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Solvability of (λ, μ)-Commuting Operator Equations for Bounded Generalization of Hyponormal Operators

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Abstract:

Recently, new generalizations have been presented for the hyponormal operators, which are (N, k)-hyponormal operators and (h, M)-hyponormal operators. Some properties of these concepts have also been proved, one of these properties is that the product of two (N, k)-hyponormal operator is also (N, k)- hyponormal operator and the product of two (h, M)-hyponormal operators is (h, M)-hyponormal operator. In our research, we will reprove these properties by using the (λ,μ)-commuting operator equations, in addition to that we will solve the (λ, μ)-commuting operator equations for (N, k)-hyponormal operators and (h, M)-hyponormal operators.

Keywords: (N, k)-hyponormal operators, (h, M)-hyponormal operators, Bounded linear operators, hyponormal operators, Hilbert spaces.

حل المعادلات الأبدالية –(λ, μ) لتعميمات المؤثرات المقيدة فوق السوية

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الخلاصة

مؤخراً تم تقديم تعميمات جديدة للمؤثرات فوق السوية وهي المؤثر فوق السوي– (N,k) و المؤثر فوق السوي (h,M), وقد تم برهنة بعض الخواص لهذه المفاهيم من هذه الخواص ان حاصل ضرب مؤثرين من نوع فوق السوي (N, k) هو مؤثر فوق السوي (N,k) و حاصل ضرب مؤثرين من نوع فوق السوي–(h, M) هو ايضا مؤثر فوق السوي (h,M). في بحثنا هذا سوف نقوم بأعادة برهنة هذه الخاصية بأستخدام معادلات المؤثرات الابدالية – (λ, μ) بالأضافة الى اننا سوف نقوم بحل المعادلات الأبدالية –(λ, μ) للمؤثرات فوق السوية –(N, k) و المؤثرات فوق السوية –(h, M).

1. Introduction:

Many mathematical researchers have presented many studies on the concept of hyponormal operators, some of them have studied the properties of hyponormal operators and their relationship to normal operators, while others have introduced new types of generalizations for hyponormal operators. In [1], B. P. Duggal and I. H. Jeon gave some results on p-quasi-hyponormal operators. In (2015), A. Gupta and K. Mamtani [2] show that

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the unbounded hyponormal operators are satisfied Weyl's theorem. In (2016), A. Tajimoti [3] solved the λ- commuting operator equation $ST = \lambda TS$ for M-hyponormal operators. In (2017), V. Lauric [4] gave the sufficient conditions that make hyponormal operators almost normal operators. In (2019), J. T. Yuan and C. H. Wang [5] proved that if T is (p, k)-quasihyponormal operator, then T satisfied the Fuglede – Putnam Theorem. In (2020), S. M. S. Nabavi Sales [6] introduced the hyponormal property. In this paper, the space H will be a complex Hilbert space and all operators defined on H are bounded linear operators.

Definition 1 [7]

Let H be a Hilbert space and $T: H \to H$ be a bounded linear operator, then T is called self-adjoint operator if $T = T^*$.

Definition 2 [3]

Let $T: H \to H$ be a bounded linear operator. A self-adjoint operator T is called a positive operator if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. It is called strictly positive if $\langle Tx, x\rangle = 0$ only if x = 0.

Definition 3 [7]

Let $T: H \to H$ be a bounded linear operator, then T is said to be unitary operator if $TT^* = T^*T = I$.

Definition 4 [8]

Let $T: H \to H$ be a bounded linear operator, then T is called hyponormal operator if $T^*T \geq TT^*$, that is $\langle T^*Tx, x\rangle \geq \langle TT^*x, x\rangle$, for every $x \in H$.

Definition 5[9] :

Let $T: H \to H$ be a bounded linear operator, and et $N: H \to H$ be a positive bounded linear operator, such that $NT = TN$. The operator T is said to be (N, k)- hyponormal operator if $NT^*T^k \geq T^kT^*$, for all positive integer k.

