DOI:10.24996/ijs.2018.59.3A.18

ISSN: 0067-2904

A new approximate solution for the Telegraph equation of space-fractional order derivative by using Sumudu method

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Abstract:
In this work, we are concerned with how to find an explicit approximate solution (AS) for the telegraph equation of space-fractional order (TESFO) using Sumudu transform method (STM). In this method, the space-fractional order derivatives are defined in the Caputo idea. The Sumudu method (SM) is established to be reliable and accurate. Three examples are discussed to check the applicability and the simplicity of this method. Finally, the Numerical results are tabulated and displayed graphically whenever possible to make comparisons between the AS and exact solution (ES).

Keywords: Fractional calculus; Caputo derivative; Sumudu transforms; Telegraph equation.

الحل التقريبي لمعادلة تلغراف ذات رتبة الفراغ الكسرية باستخدام طريقة سومودو

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الخلاصة
في هذا العمل ، كان الاهتمام هو كيفية العثور على حل تقريبي صريح لمعادلة التلغراف ذات رتبة الفراغ الكسرية باستخدام طريقة تحويل سومودو. في هذه الطريقة ، يتم وصف مشتقات رتب الفراغ الكسرية باستخدام مفهوم كابوتو.وجدنا بان طريقة سومودو لها موثوقية ودقيقة. تم مناقشة ثلاثة أمثلة للتحقق من قابلية وبساطة هذه الطريقة. وأخيرًا ، يتم جدولة النتائج العددية وعرضها بيانياً كلما أمكن ذلك لإجراء مقارنات بين الحلول التقريبية والحقيقية.

1. Introduction

Fractional calculus is a branch of mathematics that study the qrows out of the traditional concepts of calculus integral and derivative operators in much the same way fractional exponents is a conclusion of exponents with an integer value [1], many researchers have suggested several applications of fractional calculus in various areas such as, chemistry, physics, plasma physics, engineering, stochastic dynamical system, turbulence and fluid mechanics, and nonlinear control problems [2-6]. Several techniques of Numerical and analytical methods have been developed for solving the application of fractional calculus, some of these methods are, Shifted Jacobi with tau

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method [7], Sinc-Jacobi and Sinc-Legendre with Collocation method [8-9], Chebyshev wavlets [10], Legendre and Chebyshev wavlets- Collocation method [11], Chebyshev wavelets-Galerkin Method [12].variational iteration method [13], and homotopy method [14].

Telegraph equations are PDEs have recently been considered by many applications in several fields such as random walk theory, electrical signals analysis [15-17],…,etc. Various methods are developed for solving Telegraph equations of fractional order, some of these methods are given by Momani [18], Yildirim [19], Chen et al. [20], Huang [21], Mohammed et al. [22].

Watugala in 1993[23] proposed a STM which is used to solve an integral transform called the Sumudu transform (ST) and he used this method in control problems. So far, various methods are used together with ST, like homotopy analysis SM [24], Sumudu decomposition method (SDM) [25], a modified homotopy algorithm [26]. STM [27-29] which will be recently submitted to the literature will be a suitableness technobabble for solving various kinds of ordinary differential equations of fractional order (DEFO).

2. Preliminaries

The most frequently encountered definition of fractional integration and fractional derivative are the Riemann-Liouville (RL) fractional integration and Caputo fractional derivative (CFD). Comparatively, the CFD has certain advantages when trying to model real-world phenomena with DEFO. Also, we provide some basic definition and properties of STM.

Definition (2.1) [4]

The RL fractional integral operator J^v of order $v > 0$, of a function $f(t)$, is the most popular definition of fractional calculus is defined as:

$$J^v f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} f(\tau) d\tau, \ t > 0 \tag{1}$$

$$J^0 f(t) = f(t)$$

Properties of the operator J^v can be found in [4] for $v, \delta \geq 0$, and $\gamma > -1$ we have:

$J^v J^\delta f(t) = J^{v+\delta} f(t) = J^\delta J^v f(t)$;

$J^v t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(v+\gamma+1)} t^{v+\gamma}$.

Definition (2.2) [4]

The CFD operator D_t^v of order v is:

$$D_t^v f(t) = J^{n-v} D^n f(t) = \frac{1}{\Gamma(n-v)} \int_0^t (t-\tau)^{n-v-1} f^{(n)}(\tau) d\tau, \ v > 0 \tag{2}$$

For $n-1 < v \leq n$, $n \in \mathbb{N}$, $t > 0$.

Definition (2.3) [4]

The Mittage Leffler function $E_v(Z)$ with $v > 0$, is definite by the following series:

$$E_v(Z) = \sum_{n=0}^{\infty} \frac{Z^v}{\Gamma(nv+1)}, \ v > 0, Z \in \mathbb{C} \tag{3}$$

Definition (2.4) [30]

The ST over the following set of functions, such that:

$$A = \{f(x) \mid \exists\, M, \xi_1, \xi_2 > 0, \text{Such that}, |f(x)| < M \exp(\frac{|x|}{\xi_j}), \text{if } x \in (-1)^j \times [0, \infty)\} \tag{4}$$

is define as:

$$H(u) = \mathcal{S}[f(x)] = \int_0^\infty f(ux)\, e^{-x} dx, \ u \in (\xi_1, \xi_2) \tag{5}$$

The properties of the ST are given as:

1. $\mathcal{S}[1] = 1$.
2. $\mathcal{S}[\frac{t^n}{\Gamma(n+1)}] = u^n$, $n > 0$;
3. $\mathcal{S}[\exp(at)] = \frac{1}{1-au}$.
4. $\mathcal{S}[\alpha f(x) \pm \beta g(x)] = \alpha \mathcal{S}[f(x)] \pm \beta \mathcal{S}[g(x)]$.

