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The Dynamics of a Prey-Predator Model with Infectious Disease in Prey: Role of Media Coverage

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Abstract:
In this paper, an eco-epidemiological model with media coverage effect is proposed and studied. A prey-predator model with modified Leslie-Gower and functional response is studied. An SIS-type of disease in prey is considered. The existence, uniqueness and boundedness of the solution of the model are discussed. The local and global stability of this system are carried out. The conditions for the persistence of all species are established. The local bifurcation in the model is studied. Finally, numerical simulations are conducted to illustrate the analytical results.

Keywords: Media coverage, Leslie-Gower, disease, stability, persistence, bifurcation.

ديناميكية نموذج الفريسة والمفترس مع مرض معد في الفريسة: دور التغطية الاعلامية

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الخلاصه
في هذا البحث تم اقتراح ودراسة نموذج بيئي وبائي له تأثير تغطية اعلامية. مع استخدام لزلي كورالمطورة يعتبر نوع المرض من نوع SIS في الفريسة. تم مناقشة وجود ووحدانية وقيود حل النموذج المقترح، وتم تحليل الاستقرار المحلي والشامل لهذا النظام. تم وضع شروط بقاء جميع الأنواع. تمت دراسة التفرع المحلي في النموذج. واخيرا تم اجراء المحاكاة العددية لدعم النتائج التحليلية.

Introduction

The utilization of mathematical models for studying and understanding the spread and controlling infectious diseases has become a highly important tool. The scientists extensively studied the dynamics of ecological models in the existence of infectious diseases and provided important insights into complex biological processes. The study of the spread of infectious diseases within populations of ecological systems is resulting in a branch called eco-epidemiology.

This subject is rapidly growing as a branch of theoretical ecology ([1]-[3]), Later on, several researchers were proposed and studied eco-epidemiological models involving many biological factors, see ([4-13])

The modified Leslie–Gower prey-predator model, which is proposed by Leslie and Gower [14] and modified by May [15], is considered by many scientists [16-19]). In the modified

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Leslie–Gower model, the predator acts as a generalist predator because it avoids extinction by using an alternative source of food. Although in case of a severe scarcity of prey the predator population growth may still be limited by the fact that their favorite food is not available in abundance, some predator species can switch to another available food in the environment. The impact of media coverage is one of the most key factors to establish the prevention and control measure that affect the spread of infectious disease. The role of media coverage of disease outbreaks is therefore crucial and should be given prominence in the study of disease dynamics [20].

Liu and Cui [21] studied a container model that characterized the spread and control of infectious disease under the influence of media coverage. Tchuenche and Bauch [22] proposed and studied a susceptible-infected-hospitalized-recovered model with vital dynamics, where media coverage of disease incidence and prevalence can influence people to reduce their contact rates. Li and Cui [23] introduced constant and pulse vaccines in media coverage for SIS disease models. In recent years, attempts have been made to develop mathematical models for the transmission dynamics of infectious diseases within the eco-epidemiological model. Alwan and Abdul Satar [24] proposed and studied a prey-predator model having a disease in predator species and involving media coverage. They used it for describing the predation process as a Lotka-Volterra type of functional response.

In this paper, an SIS-type of disease in prey is considered, so that a modified Leslie-Gower prey-predator model is proposed and the effect of media coverage on the dynamics of a proposed eco-epidemiological model is studied. Moreover, Lotka-Volterra type of functional response is used to describe the predation process. The organization of this paper is given as follows. Section (2) deals with the model formulation. Section (3) determines the equilibrium points (EPs) and analyzes their local stability. The global stability for the EPs is studied with the help of the Lyapunov method (LM) in section (4). While, the bifurcation analysis of the system is investigated in section (5). Section (6) deals with the numerical simulation of the system. Finally, the discussion and conclusions are addressed in section (7).

2. The mathematical model

In this section, the effect of media coverage on a modified Leslie–Gower prey-predator model is formulated mathematically. An infectious disease of SIS type in prey species is included in the model. It is assumed that the prey is consumed by the predator according to Lotka-Volterra type of functional responses. Now, in order to represent the dynamics of such a real-world system, the following hypotheses are adopted.

Let the variables $S(T), I(T),$ and $Y(T)$ represent the densities at time T for the susceptible prey, infected prey, and predator, respectively. It is assumed that $S(T)$ grows logistically with $r_1 > 0$ as an intrinsic growth rate, while $I(T)$ cannot reproduce due to the disease, instead of that, it competes with the susceptible prey for environment carrying capacity $K_1 > 0$. However, the predator $Y(T)$ grows logistically with $r_2 > 0$ as a growth rate by sexual reproduction and carrying capacity, depending on the prey and given by $K_2 + S + I$, where $K_2 > 0$ represents a residual loss in predator population. The predator species $Y(T)$ consumes both the prey species $S(T)$ and $I(T)$ using Lotka-Volterra type of functional responses with maximum attack rates of $a_1 > 0$ and $a_2 > 0$, respectively. The term $\left(b_1 - \frac{nI}{m+I}\right)$ represents the infection rate due to the direct contact between $S(T)$ and $I(T)$, where $b_1 > 0$ is the contact rate before media coverage alert, while $\frac{nI}{m+I}$ represents the reduced value in the contact due to media coverage alert, so that $n > 0$ is the maximum transmission rate under the media coverage and $m > 0$ is the non-response rate of individual to the media coverage. Furthermore, since it is well known that the media coverage cannot prevent the spreading of the disease completely, then from now onward we take $b_1 \geq n$. Also, the infected individuals may recover with a rate of $\alpha > 0$. The disease-caused death rate of infected individuals is

given by $d_1 > 0$, while the parameter $p > 0$ is the maximum value in which per capita reduction rate of predator species can be attained due to intra-specific. Finally, the infected prey causes predator death due to disease when feeding on it with a probability $\varepsilon \in (0, 1)$.
According to the above hypotheses, the dynamics of the above-described system, that is consisting of a diseased prey-predator system incorporating the media coverage, can be represented by the following set of differential equations:

$$\begin{aligned}
\frac{dS}{dT} &= r_1\left(1 - \frac{S+I}{K_1}\right)S - \left(b_1 - \frac{nI}{m+I}\right)SI + \alpha I - a_1 SY,\\
\frac{dI}{dT} &= \left(b_1 - \frac{nI}{m+I}\right)SI - (\alpha + d_1)I - a_2 IY,\\
\frac{dY}{dT} &= r_2 Y\left(1 - \frac{pY}{K_2+S+I}\right) - \varepsilon a_2 IY,
\end{aligned} \tag{1}$$

with $S(0) \geq 0, I(0) \geq 0$ and $Y(0) \geq 0$ as an initial condition. Therefore, the system (1) has the domain $\mathbb{R}_+^3 = \{(S, I, Y) \in \mathbb{R}^3 | X \geq 0, S \geq 0, I \geq 0\}$.
Clearly, the system (1) contains a C^1 functions; therefore, these functions are Lipschitzain. Hence, the solution of the system (1) exists and is unique. Further, the uniformly bounded of the solutions of the system (1) is proved in the following theorem.
Theorem (1): The system (1) has uniformly bounded solutions.
Proof. Define $W_1 = S + I$, then $\frac{dW_1}{dT}$ can be written as

$$\frac{dW_1}{dT} \leq r_1\left(1 - \frac{S+I}{K_1}\right)S - d_1 I \leq 2r_1 S - \frac{r_1 S^2}{K_1} - \mu W_1,$$

where $\mu = \min\{r_1, d_1\}$. Then, direct computation shows that for T goes to ∞, we have $W_1 \leq \ell_1$, where $\ell_1 = \frac{r_1 K_1}{\mu}$.
Since the third equation of predator is a logistic growth equation, then it is easy to verify that $Y \leq \frac{r_2(K_2+\ell_1)}{4p} = \ell_2$. Therefore, all the variables are bounded.

3. Existence of EPs and Their Local Stability Analysis

The existence of EPs of the system (1) and their local stability analysis are discussed. The existence conditions for each of these EPs are established.
The trivial EP, represented by $P_0 = (0,0,0)$, always exists.
The first axial EP, represented by $P_1 = (K_1, 0,0)$, always exists as the susceptible prey population grows to carrying capacity in the absence of predation.
The second axial EP, represented by $P_2 = \left(0,0, \frac{K_2}{p}\right)$, always exists as the predator population that growth logistically grows to carrying capacity supplied by the environment in the absence of preferred prey.
The predator-free EP is denoted by $P_3 = (\bar{S}, \bar{I}, 0)$, where

$$\bar{S} = \frac{(m+\bar{I})(\alpha+d_1)}{[b_1 m + (b_1 - n)\bar{I}]}, \tag{2a}$$

where $\bar{I}$ represents a positive root for the third order polynomial equation:

$$B_1 I^3 + B_2 I^2 + B_3 I + B_4 = 0, \tag{2b}$$

where

$$\begin{aligned}
B_1 &= -(b_1 - n)[r_1(\alpha + d_1) + K_1 d_1 (b_1 - n)] < 0,\\
B_2 &= [r_1 K_1 (\alpha + d_1)(b_1 - n) - r_1(\alpha + d_1)^2 - r_1 m(\alpha + d_1)(b_1 - n) -\\
&\quad r_1(\alpha + d_1)b_1 - 2K_1 d_1 b_1 m(b_1 - n)],\\
B_3 &= [r_1 K_1 m(b_1 - n) + r_1 K_1 (\alpha + d_1)b_1 m - 2r_1 m(\alpha + d_1)^2 -\\
&\quad r_1 m^2 b_1(\alpha + d_1) - K_1 d_1 m^2 {b_1}^2],\\
B_4 &= [r_1 m(\alpha + d_1)(K_1 b_1 - m(\alpha + d_1))].
\end{aligned}$$

Therefore, $P_3 = (\bar{S}, \bar{I}, 0)$ exists if

$B_4 > 0$ and $B_3 > 0$.
OR
$B_4 > 0$ and $B_2 < 0$.

(2c) The infected prey-free EP is represented by $P_4 = (\tilde{S}, 0, \tilde{Y})$, where
$\tilde{S} = \frac{K_1(r_1 p - a_1 K_2)}{(r_1 p + a_1 K_2)}$, and $\tilde{Y} = \frac{K_2 + \tilde{S}}{p}$,

(3a) which exists provided that

$$r_1 p > a_1 K_2. \quad (3b)$$

The coexistence or positive EP is denoted by $P_5 = (S^*, I^*, Y^*)$, where
$Y^* = \frac{(r_2 - \varepsilon a_2 I^*)(K_2 + S^* + I^*)}{p}$.