Definition 6 [9]:

Let $T, h : H \to H$ be bounded linear operators, such that $hT = Th$, $hT^* = T^*h$. The operator T is said to be (h, M)-hyponormal operator if there exists a positive real number $M \geq 1$, such that $MhT^*T \geq TT^*h$.

Definition 7 [10]

Let $S, T: H \to H$ be bounded linear operators on a Hilbert space H, then S and T are said to be (λ, μ)- commuting operators if satisfy ST=λ TS and ST* = μT*S, where $\lambda, \mu \in \mathbb{C}\backslash\{0\}$.

Theorem 8 [9]

Let $S: H \to H$ be (N1, k)-hyponormal operator and $T: H \to H$ be (N2, k)-hyponormal operator on a Hilbert space H, then (ST) is (N, k)-hyponormal operator if $ST = TS$, $ST^* = T^*S$, and $SN_2 = N_2S$, then ST is (N, k)-hyponormal operator, where $N = N_1N_2$.

Theorem 9 [9]:

Let $S: H \to H$ be (h, M1)-hyponormal operator, and let $T: H \to H$ be (h, M2)-hyponormal operator, then (ST) is (h, M)-hyponormal operator if $ST = TS$, $ST^* = T^*S$, where $M = M_1M_2$.

2. Mean Results

To introduce the solution of the (λ, μ)- commuting operator equations for (N, k)-hyponormal operators, the following theorem is given.

Theorem 1: -

Let $S, T: H \to H$ be (λ, μ)-commuting operators, and $N_1, N_2 : H \to H$ be positive non-zero bounded linear operators, then

If S and S* are (N1, k)-hyponormal operators and T is (N2, k)-hyponormal operator, then $|\lambda| \leq (\|N_1\|\|N_2\|)^{\frac{1}{2}}$ and $|\mu| \geq (\|N_1\|\|N_2\|)^{-\frac{1}{2}}$.

If S and S* are (N1, k)-hyponormal operators and T* is (N2, k)-hyponormal operator, then $|\lambda| \geq (\|N_1\|\|N_2\|)^{-\frac{1}{2}}$ and $|\mu| \leq (\|N_1\|\|N_2\|)^{\frac{1}{2}}$.

Proof: -

i) Since S, T are (λ, μ)-commuting operators, then ST=λ TS and ST* = μT*S

$\|ST\| = |\lambda|\|TS\|$

$= |\lambda|\|(TS)(TS)^*\|^{\frac{1}{2}}$ (Since $\|T\| = \|TT^*\|^{\frac{1}{2}}$)

$= |\lambda|\|TSS^*T^*\|^{\frac{1}{2}}$

$= |\lambda| \left\|\frac{\bar{\mu}}{\lambda} SS^*TT^*\right\|^{\frac{1}{2}}$

$\leq |\lambda| \left\|\frac{\bar{\mu}}{\lambda} N_1 S^*STT^*\right\|^{\frac{1}{2}}$ (Since S is (N1, k)-hyponormal operator)

$\leq |\lambda|\|N_1\|^{\frac{1}{2}} \left\|\frac{\bar{\mu}}{\lambda} S^*STT^*\right\|^{\frac{1}{2}}$

$= |\lambda|\|N_1\|^{\frac{1}{2}} \left\|\frac{\mu}{\bar{\lambda}} TT^*S^*S\right\|^{\frac{1}{2}}$

$\leq |\lambda|\|N_1\|^{\frac{1}{2}} \left\|\frac{\mu}{\bar{\lambda}} N_2 T^*TS^*S\right\|^{\frac{1}{2}}$ (Since T is (N2, k)-hyponormal operator)

$\leq |\lambda|(\|N_1\|\|N_2\|)^{\frac{1}{2}} \left\|\frac{\mu}{\bar{\lambda}} T^*TS^*S\right\|^{\frac{1}{2}}$

$= |\lambda|(\|N_1\|\|N_2\|)^{\frac{1}{2}} \left\|\frac{|\mu|^2}{|\lambda|^2} (ST)^*(ST)\right\|^{\frac{1}{2}}$

$= |\mu|(\|N_1\|\|N_2\|)^{\frac{1}{2}}\|ST\|$

Therefore, $\|ST\| \leq |\mu|(\|N_1\|\|N_2\|)^{\frac{1}{2}}\|ST\|$, which implies $|\mu| \geq (\|N_1\|\|N_2\|)^{-\frac{1}{2}}$.

