant $\gamma > 0$, let $\mathcal{B}_\gamma = \{x \in PC([-\hat{a}, a], x), \|x\| \leq \gamma\}$, it is clear that $\mathcal{B}_\gamma$ is closed, convex and bounded set in $PC([-\hat{a}, a], X)$. By simple calculations, we can see that $(t-s)^{\alpha-1} \in L^{\frac{1}{\alpha_1 - 1}}([0, t])$ for $t \in [0, a]$ and $\alpha_1 \in (0, \alpha)$. We have

$$\left\|\int_0^t (t-s)^{\alpha-1} \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\, ds\right\| \leq \left\|\int_0^t (t-s)^{\alpha-1} \int\limits_0^s Gx(\zeta)d\zeta\, ds\right\| + \left\|\int_0^t (t-s)^{\alpha-1} \int\limits_0^{s-\eta(x(s))} Gx(\zeta)d\zeta\, ds\right\|$$

Using (Höider inequality) and assumption $\mathcal{A}_3$, we get that

$$\left\|\int_0^t (t-s)^{\alpha-1} \int\limits_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\, ds\right\| \leq 2\left[\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds\right]^{1-\alpha_1} \|\mathfrak{I}\|_{L^{\frac{1}{\alpha_1}}[0,t]}$$

$$\leq 2\|\mathfrak{I}\|_{L^{\frac{1}{\alpha_1}}[0,t]} \left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{(1-\alpha_1)} a^{\alpha-\alpha_1} \quad (10)$$

Hence, for all $(t_r, t_{r+1}]$, $r = 1,2, \dots, m$, we get

$$\|u_x(t)\| \le \|W^{-1}\| \left[\|x_1\| + M \left\| \varphi(0) + \left(\int_0^a \hbar(\omega, x_\omega)\, d\omega \right)(0) - g(0, x_0) \right\| + \|g(a, x_a)\| \right.$$
$$+ \left\| \sum_{r=1}^{m} \Delta x(t_r)\, \delta_r(t) \mathcal{K}_\alpha(a - t_r) \right\| + \frac{M}{\Gamma(\alpha)} \int_0^a (t-s)^{\alpha-1} \|Eg(s, x_s)\| ds$$
$$\left. + \frac{M}{\Gamma(\alpha)} \Sigma \right]$$
$$\le K \left[\|x_1\| + M\|\varphi\| + Ma \left[L_\hbar \|x\| + \sup_{\omega \in [0,a]} \|\hbar(\omega, 0)\| \right] + M\left[\|E^{-1}\| L_g \|x\| + \|E^{-1}\| L_{\bar{g}}\right] + \|E^{-1}\| L_g \|x\| \right.$$
$$\left. + \|E^{-1}\| L_{\bar{g}} + ML\|x\| + M \sum_{r=1}^{m} \|I_r(0)\| + \frac{Ma^\alpha}{\alpha\Gamma(\alpha)} \left[L_g \|x\| + L_{\bar{g}} \right] + \frac{M}{\Gamma(\alpha)} \Sigma \right]$$
$$= K \left[\|x\| \left(MaL_\hbar + M\|E^{-1}\| L_g + \|E^{-1}\| L_g + ML + \frac{Ma^\alpha}{\Gamma(\alpha+1)} Lg \right) + \|x_1\| + M\|\varphi\| \right.$$
$$+ Ma\|\hbar(\omega, 0)\| + M\|E^{-1}\| L_{\bar{g}} + \|E^{-1}\| L_{\bar{g}} + \frac{Ma^\alpha}{\Gamma(\alpha+1)} L_{\bar{g}} + M \sum_{r=1}^{m} \|I_r(0)\|$$
$$\left. + \frac{M}{\Gamma(\alpha)} \Sigma \right]$$
$$= K[C\|x\| + \|x_1\| + C_1]$$
$$C_1 = (M+1)\|E^{-1}\| L_{\bar{g}}$$
$$+ M \left[\sum_{r=1}^{m} \|I_r(0)\| + \frac{a^\alpha}{\Gamma(\alpha+1)} L_{\bar{g}} + \|\varphi\| + a \sup_{\omega \in [0,a]} \|\hbar(\omega, 0)\| + \frac{\Sigma}{\Gamma(\alpha)} \right].$$

Next, we shall prove that the operator Π has a fixed point on $\mathcal{B}_\gamma$, that means setting a specific positive constant γ_0 such that Π has a fixed point on $\mathcal{B}_{\gamma_0}$.
In fact, by choosing

$$\gamma_0 = \frac{Ma^\alpha \|B\| k[\|x_1\| + c_1]}{\Gamma(\alpha+1)\left[1 - C\left(1 + \frac{Ma^\alpha}{\Gamma(\alpha+1)} \|B\| K\right)\right]} + \frac{c_1}{1 - C\left(1 + \frac{Ma^\alpha}{\Gamma(\alpha+1)} \|B\| K\right)}$$

We can show that the operator Π has a fixed point on $\mathcal{B}_{\gamma_0}$. Our proof will be divided into the following three steps.