(4) where the point (S^*, I^*) represents the positive intersection point of the following two isoclines:

$$\begin{aligned} g_1(S,I) &= r_1\left(1 - \frac{S+I}{K_1}\right)S - \left(b_1 - \frac{nI}{m+I}\right)SI + \alpha I - a_1 S\left(\frac{(r_2 - \varepsilon a_2 I)(K_2+S+I)}{p}\right) = 0 \\ g_2(S,I) &= \left(b_1 - n\frac{I}{m+I}\right)S - (\alpha + d_1) - a_2\left(\frac{(r_2 - \varepsilon a_2 I)(K_2+S+I)}{p}\right) = 0 \end{aligned} \quad (5a)$$

Obviously, as $I \to 0$, then the isoclines become

$$g_1(S,I) = r_1\left(1 - \frac{S}{K_1}\right)S - a_1 S\frac{r_2(K_2+S)}{p} = 0.$$

$$g_2(S,I) = b_1 S - (\alpha + d_1) - a_2\frac{r_2(K_2+S)}{p} = 0.$$

(5b) Therefore, $g_1(S)$ intersects the $S-$axis at the positive point $s_1 = \frac{K_1(pr_1 - a_1 r_2 K_2)}{pr_1 + a_1 r_2 K_1}$; however, $g_2(S)$ intersects the $S-$axis at the positive point $s_2 = \frac{p(\alpha+d_1) + a_2 r_2 K_2}{b_1 p - a_2 r_2}$. Hence, the two isoclines (5a) have a unique positive intersection point and then P_5 exists uniquely in the interior of $\mathbb{R}^3_+$ if

$$\frac{p}{r_2} > \max\left\{\frac{a_1 K_2}{r_1}, \frac{a_2}{b_1}\right\}. \quad (5c)$$

$$s_2 > s_1. \quad (5d)$$

$$\frac{dI}{dS} = -\frac{\partial g_1/\partial S}{\partial g_1/\partial I} > 0. \quad (5e)$$

$$\frac{dI}{dS} = -\frac{\partial g_2/\partial S}{\partial g_2/\partial I} < 0. \quad (5f)$$

Now, to establish the local stability, the Jacobian matrix (JM) of system (1) about arbitrary point (S, I, Y) is

$$J(X,S,I) = \left[\rho_{ij}\right]_{3\times3}, \quad (6)$$

where $\rho_{11} = r_1 - \frac{2r_1 S + r_1 I}{K_1} - \left(b_1 - \frac{nI}{m+I}\right)I - a_1 Y$, $\rho_{12} = -\frac{r_1 S}{K_1} - b_1 S + \frac{2nSI}{m+I} - \frac{nSI^2}{(m+I)^2} + \alpha$, $\rho_{13} = -a_1 S$, $\rho_{21} = \left(b_1 - \frac{nI}{m+I}\right)I$, $\rho_{22} = b_1 S - \frac{2nSI}{m+I} + \frac{nSI^2}{(m+I)^2} - (\alpha + d_1) - a_2 Y$, $\rho_{23} = -a_2 I$, $\rho_{31} = \frac{r_2 p Y^2}{(K_2+S+I)^2}$, $\rho_{32} = \frac{r_2 p Y^2}{(K_2+S+I)^2} - \varepsilon a_2 Y$, $\rho_{33} = r_2 - \frac{2r_2 pY}{(K_2+S+I)} - \varepsilon a_2 I$.

It is clear that the system (1) has JM at trivial EP, $P_0 = (0,0,0)$ specified by

$$J(P_0) = \begin{bmatrix} r_1 & \alpha & 0 \\ 0 & -(\alpha + d_1) & 0 \\ 0 & 0 & r_2 \end{bmatrix},$$

(7a) Therefore, the eigenvalues of $J(P_0)$ are:

$$\lambda_{01} = r_1,\ \lambda_{02} = -(\alpha + d_1),\ \lambda_{03} = r_2. \quad (7b)$$

Hence, the trivial EP is unstable (saddle point).

The JM of the system (1) at the first axial EP, $P_1 = (K_1, 0,0)$ is

$$J(P_1)=\begin{bmatrix}-r_1 & -r_1-b_1K_1+\alpha & -a_1K_1\\ 0 & b_1K_1-(\alpha+d_1) & 0\\ 0 & 0 & r_2\end{bmatrix},$$

(8a) Therefore,, the eigenvalues of $J(P_1)$ are given by

$$\lambda_{11}=-r_1<0,\ \lambda_{12}=b_1K_1-(\alpha+d_1),\ \lambda_{13}=r_2>0. \quad (8b)$$

Hence, the first axial EP, $P_1=(K_1,0,0)$ is unstable (saddle point).

The JM of the system (1) at the second axial EP, $P_2=\left(0,0,\frac{K_2}{p}\right)$ is

$$J(P_2)=\begin{bmatrix}r_1-\frac{a_1K_2}{p} & \alpha & 0\\ 0 & -(\alpha+d_1)-\frac{a_2K_2}{p} & 0\\ \frac{r_2}{p} & \frac{r_2-\varepsilon a_2K_2}{p} & -r_2\end{bmatrix}. \quad (9a)$$

Clearly, the eigenvalues of $J(P_2)$ are given by

$$\lambda_{21}=r_1-\frac{a_1K_2}{p},\ \lambda_{22}=-(\alpha+d_1)-\frac{a_2K_2}{p},\text{ and }\lambda_{23}=-r_2. \quad (9b)$$

Hence, all the eigenvalues are negative, and the second axial EP is LAS provided that

$$r_1<\frac{a_1K_2}{p}. \quad (9c)$$

Now the JM of the system (1) at the predator-free EP, $P_3=(\bar{S},\bar{I},0)$, can be written as:

$$J(P_3)=\left[a_{ij}\right]_{3\times3}, \quad (10a)$$

where $a_{11}=r_1-\frac{2r_1\bar{S}+r_1\bar{I}}{K_1}-\left(b_1-\frac{n\bar{I}}{m+\bar{I}}\right)\bar{I}$, $a_{12}=-\frac{r_1\bar{S}}{K_1}-b_1\bar{S}+\frac{2n\bar{S}\bar{I}}{m+\bar{I}}-\frac{ns\bar{I}^2}{(m+\bar{I})^2}+\alpha$, $a_{13}=-a_1\bar{S}$, $a_{21}=\left(b_1-\frac{n\bar{I}}{m+\bar{I}}\right)\bar{I}$, $a_{22}=b_1\bar{S}-\frac{2n\bar{S}\bar{I}}{m+\bar{I}}+\frac{ns\bar{I}^2}{(m+\bar{I})^2}-(\alpha+d_1)$, $a_{23}=-a_2\bar{I}$, $a_{31}=a_{32}=0$, and $a_{33}=r_2-\varepsilon a_2\bar{I}$.

Clearly, one of the eigenvalues is $\lambda_{33}=r_2-\varepsilon a_2\bar{I}$ and the other two eigenvalues are the roots of the equation:

$$\lambda_3^2-T_2\lambda_3+D_2=0,$$

(10b) where $T_2=a_{11}+a_{22}$ and $D_2=(a_{11}a_{22}-a_{12}a_{21})$. Note that the direct computation gives that the roots (eigenvalues) of the equation (10b) can be written as

$$\lambda_{31}=\frac{T_2}{2}+\frac{1}{2}\sqrt{{T_2}^2-4D_2};\ \lambda_{32}=\frac{T_2}{2}-\frac{1}{2}\sqrt{{T_2}^2-4D_2} \quad (10c)$$

Hence, all the eigenvalues of $J(P_3)$ have negative real parts and hence P_3 is LAS if and only if

$$r_2<\varepsilon a_2\bar{I}, \quad (11a)$$

$$\frac{2n\bar{S}\bar{I}}{m+\bar{I}}+\alpha<\frac{r_1\bar{S}}{K_1}+b_1\bar{S}+\frac{ns\bar{I}^2}{(m+\bar{I})^2}, \quad (11b)$$

$$r_1<\frac{2r_1\bar{S}+r_1\bar{I}}{K_1}+\left(b_1-\frac{n\bar{I}}{m+\bar{I}}\right)\bar{I}, \quad (11c)$$

$$b_1\bar{S}+\frac{ns\bar{I}^2}{(m+\bar{I})^2}<\frac{2n\bar{S}\bar{I}}{m+\bar{I}}+(\alpha+d_1). \quad (11d)$$

Now the JM of the system (1) at the infected prey-free EP, $P_4=(\tilde{S},0,\tilde{Y})$, can be written as:

$$J(P_4)=\begin{bmatrix}r_1-\frac{2r_1\tilde{S}}{K_1}-a_1\tilde{Y} & -\frac{r_1\tilde{S}}{K_1}-b_1\tilde{S}+\alpha & -a_1\tilde{S}\\ 0 & b_1\tilde{S}-(\alpha+d_1)-a_2\tilde{Y} & 0\\ \frac{r_2}{p} & \frac{r_2}{p}-\varepsilon a_2\tilde{Y}^2 & -r_2\end{bmatrix}. \quad (12a)$$

Clearly, one of the eigenvalues is $\lambda_{42}=b_1\tilde{S}-(\alpha+d_1)-a_2\tilde{Y}$ and the other two eigenvalues are given by:

$$\lambda_{41}=\frac{T_3}{2}+\frac{1}{2}\sqrt{{T_3}^2-4D_3};\ \lambda_{43}=\frac{T_3}{2}-\frac{1}{2}\sqrt{{T_3}^2-4D_3}, \quad (12b)$$

where $T_3=r_1-\frac{2r_1\tilde{S}}{K_1}-a_1\tilde{Y}-r_2$, and $D_3=-\left(r_1-\frac{2r_1\tilde{S}}{K_1}-a_1\tilde{Y}\right)r_2+a_1\tilde{S}\frac{r_2}{p}$.

Therefore, all the eigenvalues have negative real parts, and then P_4 is LAS if and only if

$$b_1\tilde{S} < (\alpha + d_1) + a_2\tilde{Y}, \tag{13a}$$

$$r_1 < \frac{2r_1\tilde{S}}{K_1} + a_1\tilde{Y}. \tag{13b}$$

Finally the JM evaluated at the positive EP, P_5, is given by:

$$J(P_5) = [e_{ij}]_{3\times3} \tag{14a}$$

where $e_{11} = r_1 - \frac{2r_1S^* + r_1I^*}{K_1} - \left(b_1 - \frac{nI^*}{m+I^*}\right)I^* - a_1Y^*$,

$e_{12} = -\frac{r_1S^*}{K_1} - b_1S^* + \frac{2nS^*I^*}{m+I^*} - \frac{nsI^{*2}}{(m+I^*)^2} + \alpha$, $e_{13} = -a_1S^*$,

$e_{21} = \left(b_1 - \frac{nI^*}{m+I^*}\right)I^*$, $e_{22} = b_1S^* - \frac{2nS^*I^*}{m+I^*} + \frac{nsI^{*2}}{(m+I^*)^2} - (\alpha + d_1) - a_2Y^*$,

$e_{23} = -a_2I^*$, $e_{31} = \frac{r_2pY^{*2}}{(K_2+S^*+I^*)^2}$, $e_{32} = \frac{r_2pY^{*2}}{(K_2+S^*+I^*)^2} - \varepsilon a_2Y^*$.

$e_{33} = r_2 - \frac{2r_2pY^*}{(K_2+S^*+I^*)} - \varepsilon a_2I^*$.

Then the characteristic equation of $J(P_5)$ can be written as:

$$\lambda_5^3 + M_1\lambda_5^2 + M_2\lambda_5 + M_3 = 0, \tag{14b}$$

where $M_1 = -(e_{11} + e_{22} + e_{33}) = -\Gamma_1$,

$M_2 = (e_{11}e_{22} - e_{12}e_{21}) + (e_{11}e_{33} - e_{13}e_{31}) + (e_{22}e_{33} - e_{23}e_{32})$
$= \Gamma_2 + \Gamma_3 + \Gamma_4$,

$M_3 = -[e_{33}(e_{11}e_{22} - e_{12}e_{21}) + e_{23}(e_{12}e_{31} - e_{11}e_{32}) + e_{13}(e_{21}e_{32} - e_{22}e_{31})]$
$= -[e_{33}\Gamma_2 + e_{23}\Gamma_5 + e_{13}\Gamma_6]$,

with

$$\Delta = M_1M_2 - M_3 = -\Gamma_7\Gamma_2 - \Gamma_8\Gamma_3 - \Gamma_9\Gamma_4 - 2\Gamma_{11} + \Gamma_{10},$$

where $\Gamma_1 = (e_{11} + e_{22} + e_{33})$; $\Gamma_2 = (e_{11}e_{22} + e_{12}e_{21})$; $\Gamma_3 = (e_{11}e_{33} + e_{13}e_{31})$
$\Gamma_4 = (e_{22}e_{33} - e_{23}e_{32})$; $\Gamma_5 = (e_{12}e_{31} - e_{11}e_{32})$; $\Gamma_6 = (e_{21}e_{32} - e_{22}e_{31})$
$\Gamma_7 = (e_{11} + e_{22})$; $\Gamma_8 = (e_{11} + e_{33})$; $\Gamma_9 = (e_{22} + e_{33})$
$\Gamma_{10} = e_{12}e_{23}e_{31} + e_{13}e_{21}e_{32}$; $\Gamma_{11} = e_{11}e_{22}e_{33}$

Accordingly, the local stability of the positive EP can be given in the following theorem.