Now, we have to show that $|\lambda| \leq (\|N_1\|\|N_2\|)^{\frac{1}{2}}$. Since ST* = μ T*S, then

$\|ST^*\| = |\mu|\|T^*S\|$

$= |\mu|\|(T^*S)^*(T^*S)\|^{\frac{1}{2}}$

$= |\mu|\|S^*TT^*S\|^{\frac{1}{2}}$

$= |\mu| \left\|\frac{1}{\lambda\mu} S^*STT^*\right\|^{\frac{1}{2}}$

$\leq |\mu| \left\|\frac{1}{\lambda\mu} N_1 SS^*TT^*\right\|^{\frac{1}{2}}$ (Since S* is (N1, k)-hyponormal operator)

$\leq |\mu|\|N_1\|^{\frac{1}{2}} \left\|\frac{1}{\lambda\mu} SS^*TT^*\right\|^{\frac{1}{2}}$

$= |\mu|\|N_1\|^{\frac{1}{2}} \left\|\frac{1}{\bar{\mu}\bar{\lambda}} TT^*SS^*\right\|^{\frac{1}{2}}$

$\leq |\mu|\|N_1\|^{\frac{1}{2}} \left\|\frac{1}{\bar{\mu}\bar{\lambda}} N_2 T^*TSS^*\right\|^{\frac{1}{2}}$ (Since T is (N1, k)-hyponormal operator)

$\leq |\mu|(\|N_1\|\|N_2)\|)^{\frac{1}{2}} \left\|\frac{1}{\bar{\mu}\bar{\lambda}} T^*TSS^*\right\|^{\frac{1}{2}}$

$= |\mu|(\|N_1\|\|N_2)\|)^{\frac{1}{2}} \left\|\frac{1}{|\mu|^2|\lambda|^2} (ST^*)(ST^*)^*\right\|^{\frac{1}{2}}$

$= \frac{1}{|\lambda|} (\|N_1\|\|N_2)\|)^{\frac{1}{2}}\|ST^*\|$

Therefore; $\|ST^*\| \leq \frac{1}{|\lambda|} (\|N_1\|\|N_2)\|)^{\frac{1}{2}}\|ST^*\|$, this implies $|\lambda| \leq (\|N_1\|\|N_2\|)^{\frac{1}{2}}$

ii) Similarly, $|\lambda| \geq (\|N_1\|\|N_2\|)^{-\frac{1}{2}}$ and $|\mu| \leq (\|N_1\|\|N_2\|)^{\frac{1}{2}}$, if S and S* are (N1, k)-hyponormal operator and T* is (N2, k)-hyponormal operator. ■

Theorem 2: -

Let $S, T: H \to H$ be (λ, μ)-commuting operators, and $N_1, N_2: H \to H$ be positive bounded linear operators, such that S is (N1, k)-hyponormal operator and T is (N2, k)-hyponormal operator, $N_2S = SN_2$, $(ST)^k = S^kT^k$ and |μ| ≤ 1, then ST is (N, k)-hyponormal operator, where N = N1N2

Proof: -

$(ST)^k(ST)^* = S^kT^kT^*S^*$ (Since $(ST)^k = S^kT^k$)

$= \bar{\lambda}S^kT^kS^*T^*$ (Since ST=λ TS)

$= \bar{\lambda}\bar{\mu}^kS^kS^*T^kT^*$ (Since ST* = μT*S)

$\leq \bar{\lambda}\bar{\mu}^kN_1S^*S^kN_2T^*T^k$ (Since S is (N1, k)-hyponormal operator and T is (N2, k)- hyponormal operator)

$= \bar{\lambda}\bar{\mu}^kN_1N_2S^*S^kT^*T^k$ (Since $N_2S = SN_2$)

$= \bar{\lambda}\bar{\mu}^k\mu^kNS^*T^*S^kT^k$ (Since N = N1N2)

$= |\mu|^{2k}NT^*S^*S^kT^k$

$= |\mu|^{2k}N(ST)^*(ST)^k$

$\leq N(ST)^*(ST)^k$

Therefore, (ST) is (N, k)- hyponormal operator.