Step I. $\Pi x \in \mathcal{B}_{\gamma_0}$, whenever $x \in \mathcal{B}_{\gamma_0}$.
For any $x \in \mathcal{B}_{\gamma_0}$, we obtain

$$\|(\Pi x)(t)\|$$
$$\le M\,[\|\varphi\| + a\|\hbar(\omega, x_\omega)\| + \|g(0, x_0)\|] + \|g(t, x_t)\| + M \sum_{r=1}^{m} \left\| I_r\big(x(t_r)\big) \right\|$$
$$+ \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|Bu_x(s)\|\, ds + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|Eg(s, x_s\|\, ds \quad + \frac{M}{\Gamma(\alpha)} \Sigma$$

$$\leq M\|\varphi\| + MaL_{\hbar}\gamma_0 + Ma\sup_{\omega\in[0,a]}\|\hbar(\omega,0)\| + M\|E^{-1}\|L_g\gamma_0 + M\|E^{-1}\|L_{\bar{g}} + \|E^{-1}\|L_g\gamma_0$$
$$+\|E^{-1}\|L_{\bar{g}} + ML\gamma_0 + M\sum_{r=1}^{m}\|I_r(0)\| + \frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}\|B\|\|u_x(.)\|$$
$$+\frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}\left(L_g\gamma_0 + L_{\bar{g}}\right) + \frac{M}{\Gamma(\alpha)}\Sigma$$
$$=\left[M\left(L + aL_{\hbar} + \frac{a^{\alpha}L_g}{\Gamma(\alpha+1)}\right) + (M+1)\|E^{-1}\|L_g\right]\gamma_0 + (M+1)\|E^{-1}\|L_{\bar{g}}$$
$$+M\left[\sum_{r=1}^{m}\|I_r(0)\| + \frac{a^{\alpha}}{\Gamma(\alpha+1)}L_{\bar{g}} + \|\varphi\| + a\sup_{\omega\in[0,a]}\|\hbar(\omega,0)\| + \frac{M}{\Gamma(\alpha)}\Sigma\right]$$
$$+\frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}\|B\|\|u_x(.)\|$$
$$= c\gamma_0 + c_1 + \frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}\|B\|[k\|x_1\| + ck\gamma_0 + kc_1]$$
$$=\left[c + c\frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}\|B\|k\right]\gamma_0 + c_1 + \frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}\|B\|k[\|x_1\| + c_1] = \gamma_0.$$

Hence $\Pi(\mathcal{B}_{\gamma_0}) \subseteq \mathcal{B}_{\gamma_0}$.
Now, we define two operators Π_1 and Π_2 on $\mathcal{B}_{\gamma_0}$ as follows:

$$(\Pi_1 x)(t) = \mathcal{K}_{\alpha}(t)\left[\varphi(0) + \left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0) - g(0,x_0)\right] + g(t,x_t)$$
$$+\sum_{r=1}^{m}\Delta x(t_r)\mathcal{K}_{\alpha}(t-t_r), t\in(t_r,t_{r+1}],\quad t\in[0,a], r=1,2,\dots,m,$$
$$(\Pi_1 x)(\theta) = \varphi(\theta) + \left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(\theta),\qquad \theta\in[-\hat{a},0],$$
$$(\Pi_2 x)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{H}_{\alpha}(t-s)\left[Eg(s,x_s) + Bu_x(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right]ds, t \in [0,a],$$
$$(\Pi_2 x)(\theta) = 0,\qquad \theta\in[-\hat{a},0].$$

Obliviously, $\Pi = \Pi_1 + \Pi_2$ and the system (S_1) with control u_x has a mild solution if and only if the operator equation $x = \Pi_1 x + \Pi_2 x$ has mild solution x on $\mathcal{B}_{\gamma_0}$. In the following steps, we show that Π_1 is contraction mapping on $\mathcal{B}_{\gamma_0}$ and Π_2 is completely continuous.