Theorem (2). The positive EP of the system (1) is locally LAS provided that the following conditions hold:

$$r_1 < \frac{2r_1S^* + r_1I^*}{K_1} + \left(b_1 - \frac{nI^*}{m+I^*}\right)I^* + a_1Y^*, \tag{15a}$$

$$b_1S^* + \frac{nsI^{*2}}{(m+I^*)^2} < \frac{2nS^*I^*}{m+I^*} + (\alpha + d_1) + a_2Y^*, \tag{15b}$$

$$r_2 < \frac{2r_2pY^*}{(K_2+S^*+I^*)} + \varepsilon a_2I^*, \tag{15c}$$

$$\Gamma_{10} < 2\Gamma_{11}. \tag{15d}$$

Proof. According to the Routh- Hurwitz criterion, all roots of the characteristic equation given by Eq. (14b) have negative real parts roots, if and only if $M_1 > 0$, $M_3 > 0$, and $\Delta = M_1M_2 - M_3 > 0$.

Straightforward computation shows that the conditions (15a)-(15d) satisfy the Routh-Hurwitz criterion conditions, and hence all the eigenvalues of the Eq. (14b) have negative real parts. Then,, the positive EP is LAS.

The persistence of the system (1) is studied. It is well known that the biological system is persistent if and only if all its species are persistent all the time. Now, according to the system (1), if the predator individuals disappear, then

$$\begin{aligned} \frac{dS}{dT} &= r_1\left(1-\frac{S+I}{K_1}\right)S-\left(b_1-\frac{nI}{m+I}\right)SI+\alpha I \ = g_1(S,I) \\ \frac{dI}{dT} &= \left(b_1-n\frac{I}{m+I}\right)SI-(\alpha+d_1)I = g_2(S,I) \end{aligned} \tag{16}$$

Clearly, subsystem (16) is a 2D space that has a unique positive point given by $(\bar{S},\bar{I})$, which are given by Eq. (2a), and exists uniquely in the SI −plane under the condition (2c). By using Poincare Bendixon theorem, the solution of system (16) approaches either to EP $(\bar{S},\bar{I})$ or else to the periodic dynamics. Now, by using the continuous function $u(S,I)=\frac{1}{SI}$, we obtain

$$\nabla = \frac{\partial(ug_1)}{\partial S}+\frac{\partial(ug_2)}{\partial I} = -\frac{r_1}{K_1 I}-\frac{\alpha}{S^2}-\frac{nm}{(m+I)^2}<0$$

Therefore, according to the Dulac criterion, there is no periodic dynamics in the interior of the positive quadrant of SI −plane. Hence, using Poincare Bendixon theorem, the positive EP of the subsystem (16) is globally asymptotically stable (GAS) whenever it exists. Then the system (1) has no periodic dynamics in the boundary SI −plane.

The following theorem explains the conditions that guarantee the persistence of the system.

Theorem 3. The system (1) is uniformly persistent if

$$\frac{a_1K_2}{p}<r_1, \tag{17a}$$

$$\varepsilon a_2\bar{I}<r_2, \tag{17b}$$

$$\left.\begin{array}{c} (\alpha+d_1)+a_2\tilde{Y}<b_1\tilde{S} \\ or \\ \frac{2r_1\tilde{S}}{K_1}+a_1\tilde{Y}<r_1 \end{array}\right\}.$$

(17c) **Proof:** Suppose that σ is a point in the interior of $\mathbb{R}^3_+$ and $\varphi(\sigma)$ is the orbit through σ, and let $\Upsilon(\sigma)$ is the omega limit set of $\varphi(\sigma)$. Further, since $\Upsilon(\sigma)$ is bounded, due to the boundedness of the system (1), then we first show that $P_0 \notin \Upsilon(\sigma)$.

Assume the contrary, since P_0 is a saddle point, then by Butler-McGhee lemma [25], there is at least one other point σ_1 such that $\sigma_1 \in \omega^s(P_0)\cap\Upsilon(\sigma)$, where $\omega^s(P_0)$ is the stable manifold of P_0.

Now, since the stable manifold of P_0 is given by Y −direction and the entire orbit through σ_1, say $\varphi(\sigma_1)$, is contained in $\Upsilon(\sigma)$, then we obtain a contradiction to the boundedness of $\Upsilon(\sigma)$, due to the containment of an unbounded positive axis in it. This shows that $P_0 \notin \Upsilon(\sigma)$.

Now, to proof that $P_1 \notin \Upsilon(\sigma)$, we assume the converse. Since P_1 is a saddle point, then by Butler-McGhee lemma, there is another point, say σ_2, so that $\sigma_2 \in \omega^s(P_1)\cap\Upsilon(\sigma)$. Now, since the stable manifold of P_1 is given by S −direction and the entire orbit through σ_2, say $\gamma(\sigma_2)$, is contained in $\Upsilon(\sigma)$, hence we obtain a contradiction to the boundedness of $\Upsilon(\sigma)$, due to the containment of an unbounded positive axis in it. This shows that $P_1 \notin \Upsilon(\sigma)$.

Now, since the points P_2, P_3 and P_4 are saddle points under the conditions (17a), (17b), and (17c), respectively. Then by using similar argument as that given in the first part of the proof, we obtain that P_2, P_3 and $P_4 \notin \Upsilon(\sigma)$.

4. Global Stability Analysis

In this section, the global stability (GS) is studied for all LS Eps. Lyapunov method is used to investigate the GS or specify the basin of attraction of each EP.

Theorem 4. Assume that the second axial EP, $P_2=(0,0,\bar{\bar{Y}})$, of the system (1) is LAS in $\mathbb{R}^3_+$, and:

$$r_1<\frac{r_2p\bar{\bar{Y}}^2}{K_2(K_2+\ell_1)}, \tag{18a}$$

$$\frac{r_2p\bar{\bar{Y}}}{{K_2}^2}<\min\{a_1, a_2(1+\varepsilon)\}, \tag{18b}$$

$$\varepsilon a_2\bar{\bar{Y}}<d_1+\frac{r_2p\bar{\bar{Y}}^2}{K_2(K_2+\ell_1)}, \tag{18c}$$

where ℓ_1 is the upper bound of $S+I$. Then, it is GAS in $\mathbb{R}^3_+$.

Proof: Consider the following function:

$L_1 = S + I + \left(Y - \bar{\bar{Y}} - \ln\frac{Y}{\bar{\bar{Y}}}\right)$.

Then, L_1 is a C^1 real valued function, which is a positive definite. Now, the derivative $\frac{dL_1}{dT}$ can be calculated as:

$$\frac{dL_1}{dT} \le r_1 S - a_1 SY - a_2 IY - d_1 I - \frac{r_2 p}{(K_2+S+I)}\left(Y-\bar{\bar{Y}}\right)^2 + \frac{r_2 p\bar{\bar{Y}}S\left(Y-\bar{\bar{Y}}\right)}{K_2(K_2+S+I)} + \frac{r_2 p\bar{\bar{Y}}I\left(Y-\bar{\bar{Y}}\right)}{K_2(K_2+S+I)} - \varepsilon a_2 IY + \varepsilon a_2 \bar{\bar{Y}} I.$$

Hence,

$$\frac{dL_1}{dT} \le \left[r_1 - \frac{r_2 p\bar{\bar{Y}}^2}{K_2(K_2+S+I)}\right]S - \frac{r_2 p}{(K_2+S+I)}\left(Y-\bar{\bar{Y}}\right)^2 - \left[a_1 - \frac{r_2 p\bar{\bar{Y}}}{K_2(K_2+S+I)}\right]SY - \left[a_2(1+\varepsilon) - \frac{r_2 p\bar{\bar{Y}}}{K_2(K_2+S+I)}\right]IY - \left[d_1 + \frac{r_2 p\bar{\bar{Y}}^2}{K_2(K_2+S+I)} - \varepsilon a_2\bar{\bar{Y}}\right]I.$$

Therefore, by using the above set of conditions, it is obtained that:

$$\frac{dL_1}{dT} < \left[r_1 - \frac{r_2 p\bar{\bar{Y}}^2}{K_2(K_2+S+I)}\right]S - \frac{r_2 p}{(K_2+S+I)}\left(Y-\bar{\bar{Y}}\right)^2 - \left[d_1 + \frac{r_2 p\bar{\bar{Y}}^2}{K_2(K_2+S+I)} - \varepsilon a_2\bar{\bar{Y}}\right]I.$$

Obviously, $\frac{dL_1}{dT}$ is a negative definite, and since L_1 is radially unbounded function, then P_2 is GAS.

Theorem 5. Assume that the predator-free EP, P_3, of the system (1) is LAS in $\mathbb{R}^3_+$, and:

$$\left[\frac{r_1}{K_1} - \frac{nm\bar{I}}{A\bar{A}} - \frac{\alpha}{S}\right]^2 < 4\left[\frac{r_1}{K_1} + \frac{\alpha\bar{I}}{S\bar{S}}\right]\frac{nm\bar{S}}{A\bar{A}}, \quad (19a)$$

$$[a_1\bar{S} + a_2\bar{I} + r_2]\ell_2 < \left[\sqrt{\frac{r_1}{K_1} + \frac{\alpha\bar{I}}{S\bar{S}}}(S-\bar{S}) + \sqrt{\frac{nm\bar{S}}{A\bar{A}}}(I-\bar{I})\right]^2. \quad (19b)$$

where all the symbols are defined in the proof. Then it is GAS in $\mathbb{R}^3_+$.

Proof: Consider the following function:

$L_2 = \left(S - \bar{S} - \bar{S}\ln\frac{S}{\bar{S}}\right) + \left(I - \bar{I} - \bar{I}\ln\frac{I}{\bar{I}}\right) + Y$

Clearly, $L_2 : \mathbb{R}^3_+ \to \mathbb{R}$ is C^1 function that is positive definite. Then we have

$$\frac{dL_2}{dT} \le -\left[\frac{r_1}{K_1} + \frac{\alpha\bar{I}}{S\bar{S}}\right](S-\bar{S})^2 - \left[\frac{r_1}{K_1} - \frac{nm\bar{I}}{A\bar{A}} - \frac{\alpha}{S}\right](S-\bar{S})(I-\bar{I}) - \frac{nm\bar{S}}{A\bar{A}}(I-\bar{I})^2 + [a_1\bar{S} + a_2\bar{I} + r_2]\ell_2,$$

where $A = m + I$, $\bar{A} = m + \bar{I}$, and ℓ_2 is the upper bound of Y. Therefore, by using the condition (19a), it is obtained that

$$\frac{dL_2}{dT} \le -\left[\sqrt{\frac{r_1}{K_1} + \frac{\alpha\bar{I}}{S\bar{S}}}(S-\bar{S}) + \sqrt{\frac{nm\bar{S}}{A\bar{A}}}(I-\bar{I})\right]^2 + [a_1\bar{S} + a_2\bar{I} + r_2]\ell_2.$$

Obviously, under the condition (19b), we have $\frac{dL_2}{dT}$ is negative definite, and since L_2 is radially unbounded function, then P_3 is GAS.

Theorem 6. Assume that the infected prey-free EP, P_4, of the system (1) is LAS in $\mathbb{R}^3_+$, and:

$$\left[a_1 - \frac{r_2}{K_2+S+I}\right]^2 < 4\left(\frac{r_1}{K_1}\right)\left(\frac{r_2 p}{K_2+\ell_1}\right), \quad (20a)$$

$$\frac{r_2}{K_2} < a_2(1+\varepsilon), \quad (20b)$$

$$(b_1+n)\tilde{S}+\frac{r_1\tilde{S}}{K_1}+\varepsilon a_2\tilde{Y}<\frac{r_2(K_2+\tilde{S})}{p(K_2+\ell_1)}+d_1. \tag{20c}$$

Then, it is GAS in $\mathbb{R}^3_+$.