Theorem 3: -

Let $S, T: H \to H$ be (λ, μ)- commuting operators and $N_1, N_2: H \to H$ be positive bounded linear operators, such that S is (N1, k)-hyponormal operator, T is (N2, k)-hyponormal operator, SN2=N2S, $(S^*T)^k = {S^*}^kT^k$ and |λ| ≥ 1, then S*T is (N, k)-hyponormal operator, where N = N1N2.*

Proof: -

$(S^*T)^k(S^*T)^* = {S^*}^kT^kT^*S$ (Since $(S^*T)^k = {S^*}^kT^k$)

$= \frac{1}{\mu}{S^*}^kT^kST^*$

$= \frac{1}{\mu\lambda^k}{S^*}^kST^kT^*$

$\leq \frac{1}{\mu\lambda^k} N_1S{S^*}^kN_2T^*T^k$ (Since S* is (N1, k)-hyponormal operator and T is (N2, k)-hyponormal operator)

$= \frac{1}{\mu\lambda^k} N_1N_2 S{S^*}^kT^*T^k$ (Since SN2=N2S)

$= \frac{1}{\mu|\lambda|^{2k}} N ST^*{S^*}^kT^k$

$= \frac{1}{|\lambda|^{2k}} N T^*S{S^*}^kT^k$

$= \frac{1}{|\lambda|^{2k}} N (S^*T)^*(S^*T)^k$

$\leq N (S^*T)^*(S^*T)^k$ (Since |λ| ≥ 1)

Now, to solve the (λ, μ)-commuting operator equations for (h,M)-hyponormal operators , the following theorem is given.

Theorem 4: -

Let $S, T: H \to H$ be (λ, μ)-commuting operators and $h: H \to H$ be unitary bounded linear operator, then

If S and S are (h, M1)-hyponormal operators and T is (h, M2)- hyponormal operator, then $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$ and $|\mu| \geq (M_1M_2)^{-\frac{1}{2}}$.*

If S and S are (h, M1)-hyponormal operators and T* is (h, M2)-hyponormal operator, then $|\lambda| \geq (M_1M_2)^{-\frac{1}{2}}$ and $|\mu| \leq (M_1M_2)^{\frac{1}{2}}$.*

Proof: -

i)

$\|ST\| = \|(ST)(ST)^*\|^{\frac{1}{2}}$

$= \|STT^*S^*\|^{\frac{1}{2}}$

$= \left\|\bar{\lambda}\bar{\mu}SS^*TT^*I\right\|^{\frac{1}{2}}$ *(Since S and T are (λ, μ)-commuting operators)*

$= \left\|\bar{\lambda}\bar{\mu}SS^*TT^*hh^*\right\|^{\frac{1}{2}}$ *(Since h is a unitary operator)*

$\leq \left\|\bar{\lambda}\bar{\mu}\ M_2SS^*hT^*Th^*\right\|^{\frac{1}{2}}$ *(Since T is (h, M2)-hyponormal operator)*

$\leq \left\|\bar{\lambda}\bar{\mu}\ M_1M_2hS^*ST^*Th^*\right\|^{\frac{1}{2}}$ *(Since S is (h, M1)-hyponormal operator)*

$= \left\|\bar{\lambda}|\mu|^2\ M_1M_2hS^*T^*STh^*\right\|^{\frac{1}{2}}$

$= \||\mu|^2\ M_1M_2hT^*S^*STh^*\|^{\frac{1}{2}}$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|\ (STh^*)^*(STh^*)\|^{\frac{1}{2}}$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|\ STh^*\|$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|\ (STh^*)(STh^*)^*\|^{\frac{1}{2}}$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|\ STh^*hT^*S^*\|^{\frac{1}{2}}$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|\ STIT^*S^*\|^{\frac{1}{2}}$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|\ STT^*S^*\|^{\frac{1}{2}}$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|\ (ST)(ST)^*\|^{\frac{1}{2}}$