Step II. Π_1 is contraction on $\mathcal{B}_{\gamma_0}$.
For any $x, y \in \mathcal{B}_{\gamma_0}$ and $t \in J$, by using $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4$ and Eq. (10), we get

$$\|(\Pi_1 x)(t)-(\Pi_1 y)(t)\| = \left\|\mathcal{K}_\alpha(t)\left[\left(\int_0^a \hbar(\omega,x_\omega)(0)d\omega-\int_0^a \hbar(\omega,y_\omega)(0)d\omega\right) - \left(g(0,x_0)-g(0,y_0)\right)\right]+\left(g(t,x_t)-g(t,y_t)\right) + \sum_{r=1}^{m}\left(I_r(x)-I_r(y)\right)\mathcal{K}_\alpha(t-t_r)\right\|$$

$$\leq M\left[aL_\hbar\|x-y\|+\|E^{-1}\|L_g\|x-y\|\right]+\|E^{-1}\|L_g\|x-y\|+ ML\|x-y\|$$

$$= \left[M(aL_\hbar+L)+(1+M)\|E^{-1}\|L_g\right]\|x-y\|.$$

$$\|(\Pi_1 x)(\theta)-(\Pi_1 y)(\theta)\| = \left\|\int_0^a \hbar(\omega,x_\omega)(0)d\omega-\int_0^a \hbar(\omega,y_\omega)(0)d\omega\right\| \leq MaL_\hbar\|x-y\|$$

by **(9)** Π_1 contraction.

Step III. Π_2 is completely continuous.

i- Π_2 is continuous on $\mathcal{B}_{\gamma_0}$.

For each $\{x^n\}\subseteq\mathcal{B}_{\gamma_0}$, with $x^n\to x$ on $\mathcal{B}_{\gamma_0}$. Since $x_t^n\to x_t$ for $t\in J$ and by using $\mathcal{A}_1,\mathcal{A}_3,\mathcal{A}_5$ we have, $G\left(x^n(t)\right)\to G\left(x(t)\right)$, $\eta(x^n(t))\to\eta(x(t), as\ \ n\to\infty$ and $\lim_{n\to\infty}\int_0^s Gx^n(\zeta)d\zeta = \int_0^s Gx(\zeta)d\zeta$.

Noting that
$$\left\|\int_0^s Gx^n(\zeta)d\zeta-\int_0^s Gx(\zeta)d\zeta\right\| \leq 2\mathfrak{J}(s),$$

by the dominated convergence theorem, we obtain

$$\|u_x(t)-u_{x^n}(t)\| \leq k\left[MaL_\hbar\|x^n-x\|+M\|E^{-1}\|L_g\|x^n-x\|+\|E^{-1}\|L_g\|x^n-x\| + \frac{Ma^\alpha}{\alpha\Gamma(\alpha)}L_g\|x^n-x\| + \frac{M}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\left\|\left(\int_{s-\eta(x^n(s))}^{s} Gx^n(\zeta)d\zeta-\int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right)\right\| + ML\|x_n-x\|\right]$$

$$\leq k\left[MaL_{\hbar}+M\|E^{-1}\|L_g+\|E^{-1}\|L_g+\frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}L_g+ML\right]\|x_n-x\|$$
$$+k\frac{M}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\left\|\int_0^s Gx^n(\zeta)d\zeta-\int_0^s Gx(\zeta)d\zeta\right\|ds$$
$$+k\frac{M}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\left\|\int_0^{s-\eta(x^n(s))} Gx^n(\zeta)d\zeta-\int_0^{s-\eta(x(s))} Gx(\zeta)d\zeta\right\|ds$$
$$=kC\|x^n-x\|+k\frac{M}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\left\|\int_0^s Gx^n(\zeta)d\zeta-\int_0^s Gx(\zeta)d\zeta\right\|ds$$
$$+k\frac{M}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\left\|\int_0^{s-\eta(x^n(s))} Gx^n(\zeta)d\zeta-\int_0^{s-\eta(x(s))} Gx(\zeta)d\zeta\right\|ds\to 0 \quad as \quad n\to\infty.$$

Therefore, for $t\in[0,a]$,

$$\|(\Pi_2x^n)(t)-(\Pi_2x)(t)\|\leq\frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}L_g\|x^n-x\|+\frac{Ma^{\alpha}}{\alpha\Gamma(\alpha)}\|B\|\sup_{s\in[0,a]}\|u_{x^n}(s)-u_x(s)\|$$
$$+\frac{M}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\left\|\int_0^s Gx^n(\zeta)d\zeta-\int_0^s Gx(\zeta)d\zeta\right\|ds$$
$$+k\frac{M}{\Gamma(\alpha)}\int_0^a(a-s)^{\alpha-1}\left\|\int_0^{s-\eta(x^n(s))} Gx^n(\zeta)d\zeta-\int_0^{s-\eta(x(s))} Gx(\zeta)d\zeta\right\|$$