Proof: Consider the following function:

$L_3=\left(S-\tilde{S}-\tilde{S}\ln\frac{S}{\tilde{S}}\right)+I+\left(Y-\tilde{Y}-\tilde{Y}\ln\frac{Y}{\tilde{Y}}\right)$

Clearly, $L_3:\mathbb{R}^3_+\to\mathbb{R}$ is C^1 that is a positive definite real valued function.

$$\frac{dL_3}{dT}\le-\frac{r_1}{K_1}\left(S-\tilde{S}\right)^2-\left[a_1-\frac{r_2}{K_2+S+I}\right]\left(S-\tilde{S}\right)\left(Y-\tilde{Y}\right)$$
$$-\frac{r_2p}{K_2+\ell_1}\left(Y-\tilde{Y}\right)^2-\left[a_2(\varepsilon+1)-\frac{r_2}{K_2}\right]IY$$
$$-\left[\frac{r_2(K_2+\tilde{S})}{p(K_2+\ell_1)}+d_1-(b_1+n)\tilde{S}-\frac{r_1\tilde{S}}{K_1}-\varepsilon a_2\tilde{Y}\right]I,$$

where ℓ_1 is the upper bound for $S+I$. Using conditions (20a)-(20b), its obtained that:

$$\frac{dL_3}{dT}<-\left[\sqrt{\frac{r_1}{K_1}}\left(S-\tilde{S}\right)+\sqrt{\frac{r_2p}{K_2+\ell_1}}\left(Y-\tilde{Y}\right)\right]^2$$
$$-\left[\frac{r_2(K_2+\tilde{S})}{p(K_2+\ell_1)}+d_1-(b_1+n)\tilde{S}-\frac{r_1\tilde{S}}{K_1}-\varepsilon a_2\tilde{Y}\right]I.$$

Obviously, under the condition (20c), it is obtained that $\frac{dL_3}{dT}$ is negative definite. Further, since L_3 is radially unbounded function, then P_4 is GAS.

Theorem 7. Assume that the positive EP, $P_5=(S^*,I^*,Y^*)$, of the system (1) is LAS in $\mathbb{R}^3_+$, and:

$$\left.\begin{array}{l}\delta_{12}^2<\delta_{11}\delta_{22}\\ \delta_{13}^2<\delta_{11}\delta_{33}\\ \delta_{23}^2<\delta_{22}\delta_{33}\end{array}\right\} \tag{21}$$

where all the symbols are defined in the proof. Then, it is GAS in $\mathbb{R}^3_+$.

Proof: Consider the following function:

$$L_4=\left(S-S^*-S^*\ln\frac{S}{S^*}\right)+\left(I-I^*-I^*\ln\frac{I}{I^*}\right)+\left(Y-Y^*-Y\ln\frac{Y}{Y^*}\right).$$

Clearly, $L_4:\mathbb{R}^3_+\to\mathbb{R}$ is a C^1 function that is a positive definite. Then, we have

$$\frac{dL_4}{dT}=-\left[\frac{r_1}{K_1}+\frac{\alpha I^*}{SS^*}\right](S-S^*)^2-\left[\frac{r_1}{K_1}-\frac{nII^*}{AA^*}-\frac{\alpha}{S}\right](S-S^*)(I-I^*)$$
$$-\frac{nmS^*}{AA^*}(I-I^*)^2-\left[a_2(\varepsilon+1)-\frac{r_2pY^*}{BB^*}\right](I-I^*)(Y-Y^*)$$
$$-\frac{r_2p}{B}(Y-Y^*)^2-\left[a_1-\frac{r_2pY^*}{BB^*}\right](S-S^*)(Y-Y^*)$$

where $A=m+I$, $A^*=m+I^*$, $B=K_2+S+I$, and $B^*=K_2+S^*+I^*$. So, after using the given conditions (21), it is obtained that:

$$\frac{dL_4}{dT}\le-\frac{1}{2}\left[\sqrt{\delta_{11}}(S-S^*)+\sqrt{\delta_{22}}(I-I^*)\right]^2$$
$$-\frac{1}{2}\left[\sqrt{\delta_{11}}(S-S^*)+\sqrt{\delta_{33}}(Y-Y^*)\right]^2$$
$$-\frac{1}{2}\left[\sqrt{\delta_{22}}(I-I^*)-\sqrt{\delta_{33}}(Y-Y^*)\right]^2,$$

here $\delta_{11}=\left[\frac{r_1}{k_1}+\frac{\alpha I^*}{SS^*}\right]$, $\delta_{22}=\frac{nmS^*}{AA^*}$, $\delta_{33}=\frac{r_2p}{B}$, $\delta_{12}=\left[\frac{r_1}{K_1}-\frac{nII^*}{AA^*}-\frac{\alpha}{S}\right]$, $\delta_{13}=\left[a_1-\frac{r_2pY^*}{BB^*}\right]$ and $\delta_{23}=\left[a_2(\varepsilon+1)-\frac{r_2pY^*}{BB^*}\right]$.

Obviously, we have $\frac{dL_4}{dT}$ is negative definite. Also, since L_4 is radially unbounded function, then P_5 is GAS.

5. Local bifurcation

The effect of varying the parameters values on the dynamics of the system (1) is studied in this section using the local bifurcation analysis with the help of the Sotomayor's theorem.

Now, for simplifying the notations, rewrite the system (1) in the vector form as follows

$\frac{d\boldsymbol{X}}{dT} = F(X)$, with $\boldsymbol{X} = (S, I, Y)^t$ and $F = (G_1, G_2, G_3)^t$

So, according to the JM of the system (1) at the point (S, I, Y), it is easy to verify that for any vector $U = (u_1, u_2, u_3)^t$, we have that

$$D^2\boldsymbol{F}(\boldsymbol{X})(U,U) = \begin{bmatrix} \sum_{i,j=1}^{3} \frac{\partial G_1}{\partial x_i \partial x_j}(u_i)(u_j) \\ \sum_{i,j=1}^{3} \frac{\partial G_2}{\partial x_i \partial x_j}(u_i)(u_j) \\ \sum_{i,j=1}^{3} \frac{\partial G_3}{\partial x_i \partial x_j}(u_i)(u_j) \end{bmatrix} = \left[b_{ij}\right]_{3\times1},$$

(22) where $b_{11} = -\frac{2r_1}{K_1}u_1^2 - \left(\frac{2r_1}{K_1} + 2b_1 - \frac{4nI}{m+I} + \frac{2nI^2}{(m+I)^2}\right)u_1u_2 + \left(\frac{2mnS}{(m+I)^2} - \frac{2mnSI}{(m+I)^3}\right)u_2^2 - 2a_1u_1u_3$

$$b_{21} = \left(2b_1 - \frac{4nI}{m+I} + \frac{2nI^2}{(m+I)^2}\right)u_1u_2 - \left(\frac{2mnS}{(m+I)^2} - \frac{2mnSI}{(m+I)^3}\right)u_2^2 - 2a_2u_2u_3$$

$$b_{31} = -\frac{2r_2pY^2}{(K_2+S+I)^3}u_1^2 - \frac{4r_2pY^2}{(K_2+S+I)^3}u_1u_2 + \frac{4r_2pY}{(K_2+S+I)^2}u_1u_3 - \frac{2r_2pY^2}{(K_2+S+I)^3}u_2^2 - \frac{2r_2p}{K_2+S+I}u_3^2 + \left(\frac{4r_2pY}{(K_2+S+I)^2} - 2\varepsilon a_2\right)u_2u_3.$$

We also have

$$D^3\boldsymbol{F}(\boldsymbol{X})(U,U,U) = \left[c_{ij}\right]_{3\times1}, \qquad (23)$$

where

$$c_{11} = \left[\frac{2mn}{(m+I)^2}\left(3 - \frac{I}{(m+I)}\right) - \frac{2nI}{(m+I)^2}\left(1 - \frac{2I}{(m+I)}\right)\right]u_1u_2^2 - \frac{6mnS}{(m+I)^2}\left(1 - \frac{2I}{(m+I)^2}\right)u_2^3$$
$$c_{21} = \left[\frac{-2mn}{(m+I)^2}\left(3 - \frac{I}{(m+I)}\right) + \frac{2nI}{(m+I)^2}\left(1 - \frac{2I}{(m+I)}\right)\right]u_1u_2^2 + \frac{6mnS}{(m+I)^2}\left(1 - \frac{2I}{(m+I)^2}\right)u_2^3$$
$$c_{31} = \frac{6r_2pY^2}{(K_2+S+I)^4}(u_1+u_2)^3 - \frac{12r_2pYu_3}{(K_2+S+I)^3}(u_1+u_2)^2 + \frac{6r_2p}{(K_2+S+I)^2}(u_1+u_2)$$

The occurrence of LB around the EPs, P_2, P_3, P_4 and P_5, is investigated respectively.

Theorem 8. Assume that the parameter r_1 satisfies that

$$r_1 \equiv r_1^* = \frac{a_1K_2}{p}$$

(24) Then, the system (1) near the second axial EP, P_2, has a transcritical bifurcation (TB) but saddle – node bifurcation (SNB) and a pitchfork bifurcation (PB) cannot occur.

Proof: Note that, when $r_1 = r_1^*$, then the JM of the system (1) at P_2 can be written as

$$J_1 = J(P_2, r_1^*) = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & -(\alpha+d_1) - \frac{a_2K_2}{p} & 0 \\ \frac{r_2}{p} & \frac{r_2 - \varepsilon a_2K_2}{p} & -r_2 \end{bmatrix}.$$

So, J_1 has the following eigenvalues: $\lambda_{11}^* = 0, \lambda_{12}^* = -(\alpha+d_1) - \frac{a_2K_2}{p} < 0$ and $\lambda_{12}^* = -r_2$. Hence, the second axial EP is a non-hyperbolic point, and then the necessary but not sufficient condition for bifurcation is satisfied.

Let $\boldsymbol{U_1} = (u_{11}, u_{12}, u_{13})^t$ be the eigenvectors of J_1 of $\lambda_{11}^* = 0$. Then, simple computation gives that $\boldsymbol{U_1} = (u_{11}, 0, \gamma_1u_{11})^t$, where $u_{11} \neq 0$ and $\gamma_1 = \frac{1}{p} > 0$.

Also, let $\boldsymbol{\Psi_1} = (\psi_{11}, \psi_{12}, \psi_{13})^T$ represents the eigenvectors of J_1^T that of $\lambda_{11}^* = 0$. Then again, simple calculation shows that $\boldsymbol{\Psi_1} = (\psi_{11}, \gamma_2\psi_{11}, 0)^T$, where $\psi_{11} \neq 0$ and $\gamma_2 = \frac{\alpha p}{(\alpha+d_1)p + a_2K_2} > 0$.

Since the partial derivative of vector field $\boldsymbol{F}$ w.r.t the parameter r_1 is given by $\frac{\partial \boldsymbol{F}}{\partial r_1} = \left(S\left(1-\frac{S+I}{K_1}\right),0,0\right)^t$, hence, by substituting P_2 and $r_1{}^*$ in this derivative, we obtain that $\boldsymbol{F}_{r_1}(P_2,r_1{}^*) = (0,0,0)^t$. Therefore, $\boldsymbol{\Psi_1}{}^t\left[\boldsymbol{F}_{r_1}(P_2,r_1{}^*)\right] = 0$.
Thus the system (1) at P_2 with $r_1 = r_1{}^*$ does not experience SNB in view of Sotomayor theorem. Moreover, since

$$\boldsymbol{\Psi_1}{}^t\left[D\boldsymbol{F}_{r_1}(P_2,r_1{}^*)\,\boldsymbol{U_1}\right] = u_{11}\psi_{11} \neq 0,$$

where $D\boldsymbol{F}_{r_1}$ represents the derivative of $\boldsymbol{F}_{r_1}$ w.r.t $\boldsymbol{X}$,
then

$$\boldsymbol{\Psi_1}{}^t[D^2\boldsymbol{F}(P_2,r_1{}^*)(\boldsymbol{U_1},\boldsymbol{U_1})] = -2a_1\psi_{11}u_{11}{}^2\left(\frac{K_2}{pK_1}+\gamma_1\right) \neq 0,$$

where $D^2\boldsymbol{F}$ represents the second derivative of $\boldsymbol{F}$ w.r.t. $\boldsymbol{X}$ that is given by equation (22). Accordingly, by Sotomayor theorem [26], the system (1) near the EP, P_2, with $r_1 = r_1{}^*$ possesses a TB but not PB.