$= |\mu|(M_1M_2)^{\frac{1}{2}}\|ST\|$

Therefore, we have $\|ST\| \leq |\mu|(M_1M_2)^{\frac{1}{2}}\|ST\|$. *Hence,* $|\mu| \geq (M_1M_2)^{-\frac{1}{2}}$.

Now, we have to show that $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$,

$\|ST\| = \|(ST)^*(ST)\|^{\frac{1}{2}}$

$= \|T^*S^*ST\|^{\frac{1}{2}}$

$= \|T^*S^*SIT\|^{\frac{1}{2}}$

$= \|T^*S^*Shh^*T\|^{\frac{1}{2}}$

$\leq \|M_1T^*hSS^*h^*T\|^{\frac{1}{2}}$

$= M_1^{\frac{1}{2}}\|(T^*hS)(T^*hS)^*\|^{\frac{1}{2}}$

$= M_1^{\frac{1}{2}}\|(T^*hS\|$

$= M_1^{\frac{1}{2}}\|(T^*hS)^*(T^*hS)\|^{\frac{1}{2}}$

$= M_1^{\frac{1}{2}}\|S^*h^*TT^*hS\|^{\frac{1}{2}}$

$\leq M_1^{\frac{1}{2}}\|M_2S^*h^*hT^*TS\|^{\frac{1}{2}}$

$= (M_1M_2)^{\frac{1}{2}}\|S^*IT^*TS\|^{\frac{1}{2}}$

$= (M_1M_2)^{\frac{1}{2}}\|S^*T^*TS\|^{\frac{1}{2}}$

$= (M_1M_2)^{\frac{1}{2}}\left\|\frac{1}{|\lambda|^2}T^*S^*ST\right\|^{\frac{1}{2}}$

$= (M_1M_2)^{\frac{1}{2}}\ \frac{1}{|\lambda|}\ \|(ST)^*(ST)\|^{\frac{1}{2}}$

$= (M_1M_2)^{\frac{1}{2}}\ \frac{1}{|\lambda|}\ \|ST\|$

Therefore, we get $\|ST\| \leq (M_1M_2)^{\frac{1}{2}}\ \frac{1}{|\lambda|}\ \|ST\|$, *which implies* $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$.

ii) By similar way, we can obtain $|\lambda| \geq (M_1M_2)^{-\frac{1}{2}}$ *and* $|\mu| \leq (M_1M_2)^{\frac{1}{2}}$*, if S and S* are (h, M1)-hyponormal operator and T* is (h, M2)-hyponormal operator.* ■

The following theorem shows that the product of (h, M)-hyponormal operators is (h, M)-hyponormal operator by using the (λ, μ)- commuting operator equations.

Theorem 5: -

Let $S: H \to H$ *be (h, M1)-hyponormal operator and* $T: H \to H$ *be (h, M2)-hyponormal operator, such that S and T are (λ, μ)-commuting operators. If |μ| ≤ 1, then ST and TS are (h, M)-hyponormal operators, where M = M1M2.*

Proof: -

$(ST)(ST)^*h = STT^*S^*h$

$= \lambda TST^*S^*h$ *(Since ST=λ TS)*

$= \lambda\mu TT^*SS^*h$ *(Since ST*=μ T*S)*

$\leq \lambda\mu M_1TT^*hS^*S$ *(Since S is (h, M1)-hyponormal operator)*

$\leq \lambda\mu M_1M_2hT^*TS^*S$ *(Since T is (h, M2)-hyponormal operator)*

$= \lambda|\mu|^2MhT^*S^*TS$

$= |\mu|^2MhT^*S^*ST$

$\leq Mh(ST)^*(ST)$ *(Since |μ| ≤ 1)*

Therefore; ST is (h, M)-hyponormal operator.