Which implies $\|\Pi_2x^n-\Pi_2x\|\to 0$ $as\ n\to\infty$. This means that Π_2 is continuous.
ii- Π_2 is compact operator.
According to Arzela-Ascoli's theorem, we just need to prove that the family $\{\Pi_2x,\ x\in\mathcal{B}_{\gamma_0}\}$ is equicontinuous and uniformly bounded, and for each $t\in[0,a]$, $\{(\Pi_2x)(t),\ x\in\mathcal{B}_{\gamma_0}\}$ is relatively compact in X.
Since $\|\Pi_2x\|\leq\gamma_0$ for each $x\in\gamma_0$, then the family $\{\Pi_2x,\ x\in\mathcal{B}_{\gamma_0}\}$ is uniformly bounded set, we will prove that the $\Pi_2(\mathcal{B}_{\gamma_0})\subseteq PC([-\hat{a},a],X)$ is a family of equicontinuous functions. For any $x\in\mathcal{B}_{\gamma_0}$, let $0\leq t_1<t_2\leq a$, we have

$$\|(\Pi_2x)(t_2)-(\Pi_2x)(t_1)\|$$
$$=\left\|\int_0^{t_2}(t_2-s)^{\alpha-1}\mathcal{H}_{\alpha}(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\right.$$
$$\left.-\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{H}_{\alpha}(t_1-s)\left[Eg(s,x_s)+Bu(s)+\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\right\|$$
$$=\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{H}_{\alpha}(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\right.$$
$$+\int_0^{t_1}(t_2-s)^{\alpha-1}\mathcal{H}_{\alpha}(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds$$

$$+\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{H}_\alpha(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds$$
$$-\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{H}_\alpha(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds$$
$$-\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{H}_\alpha(t_1-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\Bigg\|$$
$$\leq\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{H}_\alpha(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\right\|$$
$$+\Bigg\|\int_0^{t_1}(t_2-s)^{\alpha-1}\mathcal{H}_\alpha(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds$$
$$-\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{H}_\alpha(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\Bigg\|$$
$$+\Bigg\|\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{H}_\alpha(t_2-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds$$
$$-\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{H}_\alpha(t_1-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\Bigg\|$$
$$=\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{H}_\alpha(t_1-s)\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\right\|$$
$$+\Bigg\|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathcal{H}_\alpha(t_2-s)\Bigg[Eg(s,x_s)+Bu(s)$$
$$+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\Bigg]ds\Bigg\|$$
$$+\Bigg\|\int_0^{t_1}(t_1-s)^{\alpha-1}[\mathcal{H}_\alpha(t_2-s)-\mathcal{H}_\alpha(t_1-s)]\Bigg[Eg(s,x_s)+Bu(s)$$
$$+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\Bigg]ds\Bigg\|$$
$$=P_1+P_2+P_3$$

Where,

$$P_1 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{H}_\alpha(t_1 - s) \left[Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right] ds \right\|$$

$$P_2 = \left\| \int_{0}^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{H}_\alpha(t_2 - s) \left[Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right] ds \right\|$$

$$P_3 = \left\| \int_{0}^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{H}_\alpha(t_2 - s) - \mathcal{H}_\alpha(t_1 - s)] \left[Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right] ds \right\|$$

Now, we show that P_1, P_2 and P_3 tends to 0 a uniformly for all $x \in \mathcal{B}_{\gamma_0}$, when $t_2 \to t_1$.
For P_1, we have

$$\begin{aligned} P_1 &\le \frac{M}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left\| Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right\| ds \\ &\le \frac{M}{\Gamma(\alpha)} \left(L_g\gamma_0 + L_{\bar{g}}\right)(t_2 - t_1)^\alpha + \frac{M}{\Gamma(\alpha)} \|B\|\|u\|(t_2 - t_1)^\alpha \\ &\quad + \frac{2M}{\Gamma(\alpha)} \|\mathfrak{I}\| \left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{(1-\alpha_1)} (t_2 - t_1)^{\alpha-\alpha_1} \\ &= \frac{M}{\Gamma(\alpha)} \left(\left[\frac{L_g\gamma_0 + L_{\bar{g}} + \|B\|\|u\|}{\alpha}\right] (t_2 - t_1)^\alpha + 2\|\mathfrak{I}\| \left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{(1-\alpha_1)} (t_2 - t_1)^{\alpha-\alpha_1} \right) \end{aligned}$$