Theorem 9. Assume that the conditions (11b)-(11d) hold and the parameter ε satisfies

$$\varepsilon \equiv \varepsilon^* = \frac{r_2}{a_2\bar{I}}. \tag{25}$$

Then, the system (1) near the predator-free EP, P_3, has a TB provided that the following condition holds, otherwise it has a PB:

$$\frac{\beta_2}{\bar{I}} + \frac{p}{(K_2+\bar{S}+\bar{I})} \neq 0, \tag{26}$$

Proof: Note that, when $\varepsilon = \varepsilon^*$, then the JM of system (1) at P_3 can be written as

$$J_2 = J(P_3,\varepsilon^*) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

where $a_{ij}; i = 1,2; j = 1,2,3$ are given in equation (10a). Clearly, J_2 has zero eigenvalue $\lambda_{23}{}^* = 0$ with two other eigenvalues, that are given by Eq. (10c), having negative real parts due to conditions (11b)-(11d).
Let $\boldsymbol{U_2} = (u_{21},u_{22},u_{23})^t$ be the eigenvectors of J_2 of $\lambda_{23}{}^* = 0$.
So, direct computation shows that $\boldsymbol{U_2} = (\beta_1 u_{23},\beta_2 u_{23},u_{23})^t$, where $u_{23} \neq 0$ (any real number) and $\beta_1 = \frac{a_{12}a_{23}-a_{22}a_{13}}{a_{11}a_{22}-a_{12}a_{21}}$ and $\beta_2 = \frac{a_{23}a_{11}-a_{21}a_{13}}{a_{11}a_{22}-a_{12}a_{21}}<0$.
Let $\boldsymbol{\Psi_2} = (\psi_{21},\psi_{22},\psi_{23})^t$ represents the eigenvectors of $J_2{}^t$ that $\lambda_{23}{}^* = 0$. Then, a straightforward calculation shows that $\boldsymbol{\Psi_2} = (0,0,\psi_{23})^t$, where $\psi_{23} \neq 0$ (any real number).
Since $\frac{\partial \boldsymbol{F}}{\partial \varepsilon} = (0,0,-a_2IY)^t$, hence we obtain that $\boldsymbol{F}_\varepsilon(P_3,\varepsilon^*) = (0,0,0)^T$.
Therefore, $\boldsymbol{\Psi_2}{}^t[\boldsymbol{F}_\varepsilon(P_3,\varepsilon^*)] = 0$. Thus, system (1) at EP P_3 with $\varepsilon = \varepsilon^*$ does not undergo SNB in view of Sotomayor theorem.
Now, since

$$\boldsymbol{\Psi_2}{}^t[D\boldsymbol{F}_\varepsilon(P_3,\varepsilon^*)\boldsymbol{U_2}] = -a_2\bar{I}u_{23}\psi_{23} \neq 0,$$

and

$$\boldsymbol{\Psi_2}{}^t[D^2\boldsymbol{F}(P_3,\varepsilon^*)(\boldsymbol{U_2},\boldsymbol{U_2})] = -2r_2\psi_{23}u_{23}{}^2\left[\frac{\beta_2}{\bar{I}} + \frac{p}{(K_2+\bar{S}+\bar{I})}\right],$$

then clearly, $\boldsymbol{\Psi_2}{}^t[D^2\boldsymbol{F}(P_3,\varepsilon^*)(\boldsymbol{U_2},\boldsymbol{U_2})] \neq 0$ due to condition (26) and hence the system (1) undergoes a TB near P_2 when $\varepsilon = \varepsilon^*$. However, violating condition (26) leads to $\boldsymbol{\Psi_2}{}^t[D^2\boldsymbol{F}(P_3,\varepsilon^*)(\boldsymbol{U_2},\boldsymbol{U_2})] = 0$. Furthermore, using equation (23) gives
$\boldsymbol{\Psi_2}{}^t[D^3\boldsymbol{F}(P_2,\alpha^*)(\boldsymbol{V_2},\boldsymbol{V_2},\boldsymbol{V_2})] \neq 0$.
Hence, system (1) undergoes PB.

Theorem 10. Assume that the condition (13b) holds and the parameter d_1 satisfies that

$$d_1 \equiv d_1{}^* = b_1\tilde{S} - \alpha - a_2\tilde{Y}. \tag{27a}$$

Then, system (1) near the infected prey-free EP, P_4, has a TB, provided that the following condition holds, otherwise it has a PB:

$$b_1\rho_1 - a_2\rho_2 - \frac{n\tilde{S}}{m} \neq 0 \qquad (27b)$$

Proof: Note that, when $d_1 = {d_1}^*$, then the JM of the system (1) at P_4 can be written as

$$J_3 = J(P_4, {d_1}^*) = \begin{bmatrix} r_1 - \frac{2r_1\tilde{S}}{K_1} - a_1\tilde{Y} & -\frac{r_1\tilde{S}}{K_1} - b_1\tilde{S} + \alpha & -a_1\tilde{S} \\ 0 & 0 & 0 \\ \frac{r_2}{p} & \frac{r_2}{p} - \varepsilon a_2\tilde{Y}^2 & -r_2 \end{bmatrix} = (m_{ij}).$$

Clearly, J_3 has ${\lambda_{32}}^* = 0$ with two other eigenvalues, given by Eq. (12b), having negative real parts due to condition (13b).

Clearly, the necessary but not sufficient condition for bifurcation is satisfied and P_4 is a non-hyperbolic point.

Let $\boldsymbol{U_3} = (u_{31}, u_{32}, u_{33})^t$ be the eigenvectors of J_3 of $\lambda_{32}^* = 0$. Then, simple computation gives that $\boldsymbol{U_3} = (\rho_1 u_{32}, u_{32}, \rho_2 u_{32})^t$, where $u_{32} \neq 0$, $\rho_1 = \frac{m_{32}m_{13} - m_{12}m_{11}}{m_{11}m_{33} - m_{13}m_{31}}$, and $\rho_2 = \frac{m_{12}m_{31} - m_{32}m_{11}}{m_{11}m_{33} - m_{13}m_{31}}$.

Also, let $\boldsymbol{\Psi_3} = (\psi_{31}, \psi_{32}, \psi_{33})^t$ represents the eigenvectors of ${J_3}^t$ that of $\lambda_{32}^* = 0$. Then again, simple calculation shows that $\boldsymbol{\Psi_3} = (0, \psi_{32}, 0)^t$, where $\psi_{11} \neq 0$.

Since the partial derivative of vector field $\boldsymbol{F}$ w.r.t the parameter d_1 is given by $\frac{\partial \boldsymbol{F}}{\partial d_1} = (0, -I, 0)^t$, hence by substituting P_4 and ${d_1}^*$ in this derivative we obtain that $\boldsymbol{F}_{d_1}(P_4, {d_1}^*) = (0,0,0)^t$. Therefore, ${\boldsymbol{\Psi_3}}^t[\boldsymbol{F}_{d_1}(P_4, {d_1}^*)] = 0$.

Thus, the system (1) at P_4 with $d_1 = {d_1}^*$ does not experience SNB in view of Sotomayor theorem. Moreover, since

$${\boldsymbol{\Psi_3}}^t[D\boldsymbol{F}_{d_1}(P_4, {d_1}^*)\,\boldsymbol{U_3}] = -u_{32}\psi_{32} \neq 0,$$

then,

$${\boldsymbol{\Psi_3}}^t[D^2\boldsymbol{F}(P_4, {d_1}^*)(\boldsymbol{U_3}, \boldsymbol{U_3})] = 2\psi_{32}{u_{32}}^2(b_1\rho_1 - a_2\rho_2 - \frac{n\tilde{S}}{m}).$$

Clearly, ${\boldsymbol{\Psi_3}}^t[D^2\boldsymbol{F}(P_4, {d_1}^*)(U_3, \boldsymbol{U_3})] \neq 0$ due to condition (27b) and hence the system (1) undergoes a TB near P_4 when $d_1 = {d_1}^*$. However, violating condition (27b) leads to ${\boldsymbol{\Psi_3}}^t[D^2\boldsymbol{F}(P_4, {d_1}^*)(U_3, \boldsymbol{U_3})] = 0$. Furthermore, using equation (23) gives

$${\boldsymbol{\Psi_2}}^t[D^3\boldsymbol{F}(P_2, \alpha^*)(\boldsymbol{V_2}, \boldsymbol{V_2}, \boldsymbol{V_2})] = -2\frac{n}{m}\psi_{32}{u_{32}}^3(3\rho_1 + \frac{2\tilde{S}}{m^2}) \neq 0$$

Hence, system (1) undergoes PB.

Theorem 11. Assume the conditions (15a)-(15b) along with the

$$\frac{2r_2pY^*}{(K_2 + S^* + I^*)} + \varepsilon a_2 I^* < r_2, \qquad (28a)$$

$$2\Gamma_{11} < \Gamma_{10}. \qquad (28b)$$

Then, as the parameter a_1 passes through the value

$${a_1}^* = \frac{1}{S^*}\left[\frac{e_{33}(e_{11}e_{22} - e_{12}e_{21}) + e_{23}(e_{12}e_{31} - e_{11}e_{32})}{(e_{21}e_{32} - e_{22}e_{31})}\right] , \qquad (28c)$$

Then, the system (1) near the coexistence EP, P_5, has a SNB provided that

$${\xi_1}^2{q_{11}}^* + (\xi_3 + \xi_4){q_{21}}^* + {q_{31}}^* + {q_{41}}^* \neq 0, \qquad (29)$$

where all the symbols are given in the proof.

Proof: Straightforward computation shows that under the conditions (15a), (15b), (27a), and (28b) with $a_1 = {a_1}^*$, the coefficients of the characteristic equation given by Eq. (14b) are $M_1 > 0$, $M_2 > 0$ and $M_3 = 0$. Hence, Eq. (14b) has three roots (eigenvalues of $J(P_5)$) given by

$${\lambda_{41}}^* = 0, {\lambda_{42}}^* = -\frac{M_1}{2} + \frac{1}{2}\sqrt{{M_1}^2 - 4M_2}, {\lambda_{43}}^* = -\frac{M_1}{2} - \frac{1}{2}\sqrt{{M_1}^2 - 4M_2},$$

where M_1, M_2 and M_3 are defined in equation (14b).

Clearly, the eigenvalues ${\lambda_{42}}^*$ and ${\lambda_{43}}^*$ have negative real parts due to conditions (15a) and (15b). Hence, the Jacobian matrix of the system (1) of P_5 and $a_1 = {a_1}^*$ can be written as $J_4 = J(P_5, {a_1}^*) = ({e_{ij}}^*)_{3\times 3}$ with ${e_{ij}}^* = e_{ij}$; $\forall i,j$ with ${e_{11}}^* = e_{11}({a_1}^*)$, and ${e_{13}}^* = e_{13}({a_1}^*)$, where e_{ij} are given in Eq. (14a). We will drop the star for simplification.

Let $\boldsymbol{U_4} = (u_{41}, u_{42}, u_{43})^t$ be the eigenvector of J_4 corresponding to ${\lambda_{41}}^* = 0$. Then, direct computation shows that $\boldsymbol{U_4} = (\xi_1 u_{43}, \xi_2 u_{43}, u_{43})^t$, where $u_{43} \neq 0$, $\xi_1 = \frac{e_{12}e_{23} - e_{13}e_{22}}{e_{11}e_{22} - e_{12}e_{21}}$, and $\xi_2 = \frac{e_{13}e_{21} - e_{23}e_{11}}{e_{11}e_{22} - e_{12}e_{21}}$.