Similarly, TS is (h, M)-hyponormal operator. ■

Theorem 6: -

Let $S, T: H \to H$ *be (λ, μ)-commuting operators, such that S* is (h, M1)-hyponormal operator, T is (h, M2)-hyponormal operator and |λ| ≥ 1, then S*T and TS* are (h, M)-hyponormal operators, where M =M1M2 .*

Proof: -

$(S^*T)(S^*T)^*h = S^*TT^*Sh$

$= \frac{1}{\bar{\mu}}TS^*T^*Sh$ *(Since ST*=μ T*S)*

$= \frac{1}{\bar{\mu}\bar{\lambda}}TT^*S^*Sh$ *(Since ST=λ TS)*

$\leq \frac{1}{\bar{\mu}\bar{\lambda}}M_1TT^*hSS^*$ *(Since S* is (h, M1)-hyponormal operator)*

$\leq \frac{1}{\bar{\mu}\bar{\lambda}}M_1M_2hT^*TSS^*$ *(Since T is (h, M2)-hyponormal operator)*

$= \frac{1}{\bar{\mu}|\lambda|^2}MhT^*STS^*$

$= \frac{1}{|\lambda|^2}Mh(S^*T)^*(S^*T)$

$\leq Mh(S^*T)^*(S^*T)$ *(Since |λ| ≥ 1)*

*Hence, S*T is (h, M)-hyponormal operator.*

Similarly, we can get TS is (h, M)-hyponormal operator.* ■

Conclusions

in this article, the Solvability of (λ, μ)-commuting operator equations for (N, k)-hyponormal operators and (h, M)-hyponormal operators have been found. The (N, k)-hyponormal operators and (h, M)-hyponormal operators are new generalizations of the hyponormal operators. Further, we use the (λ,μ)-commuting operator equations for having some properties of these concepts. We show that the product of two (N, k)-hyponormal operator is (N,k)-hyponormal operator and the product of two (h, M)-hyponormal operators is (h, M)-hyponormal operator.

References

[1] B. P. Duggal and I. H. Jeon, “On p-quasi-hyponormal operators,” *Linear Algebra Appl.*, vol. 422, no. 1, pp. 331–340, 2007.

[2] A. Gupta and K. Mamtani, “Weyl type theorems for unbounded hyponormal operators,” *Kyungpook Math. J.*, vol. 55, no. 3, pp. 531–540, 2015.

[3] A. Tajmouati, A. El Bakkali, and M. B. Ahmed, “ON {\lambda}-Commuting Operators,” *arXiv Prepr. arXiv1607.06747*, 2016.

[4] V. Lauric, “Remarks on hyponormal operators and almost normal operators.,” *Le Mat.*, vol. 72, no. 1, pp. 3–8, 2017.

[5] J.-T. Yuan and C.-H. Wang, “Fuglede–Putnam type theorems for (p, k) $(p, k) $-quasihyponormal operators via hyponormal operators,” *J. Inequalities Appl.*, vol. 2019, no. 1, pp. 1–11, 2019.

[6] I. F. Z. Bensaid, S. Dehimi, B. Fuglede, and M. H. Mortad, “The Fuglede theorem and some intertwining relations,” *Adv. Oper. Theory*, vol. 6, no. 1, pp. 1–8, 2021.

[7] E. Kreyszig, “Introductory functional analysis with applications. Jonh Wiley & Sons,” *Inc., New York*, 1978.

[8] M. H. Mortad, *An operator theory problem book*. World Scientific, 2018.

[9] A. Benali and M. H. Mortad, “Generalizations of Kaplansky’s Theorem Involving Unbounded Linear Operators,” *Bull. Polish Acad. Sci. Math.*, vol. 62, no. 2, pp. 181–186, 2014.

[10] K. Rasimi, A. Ibraimi, and L. Gjoka, “Notes on λ-commuting operators,” *Int. J. Pure. Appl. Math*, vol. 91, pp. 191–196, 2014.