Which implies that $\lim_{t_2 \to t_1} P_1 = 0$.
For P_2, we have

$$\begin{aligned} P_2 &\le \frac{M}{\Gamma(\alpha)} \left\| \int_{0}^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \left[Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right] ds \right\| \\ &\le \frac{M}{\Gamma(\alpha)} \int_{0}^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \left\| \left[Eg(s, x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right] ds \right\| \\ &\le \frac{M}{\alpha\Gamma(\alpha)} \left(L_g\gamma_0 + L_{\bar{g}}\right)(t_2 - t_1)^\alpha + \frac{M}{\alpha\Gamma(\alpha)} \|B\|\|u\|(t_2 - t_1)^\alpha \\ &\quad + \frac{2M}{\alpha\Gamma(\alpha)} \|\mathfrak{I}\| \left[\int_{0}^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}} ds \right]^{1-\alpha_1} \end{aligned}$$

Since

$$[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}} \le (t_2 - s)^{\frac{\alpha-1}{1-\alpha_1}} - (t_1 - s)^{\frac{\alpha-1}{1-\alpha_1}},$$

then,

$$\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}}\,ds \le \frac{1-\alpha_1}{\alpha-\alpha_1}\left[t_1^{\frac{\alpha-1}{1-\alpha_1}}-t_2^{\frac{\alpha-1}{1-\alpha_1}}+(t_2-t_1)^{\frac{\alpha-1}{1-\alpha_1}}\right].$$

So, we get

$$P_2 \le \frac{M}{\Gamma(\alpha)}\left[\frac{L_g\gamma_0+L_{\bar{g}}+\|B\|\|u\|}{\alpha}(t_2-t_1)^{\alpha}+2\|\mathfrak{I}\|\left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{(1-\alpha_1)}(t_2-t_1)^{\alpha-\alpha_1}\right].$$

Hence, $\lim_{t_2\to t_1} P_2 = 0$.

Concerning P_3 , if $t_1 = 0,\ 0 < t_2 \le a$, it is clear that $P_3 = 0$.

For $t_1 > 0$, and w is sufficiently small, we get that

$$\begin{aligned}
P_3 \le& \int_0^{t_1-w}(t_1-s)^{\alpha-1}\|\mathcal{H}_\alpha(t_2-s)-\mathcal{H}_\alpha(t_1-s)\|\left\|Eg(s,x_s)+Bu(s)\right.\\
&\left.+\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right\|ds\\
&+\int_{t_1-w}^{t_1}(t_1-s)^{\alpha-1}\|\mathcal{H}_\alpha(t_2-s)-\mathcal{H}_\alpha(t_1-s)\|\left\|Eg(s,x_s)+Bu(s)\right.\\
&\left.+\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right\|ds\\
\le& \int_0^{t_1-w}(t_1-s)^{\alpha-1}\|\mathcal{H}_\alpha(t_2-s)-\mathcal{H}_\alpha(t_1-s)\|\left[\|Eg(s,x_s)\|+\|Bu(s)\|\right.\\
&\left.+\left\|\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right\|\right]ds\\
&+\int_{t_1-w}^{t_1}(t_1-s)^{\alpha-1}\|\mathcal{H}_\alpha(t_2-s)-\mathcal{H}_\alpha(t_1-s)\|\left[\|Eg(s,x_s)\|+\|Bu(s)\|\right.\\
&\left.+\int_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]ds\\
\le& \left[\frac{L_g\gamma_0+L_{\bar{g}}+\|B\|\|u\|}{\alpha}(t_1{}^{\alpha}-w^{\alpha})+2\|\mathfrak{I}\|\left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{(1-\alpha_1)}(t_1{}^{\alpha-\alpha_1}-w^{\alpha-\alpha_1})\right]\\
&\times\sup_{t\in[0,t_{1-w}]}\|\mathcal{H}_\alpha(t_2-s)-\mathcal{H}_\alpha(t_1-s)\|\\
&+\frac{2M}{\Gamma(\alpha)}\left[\frac{L_g\gamma_0+L_{\bar{g}}+\|B\|\|u\|}{\alpha}w^{\alpha}+2\|\mathfrak{I}\|\left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{(1-\alpha_1)}w^{\alpha-\alpha_1}\right].
\end{aligned}$$

The compactness of $T(t)$ $(t > 0)$ and Lemma (2.7) implies the continuity of $\mathcal{H}_\alpha$ in the uniform operator topology, this yields that $P_3 \to 0,\ as\ t_2 \to t_1$ and $w \to 0$.