Let $\boldsymbol{\Psi_4} = (\psi_{41}, \psi_{42}, \psi_{43})^t$ represents the eigenvector of ${J_4}^t$ that of ${\lambda_{41}}^* = 0$. Then, simple calculation shows that $\boldsymbol{\Psi_4} = (\xi_3 \psi_{43}, \xi_4 \psi_{43}, \psi_{43})^t$, where $\psi_{43} \neq 0$, $\xi_3 = \frac{e_{12}e_{32} - e_{22}e_{31}}{a_{11}a_{22} - a_{12}a_{21}}$, and $\xi_4 = \frac{e_{12}e_{31} - e_{32}e_{11}}{a_{11}a_{22} - a_{12}a_{21}}$.

We have that $\frac{\partial \boldsymbol{F}}{\partial a_1} = (-SY, 0,0)^t$, hence we obtain that $\boldsymbol{F}_{a_1}(P_5, {a_1}^*) = (-S^*Y^*, 0,0)^t$. Therefore, we obtain that

$${\boldsymbol{\Psi_4}}^t\left[\boldsymbol{F}_{a_1}(P_5, {a_1}^*)\right] = -\xi_3 \psi_{43} S^* Y^* \neq 0.$$

Consequently, the first condition of SNB in view of Sotomayor theorem is satisfied. Now, since:

$${\boldsymbol{\Psi_4}}^t[D^2\boldsymbol{F}(P_5, {a_1}^*)(\boldsymbol{U_4}, \boldsymbol{U_4})] = {u_{43}}^2 \psi_{43}\left[{\xi_1}^2 {q_{11}}^* + (\xi_3 + \xi_4){q_{21}}^* + {q_{31}}^* + {q_{41}}^*\right],$$

where ${q_{11}}^* = \frac{2r_1}{K_1}\xi_3$, ${q_{21}}^* = \left[\left(2b_1 - \frac{4nI^*}{m+I^*} + \frac{2n{I^*}^2}{(m+I^*)^2}\right)\xi_1\xi_2 + \left(\frac{2nmS^*}{(m+I^*)^2} - \frac{2nmS^*}{(m+I^*)^3}\right){\xi_2}^2\right]$,

$${q_{31}}^* = -2\left[\left(\frac{r_1}{K_1} + a_1\xi_2\right)\xi_1\xi_3 + a_2\xi_2(\xi_4 + \varepsilon)\right],$$

$${q_{41}}^* = \frac{-2r_2 p}{(K_2 + S^* + I^*)}\left[\frac{{Y^*}^2}{(K_2 + S^* + I^*)}(\xi_1 + \xi_2)^2 + \frac{2Y}{(K_2 + S^* + I^*)}(\xi_1 + 1) - 1\right].$$

Clearly, ${\boldsymbol{\Psi_4}}^T[D^2\boldsymbol{F}(P_5, {a_1}^*)(\boldsymbol{U_4}, \boldsymbol{U_4})] \neq 0$ under the condition (28b), and hence the system (1) undergoes SNB near the coexistence equilibrium.

6. Numerical Simulation

In this section, the global dynamics of the system (1) is further investigated. To specify the control set of parameters, the system is solved numerically using Runge-Kutta of ordered six, followed by forth steps Predictor- Corrector method. Then, all the obtained numerical results are drawn in the form of $3D$ phase portraits and $2D$ time series using Matlab version 6. Therefore, in order to run simulations, the following hypothetical set of biological data is used in this section:

$$\begin{gathered} r_1 = 5,\ K_1 = 20,\ r_2 = 2,\ K_2 = 50,\ a_1 = 0.3,\ a_2 = 0.1, \\ b_1 = 0.15,\ b_2 = 0.1, p = 4,\ m = 20,\ \varepsilon = 0.1,\ d_1 = 0.05,\ \alpha = 0.5 \end{gathered} \tag{30}$$

It is observed, for this set of data, that the system (1) approaches asymptotically to the unique coexistence EP, $P_5 = (8.67, 2.66, 6.49)$, starting from five different initial values, as shown in Figures 1 and 2.

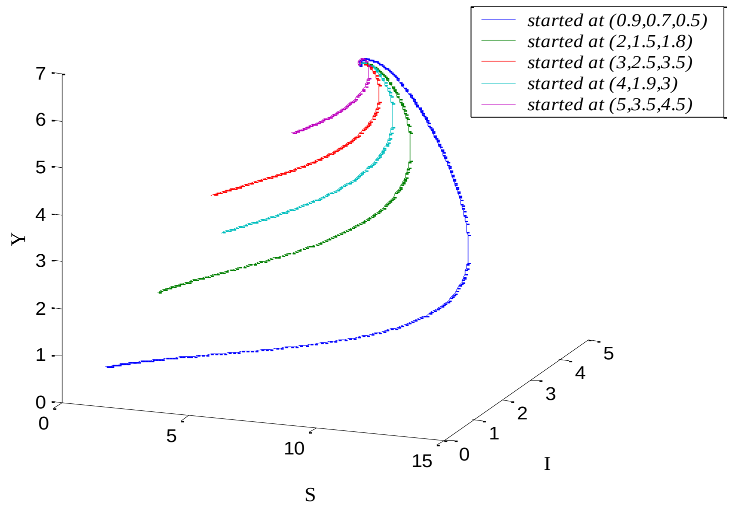
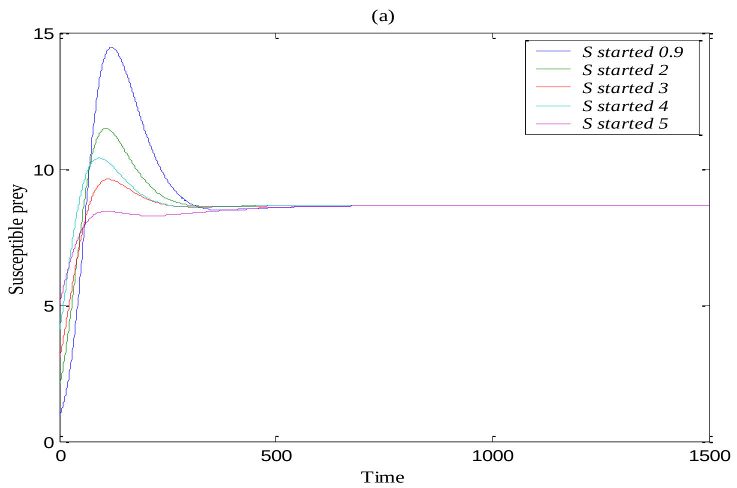
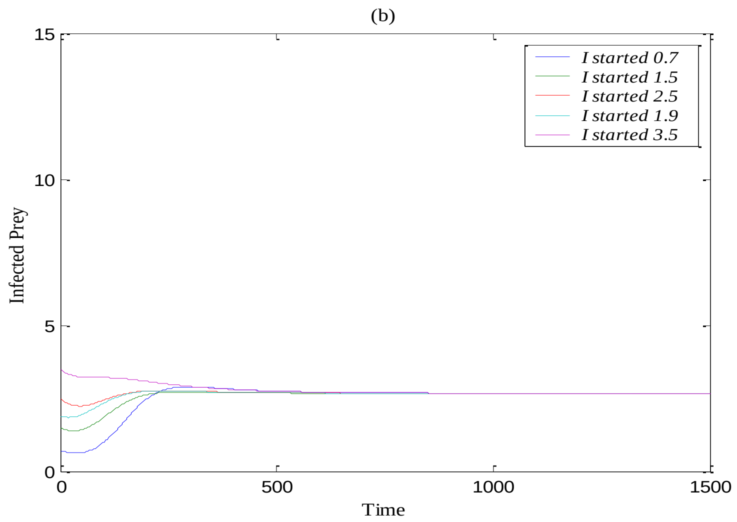


Figure 1-3D phase portrait of the system (1) using the parameters given by Eq. (30) in which the solution approaches asymptotically to the $P_5 = (8.67, 2.66, 6.49)$.





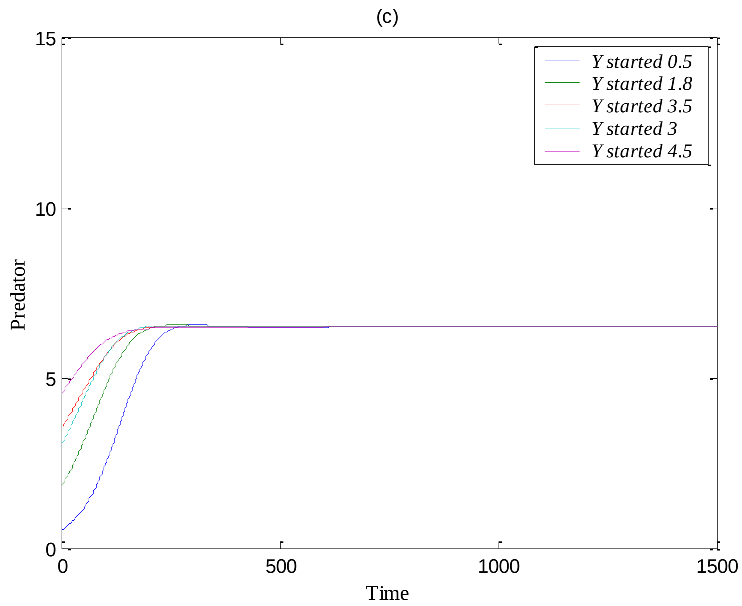
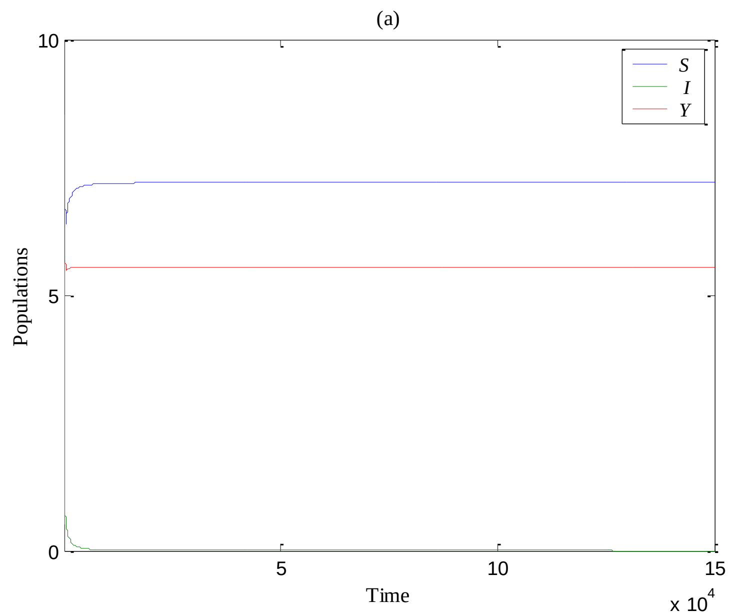


Figure 2-The solution of system (1) approaching asymptotically to $P_5 = (8.67, 2.66, 6.49)$ for the data given by Eq. (30). (a) The trajectory of susceptible prey versus time. (b) The trajectory of infected prey versus time. (c) The trajectory of predator versus time.

According to these two figures, the system (1) persists at the coexistence point in $\mathbb{R}^3_+$. Now, in order to discuss the effect of varying the values of parameters on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (30), with varying a specific parameter each time and then the obtained solutions are drawn as shown below. It is observed that, for the values of parameter r_1 in the range $r_1 \leq 2.6$ with the other parameters as in Eq. (30), the system (1) approaches asymptotically to infected prey-free EP in the interior of of SY $-$plane; otherwise, it has a GAS coexistence EP; see Figures 3a and 3b for typical values of r_1.



(b)

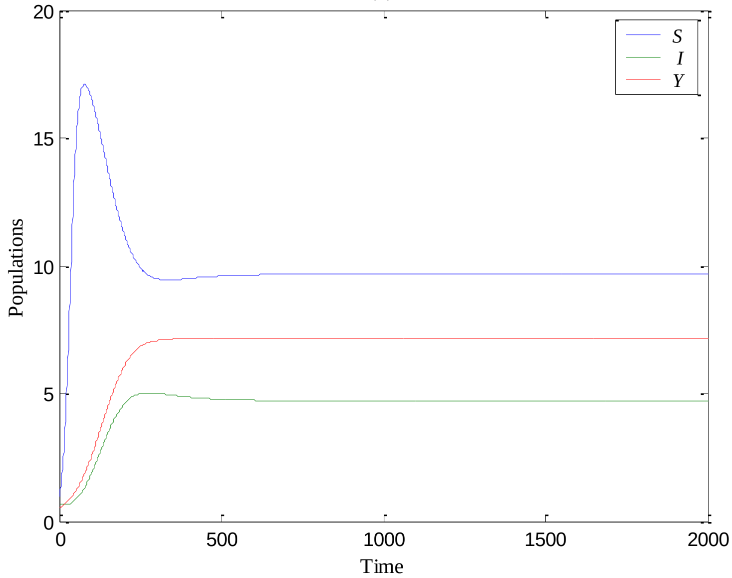
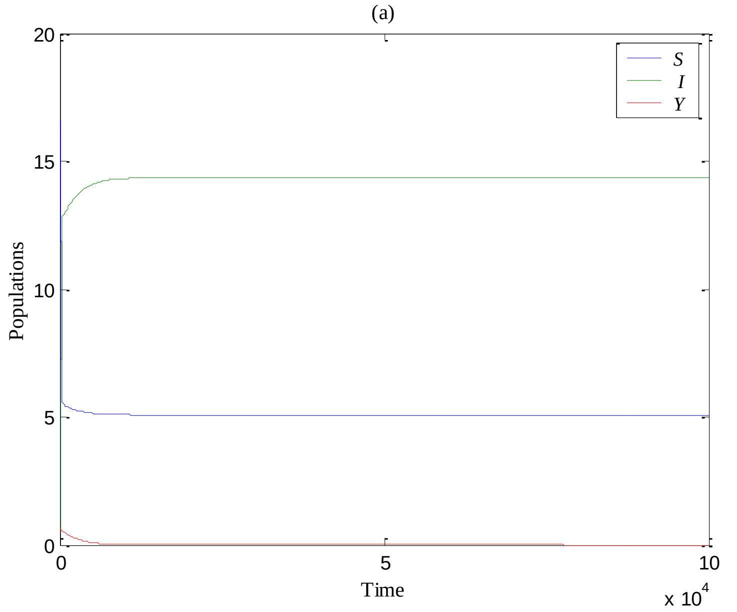


Figure 3-The trajectories of system (1) versus time for the data given by Eq. (30) with different values of r_1. (a) The system approaches asymptotically to $P_4 = (7.19,0,5.54)$ when $r_1 = 2.6$. (b)The system approaches asymptotically to $P_5 = (8.67,2.66,6.49)$ when $r_1 = 5$.

It is observed that varying the parameters K_1 and b_1 has a similar effect to that shown with varying r_1. Now, for the parameter r_2 in the range $r_2 \leq 0.1$, it is observed that the system (1) approaches asymptotically to predator-free EP in the interior of SI −plane, as shown in the below typical figures given by Figure 4. However, it approaches to P_5 otherwise.

(a)



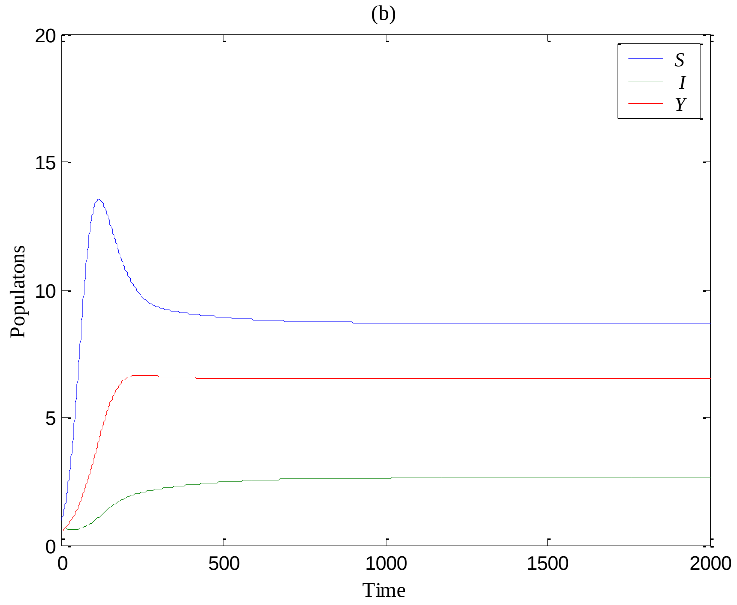
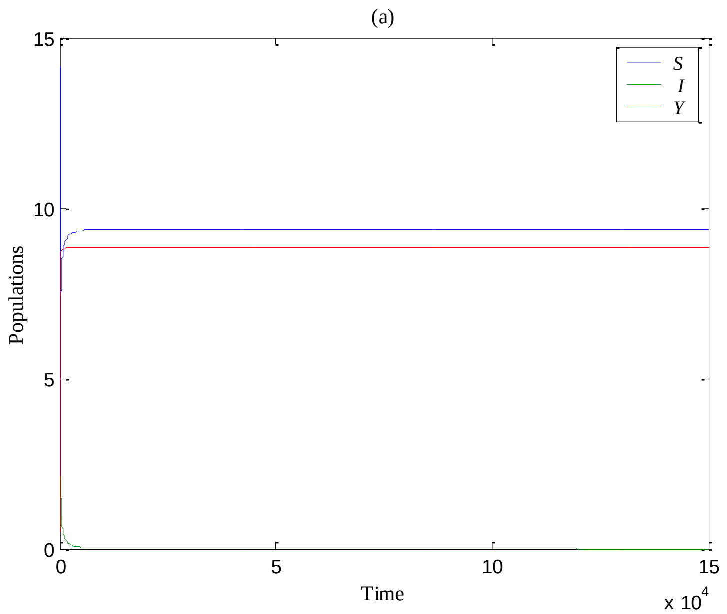


Figure 4-The trajectories of the system (1) versus time for the data given by Eq. (30) with different values of r_2. (a) The system approaches asymptotically to $P_3 = (5.08,14.35,0)$ when $r_2 = 0.1$. (b)The system approaches asymptotically to $P_5 = (8.68,2.63,6.52)$ when $r_2 = 2.6$.

It is observed that varying the parameters ε has a similar effect to that shown with varying r_2. Now, for the parameter K_2 in the range $K_2 \leq 26$, it is observed that the system (1) approaches asymptotically to the coexistence EP, as shown typically in Figure 5. However, it approaches to P_4 otherwise.



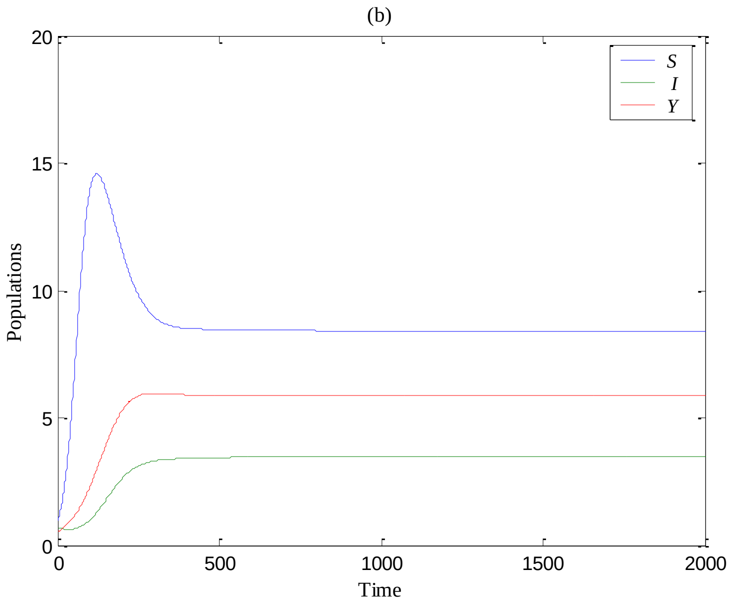
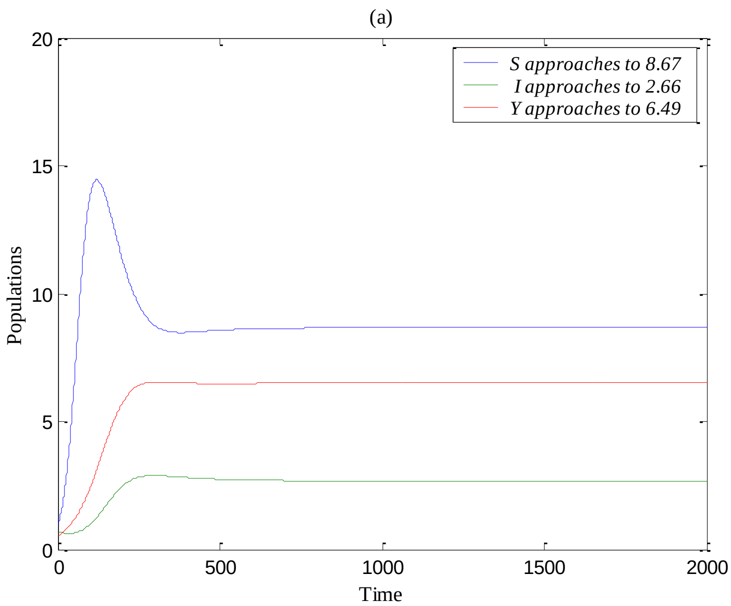


Figure 5-The trajectories of the system (1) versus time for the data given by Eq. (30) with different values of K_2. (a) The system approaches asymptotically to $P_4 = (9.38,0,8.84)$ when $K_2 = 26$. (b)The system approaches asymptotically to $P_5 = (8.41,3.48,5.87)$ when $K_2 = 12$.

It is observed that varying the parameters a_1, α, and d_1 has similar effects as those shown with varying K_2. On the other hand, varying the parameters of the infection rate of the system (1) is also studied. It is observed that, for $b_1 \geq 0.1$, the system (1) approaches asymptotically to the infected prey-free EP; otherwise, it has a GAS coexistence EP that has a GAS at P_5 . However, for $n = 0.15$ (maximum transmission rate under the media coverage alert), with increasing the response of individuals to the media coverage alert or decreasing the parameter m, the system (1) approaches gradually to P_4, as shown in Figure 6 for the values $m = 20,10,5,0$, respectively.