Thus, $\|(\Pi_2x)(t_2)-(\Pi_2x)(t_1)\| \to 0, as\ \ t_2 \to t_1,$ is independently of $x \in \mathcal{B}_{\gamma_0}$.From this, we conclude that the family $\{\Pi_2x,\ x \in \mathcal{B}_{\gamma_0}\}$ is equicontinuous. Finally, we will show that the family $\mathcal{Z}(t) := \{(\Pi_2x)(t),\ x \in \mathcal{B}_{\gamma_0}\}$ is relatively compact in X for any $t \in [-\hat{a}, a]$. For $t \in [-\hat{a}, 0]$, $\mathcal{Z}(t)$ is relatively compact in X. Let $t \in (0, a]$ be a fixed, $\varrho \in (0, t)$ and m is a positive real number. Define an operator Π_ϱ^m on $\mathcal{B}_{\gamma_0}$ by

$$\left(\Pi_{\varrho}^{m}x\right)(t)=\int_{0}^{t-\varrho}\int_{m}^{\infty}\alpha(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds$$

$$=T(\varrho^{\alpha}m)\int_{0}^{t-\varrho}\int_{m}^{\infty}\alpha(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}-\varrho^{\alpha}m]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds$$

$x\in\mathcal{B}_{\gamma_0}$. According to the compactness of $T\ (\varrho^{\alpha}m)$ $(\varrho^{\alpha}m>0)$ and boundedness of

$$\int_{0}^{t-\varrho}\int_{m}^{\infty}\alpha(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}-\varrho^{\alpha}m]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds,$$

for each $t\in[0,a]$, the set $\left\{\left(\Pi_{\varrho}^{m}x\right)(t),\ x\in\mathcal{B}_{\gamma_0}\right\}$ is relatively compact in X. In addition, for any $x\in\mathcal{B}_{\gamma_0}$, we get that

$$\left\|(\Pi_2x)(t)-\left(\Pi_{\varrho}^{m}x\right)(t)\right\|$$

$$=\alpha\left\|\int_{0}^{t}\int_{0}^{m}(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds\right.$$
$$+\int_{0}^{t}\int_{m}^{\infty}(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds-$$
$$\left.\int_{0}^{t-\varrho}\int_{m}^{\infty}(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds\right\|$$

$$\leq\alpha\left\|\int_{0}^{t}\int_{0}^{m}(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds\right\|$$
$$+\ \alpha\left\|\int_{t-\varrho}^{t}\int_{m}^{\infty}(t-s)^{\alpha-1}\mu\Psi_{\alpha}(\mu)T[\mu(t-s)^{\alpha}]\left[Eg(s,x_s)+Bu(s)+\int\limits_{s-\eta(x(s))}^{s}Gx(\zeta)d\zeta\right]d\mu ds\right\|$$

$$\leq \alpha M \int_0^t (t-s)^{\alpha-1} \left\| Eg(s,x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right\| ds \int_0^m \mu\Psi_\alpha(\mu)d\mu +$$

$$\alpha M \int_{t-\varrho}^t (t-s)^{\alpha-1} \left\| Eg(s,x_s) + Bu(s) + \int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta \right\| ds \int_m^\infty \mu\Psi_\alpha(\mu)d\mu$$

$$\leq \alpha M \left[\frac{L_g\gamma_0 + L_{\bar{g}} + \|B\|\|u\|}{\alpha} a^\alpha + \Sigma \right] \int_0^m \mu\Psi_\alpha(\mu)d\mu$$

$$+\alpha M \left[\frac{L_g\gamma_0 + L_{\bar{g}} + \|B\|\|u\|}{\alpha} \varrho^\alpha + 2\|\Im\| \left(\frac{1-\alpha_1}{\alpha-\alpha_1}\right)^{(1-\alpha_1)} \varrho^{\,\alpha-\alpha_1} \right].$$

Which implies $\left\|(\Pi_2 x)(t) - \left(\Pi_\varrho^m x\right)(t)\right\| \to 0, as\ \varrho, m \to 0.$ Therefore there are relatively compact sets arbitrary close to the set $\mathcal{Z}(t)$ and so $\mathcal{Z}(t)$ is relatively compact in X. Consequently, Π_2 is completely continuous operator on $\mathcal{B}_{\gamma_0}$. From the Krasnoselskii fixed point theorem, Π has a fixed point in $\mathcal{B}_{\gamma_0}$ satisfying $(\Pi x)(t) = x(t)$. Therefore, the system (S_1) is controllable on J. The proof is complete. □