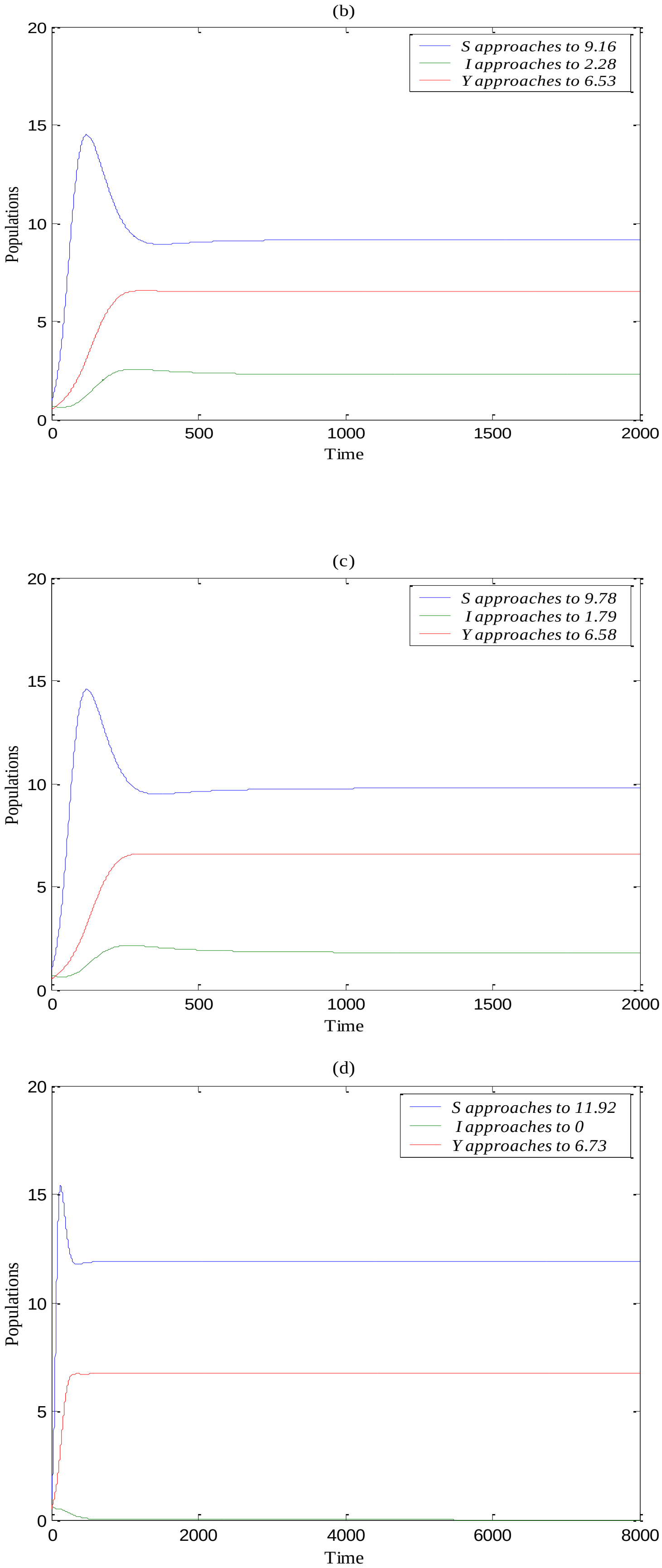


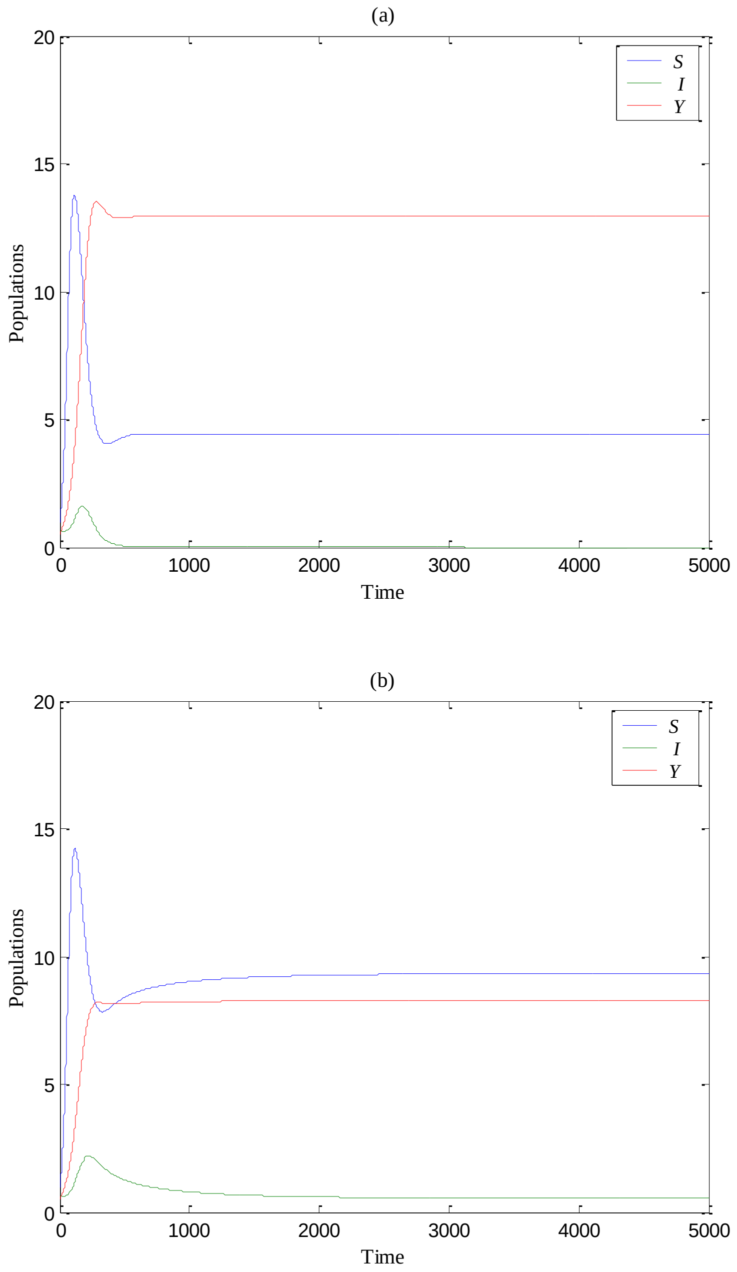
Figure 6-The trajectories of system (1) versus time for the data given by Eq. (30) with $n = 0.15$ and different values of m. (a) The system approaches asymptotically to $P_5 = (8.67, 2.66, 6.49)$ when $m = 20$. (b) The system approaches asymptotically to $P_5 =$

$(9.16, 2.28, 6.53)$ when $m = 10$. (c) The system approaches asymptotically to $P_5 = (9.78, 1.79, 6.58)$ when $m = 5$. (d) The system approaches asymptotically to $P_4 = (11.92, 0, 6.73)$ when $m = 0$.

Now, for the the parameter p in the range $0.9 \leq p \leq 2.7$ with the rest of parameters as in Eq. (30), the system (1) approaches asymptotically to P_4. However, the system (1) approaches asymptotically to the coexistence equilibrium points P_5 and P_2 for the ranges $p > 2.7$ and $p < 0.9$, respectively, as shown typically in Figure 7.

(a)

(b)



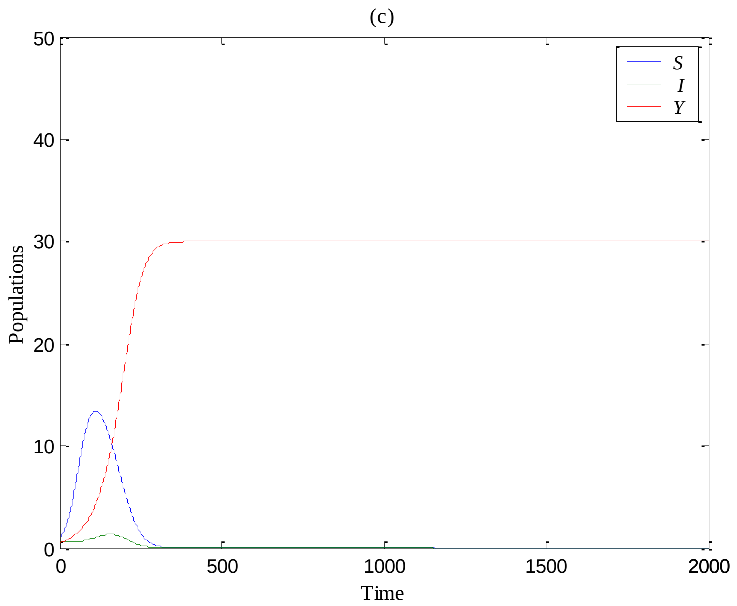
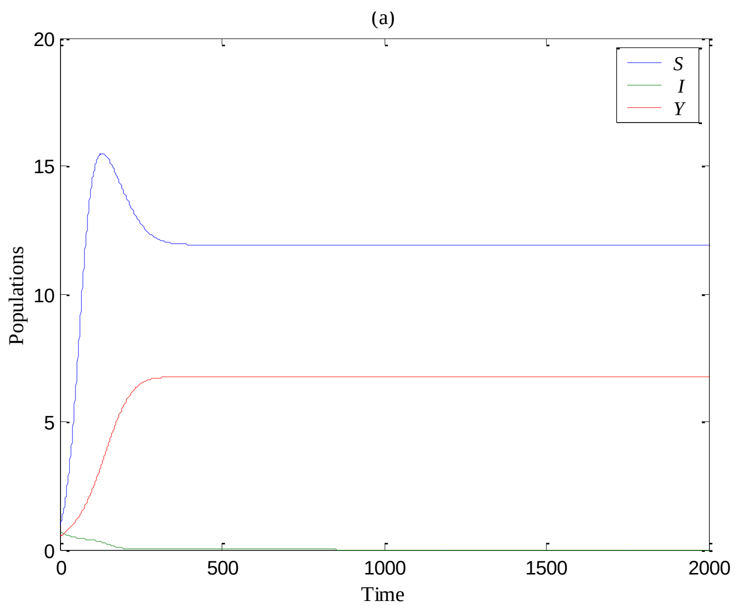


Figure 7-The trajectories of system (1) versus time for the data given by Eq.(30) with different values of a_1. (a) The system approaches asymptotically to $P_4 = (4.44,0,12.96)$ when $p = 1.5$. (b) The system approaches asymptotically to $P_5 = (9.39,0.53,8.26)$ when $p = 3$. (c) The system approaches asymptotically to $P_2 = (0,0,30)$ when $p = 0.5$.

Finally, for the parameter a_2 in the range $0.2 \leq a_2 \leq 2.2$, with the rest of parameters as in Eq.(30), the system (1) approaches asymptotically to P_4, as shown in Figure 8. Otherwise, the system (1) still approaches to P_5 in the interior of $\mathbb{R}^3_+$.



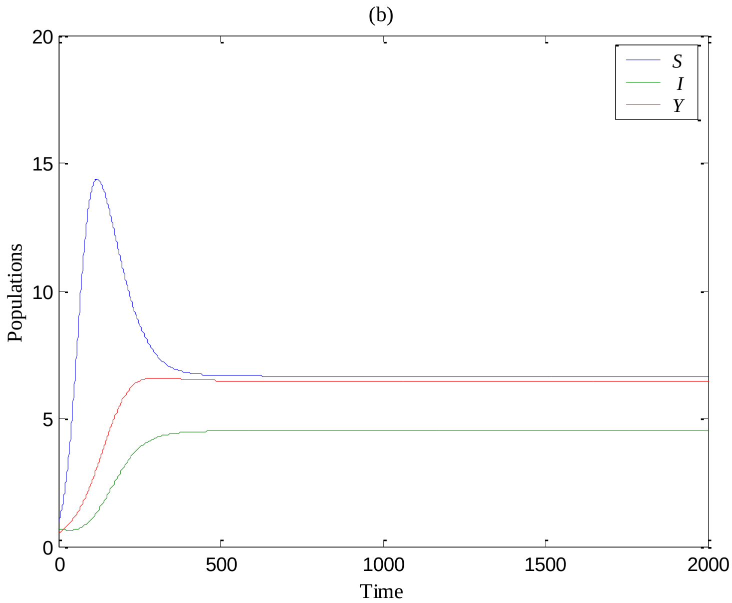


Figure 8-The trajectories of system (1) versus time for the data given by Eq. (30) with different values of a_2. (a) The system approaches asymptotically to $P_4 = (11.92,0,6.73)$ when $a_2 = 0.9$. (b) The system approaches asymptotically to $P_5 = (6.64,4.55,6.47)$ when $a_2 = 0.05$.

7. Discussion and Conclusion

In this paper, the effect of media coverage alert on the dynamical behavior of the diseased modified Leslie–Gower prey-predator model involving disease in the prey is considered. The system is studied theoretically as well as numerically. It is observed that the system has at most six non-negative equilibrium points. Since the solution of the system is proved to be uniformly bounded, it is observed that the solution approaches asymptotically to one of its equilibrium points depending on determined conditions. According to the numerical simulation, it is observed that media coverage works as a control parameter for the spread of disease.

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