4. Application

In this section, we establish an application of our main result. Consider the following impulsive fractional partial differential equation:

$$\begin{cases} \dfrac{\partial^\alpha}{\partial t^\alpha}\left[v(t,z) - \int_0^z \sin v_t(\theta,\mathfrak{B})\, d\mathfrak{B}\right] = \dfrac{\partial^2}{\partial z^2} v(t,z) + Bu(t) + \displaystyle\int_{t-\eta(v(t,z))}^{t} \lambda v(\zeta,z)\, e^{-\gamma(t-\zeta)} d\zeta, \\ \qquad t \in [0,1]/\{t_1\}, z \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \qquad t \in [0,1], \\ \Delta v(t,z)|_{t=\frac{1}{2}} = \dfrac{\left|v\left(\frac{1}{2}^-, z\right)\right|}{1 + \left|v\left(\frac{1}{2}^-, z\right)\right|}, \quad z \in [0,\pi], t_1 = \left\{\frac{1}{2}\right\} \\ v(\theta,z) = \varphi(\theta,z) + \displaystyle\int_0^1 \sin(v(\omega+\theta,z))\, d\omega, \qquad \theta \in [-\hat{a}, 0], \end{cases} \qquad (S_4)$$

where $\alpha \in (0,1)$. Set $X = L^2([0,\pi], R)$ and the operator $E: D(E) \subseteq X \longrightarrow X$ defined by $Ev(t,z) = \frac{\partial^2}{\partial z^2} v(t,z)$ with domain $\left\{v \in X: \frac{\partial v}{\partial z}, \frac{\partial^2 v}{\partial z^2} \in X, v(t,0) = v(t,\pi) = 0\right\}$. Then, E can be written as $Ev = \sum_{j=1}^\infty j^2 \langle v, v_j\rangle\, v_j, v \in D(E)$ where $v_j(z) = \sqrt{\frac{2}{\pi}} \sin jz, j = 1,2,3, \dots$ and its is generates of a compact C_0 −semigroup $T(t)$ in X given by $T(t)v = \sum_{j=1}^\infty e^{-j^2 t} \langle v, v_j\rangle\, v_j$, $v \in D(E)$. Several authors are used this semigroup in their examples, see[16,20,32] .etc. The system (S_4) can be redrafted as the following impulsive fractional control system

$$\begin{cases} {}^c D^\alpha[x(t) - g(t,x_t)] = Ex(t) + Bu(t) + \displaystyle\int_{t-\eta(x(t))}^{t} Gx(\zeta)\, d\zeta, \qquad t \in J = [0,a]/\{t_1, t_2, t_3, \dots, t_m\} \\ \Delta x(t_r) = I_r\left(x(t_r^-)\right), \quad r = 1,2,3, \dots, m \\ x_0(\theta) = \varphi(\theta) + \left(\displaystyle\int_0^a \hbar(\omega, x_\omega)\, d\omega\right)(\theta), \qquad \theta \in [-\hat{a}, 0], \qquad \hat{a} > 0. \end{cases}$$

Where $x(t) = v(t,.)$, that is $x(t)(z) = v(t,z)$ and $x_t(\theta)(z) = v_t(\theta, z) = v(t+\theta, z), t \in J, z \in [0,\pi]$, $\theta \in [-\hat{a}, 0]$, $m = 1$. Now we define the functions $g, G, \eta, \hbar, I_1$ by

i- $g: J \times D \to D(E)$ is given by

$$g(t, v_t(\theta, z)) = \int_0^z \sin v_t(\theta, \mathfrak{V})\, d\mathfrak{V}, \qquad t \in J, z \in [0,\pi], \theta \in [-\hat{a}, 0].$$

ii- $G: PC([-\hat{a}, a], X) \to X$ is given by

$$Gv(t,z) = \lambda v(\zeta, z) e^{-\gamma(t-\zeta)}, \qquad \lambda, \gamma > 0, t \in J, \zeta \in [0,t], z \in [0,\pi].$$

iii- $\eta: PC([0,1], X) \to J$ is given by

$$\eta(v(t,z)) = \frac{t}{1+t}, \qquad t \in J.$$

iv- $\hbar: J \times D \to D$ is given by

$$\hbar(\omega, v(\omega, z))(\theta) = \sin v(\omega + \theta, z), \qquad t \in J, z \in [0,\pi], \theta \in [-\hat{a}, 0].$$

v- $I_1: X \to X$ is given by

$$I_1(v(t_1, z)) = \frac{\left|v\left(\frac{1}{2}^-, z\right)\right|}{2 + \left|v\left(\frac{1}{2}^-, z\right)\right|}, \qquad t \in J, z \in [0,\pi].$$

For $u, v \in X$, we have

$$\left\|Eg(t, v_t(\theta, z)) - Eg(t, u_t(\theta, z))\right\| = \left\|\frac{\partial^2}{\partial z^2}\int_0^z \sin v_t(\theta, \mathfrak{V})\, d\mathfrak{V} - \frac{\partial^2}{\partial z^2}\int_0^z \sin u_t(\theta, \mathfrak{V})\, d\mathfrak{V}\right\|$$

$$= \left\|\cos v_t(\theta, z)\frac{\partial v_t}{\partial z} - \cos u_t(\theta, z)\frac{\partial u_t}{\partial z}\right\| \le \mathfrak{M}\|v - u\|.$$

where $\mathfrak{M} = \sup\left\{\left\|\frac{\partial^2}{\partial z^2}\sin v_t(\theta, z)\right\| : v \in X, t \in J, z \in [0,\pi], \theta \in [-\hat{a}, 0]\right\}$. And it is easy to see that

$$\left\|\hbar(\omega, v(\omega, z))(\theta) - \hbar(\omega, v(\omega, z))(\theta)\right\| \le \|v - u\|$$

and

$$\left\|I_1(v(t_1, z)) - I_1(u(t_1, z))\right\| \le \frac{1}{2}\|v - u\|.$$

It is also obvious that

$$\left\|\int_0^t Gv(\pi\zeta, z)\, d\zeta\right\| \le \frac{\lambda}{\gamma}\mathfrak{N}(1 - e^{-\gamma t}) \coloneqq \mathfrak{I}(t) \in L^{\frac{1}{\alpha_1}}(J, R^+), \alpha_1 \in (0, \alpha), t \in J,$$

where $\mathfrak{N} = \sup\{\|v(t,z)\| : t \in J, z \in [0,\pi], \theta \in [-\hat{a}, 0]\}$. Therefore, the assumptions of Theorem (3.2) are satisfied, and we deduce that the control system (S_4) is controllable.

Remark: If we replace the control function $u(t)$ by the control delay function $u(\delta(t))$ in system (S_1), where $\delta: J \to [0, \infty)$ is twice continuously differentiable and strictly increasing function in J, satisfies $\delta(t) \le t$, then the PC −mild solution of control delay system is as follows

$$x(t) = \begin{cases} \mathcal{K}_\alpha(t)\left[\varphi(0)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0)-g(0,x_0)\right]+g(t,x_t) \\ \quad +\int_0^t (t-s)^{\alpha-1}\mathcal{H}_\alpha(t-s)\left[Eg(s,x_s)+\int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right]ds, \\ \quad +\int_{\delta(0)}^{\delta(t)} \left(t-\varepsilon(s)\right)^{\alpha-1}\mathcal{H}_\alpha\left(t-\varepsilon(s)\right)B\dot{\varepsilon}(s)u(s)ds, \qquad t\in[0,t_1], \\ \mathcal{K}_\alpha(t)\left[\varphi(0)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(0)-g(0,x_0)\right]+g(t,x_t)+\sum_{r=1}^{m}\Delta x(t_r)\mathcal{K}_\alpha(t-t_r) \\ +\int_0^t (t-s)^{\alpha-1}\mathcal{H}_\alpha(t-s)\left[Eg(s,x_s)+\int_{s-\eta(x(s))}^{s} Gx(\zeta)d\zeta\right]ds \\ +\int_{\delta(0)}^{\delta(t)} \left(t-\varepsilon(s)\right)^{\alpha-1}\mathcal{H}_\alpha\left(t-\varepsilon(s)\right)B\dot{\varepsilon}(s)u(s)ds, t\in(t_r,t_{r+1}], r=1,2,3,\dots,m \end{cases}$$

$$x_0(\theta)=\varphi(\theta)+\left(\int_0^a \hbar(\omega,x_\omega)\,d\omega\right)(\theta), \qquad \theta\in[-\hat{a},0],$$

where $\varepsilon(s):[\delta(0),\delta(a)]\to J$ is the time lead function, such that $\varepsilon(\delta(t))=\delta(\varepsilon(t))=t,$ for $t\in J$.

To study the controllability of the control delay system, we need to redefine the linear operator that defined in the hypothesis $\mathcal{A}_6$ as follows:

$$Wu=\int_0^a \left(a-\varepsilon(s)\right)^{\alpha-1}\mathcal{H}_\alpha\left(a-\varepsilon(s)\right)B\dot{\varepsilon}(s)u(s)ds.$$

By the same technique used in this article, one can show that the control delay system is completely controllable.

5.Conclusion

This paper has investigated the completely controllable nonlinear impulsive integro-differential fractional nonlocal control system with state-dependent delay in a Banach space. The mild solutions of the control system(S_1) were obtained by using fractional calculus, the Laplace transform, semigroup theory, and probability density function. With the use of the control function has been constructed, together with a compact strongly continuous semigroup $\{T(t),t\geq 0\}$ has helped us to establish sufficient conditions for controllability of the control system (S_1) via Krasnoselskii fixed point theorem. Finally, an example has been given to illustrate the effectiveness of the main results.

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