Theorem (2.1) [30]

Let $H(u)$ be the ST of $f(x)$, s.t

1. $\frac{H(1/s)}{s}$, is a meromorph function, with singularities $\text{Re}(s) < \gamma$,
2. There exists a circular region Γ with radius R and positive constants, $M, k > 0$, with $|\frac{H(1/s)}{s}| < M R^{-k}$,

So, the Sumudu inverse (SI) of the function f(x) is introduced by:

$$\mathcal{S}^{-1}[H(s)] = f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp(sx)H(\tfrac{1}{s})\frac{ds}{s} = \sum residues[\exp(sx)\frac{H(1/s)}{s}]. \quad (6)$$

Definition (2.5) [30]

The ST $\mathcal{S}\,[D_x^v f(x)]$ of the fractional derivative using the Caputo idea of the function $f(x)$ is given by:

$$\mathcal{S}\,[D_x^v f(x)] = \frac{H(u)}{u^v} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{v-k}}, \text{ where } H(u) = \mathcal{S}[f(x)] \quad (7)$$

It is easy to understand that:

$$\mathcal{S}\,[D_t^v f(x,t)] = \frac{\mathcal{S}\,[f(x,t)]}{u^v} - \sum_{k=0}^{n-1}\frac{f^{(k)}(x,0)}{u^{v-k}}, \; n-1 < v \le n \quad (8)$$

3. The direct approach for solving linear TESFO using SDM.

We consider the following linear TESFO of the form:

$$D_x^v u(x,t) = au_t + u_{tt} + bu(x,t) + g(x,t) \quad 0 < x < 1 \quad (9)$$

$t \ge 0, 0 < v \le 2$.

Where $g(x,t)$ is the source term and a, b are constants.

With Initial Condition (I.C)

$$\frac{\partial^{(r)}u(0,t)}{\partial x^r} = u^{(r)}(0,t)|_{t=0} = f_r(t), \; r = 0, 1, 2,\ldots, -1. \quad (10)$$

Now applying the ST into Eq.(9) we have:

$$\mathcal{S}[D_x^v u(x,t)] = \mathcal{S}[au_t + u_{tt} + bu(x,t)] + \mathcal{S}[g(x,t)] \quad (11)$$

Substituting Eq.(8) into Eq.(11) we get:

$$u^{-v}\mathcal{S}[u(x,t)] - \sum_{k=0}^{m-1} u^{-(v-k)}\, u^{(k)}(0,t) = \mathcal{S}[au_t + u_{tt} + bu(x,t)] + \mathcal{S}[g(x,t)] \quad (12)$$

$$\mathcal{S}[u(x,t)] = \sum_{k=0}^{m-1} u^k f_k(t) + u^v\mathcal{S}[au_t + u_{tt} + bu(x,t)] + u^v\mathcal{S}[g(x,t)] \quad (13)$$

So, according to SDM we can obtain the solution result $u(x,t)$ as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (14)$$

Now, substituting Eq.(14) into Eq.(13) gives:

$$\mathcal{S}[\sum_{n=0}^{\infty} u_n(x,t)] = \sum_{k=0}^{m-1} u^k f_k(t) + u^v\mathcal{S}[a\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_t + \left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{tt} + b\sum_{n=0}^{\infty} u_n(x,t)] + u^v\mathcal{S}[g(x,t)] \quad (15)$$

From Eq.(15) we can define all the coefficients of $u_{n+1}(x,t)$

So we get the zero coefficients $u_0(x,t)$ as:

$$\mathcal{S}\;[u_0(x,t)] = \sum_{k=0}^{m-1} u^k f_k(t)$$

The first component $u_1(x,t)$ as:

$$\mathcal{S}\;[u_1(x,t)] = u^v\mathcal{S}[a(u_0(x,t))_t + (u_0(x,t))_{tt} + bu_0(x,t) + g(x,t)]$$

Finally the remaining coefficients of u_n(x, t) can be find in a way like each coefficients is found by using the coming before components.

$$\mathcal{S}\,[u_{n+1}(x,t)] = u^v\mathcal{S}[a(u_n(x,t))_t + (u_n(x,t))_{tt} + bu_n(x,t)], n \ge 1.$$

Applying the SI to the above equations yields the following:

$$u_0(x,t) = \mathcal{S}^{-1}(\sum_{k=0}^{m-1} u^k f_k(t))$$

$$u_1(x,t) = \mathcal{S}^{-1}(u^v\mathcal{S}[a(u_0(x,t))_t + (u_0(x,t))_{tt} + bu_0(x,t) + g(x,t)])$$

$$u_{n+1}(x,t) = \mathcal{S}^{-1}(u^v\mathcal{S}[a(u_n(x,t))_t + (u_n(x,t))_{tt} + bu_n(x,t)]), n \ge 1.$$

So that, the AS $u_n(x,t;v)$ is given as:

$$\mathrm{u}_n(x,t) = \sum_{j=0}^{n-1} u_j(x,t) \quad (16)$$

Such that

$$\lim_{n\to\infty} \mathrm{u}_n(x,t) = u(x,t) \quad (17)$$

4. Numerical Examples

In this section we shall test three examples using the STM to solve the TESFO and the solutions we got it by using the present procedure will be comparing with original ES.

Example (1) [22]: consider the following homogeneous TESFO

$$D_x^v u(x,t) = u_{tt} + u_t + u, \; x,t \ge 0, \; 0 < v \le 2, \quad (18)$$

With the I. C.

$$\begin{cases} u(0,t) = e^{-t}, & t \ge 0 \\ u_x(0,t) = e^{-t}, & t \ge 0 \end{cases} \quad (19)$$

Now applying the ST with Eq.(8) into Eqs.(18-19) we get:

$$\mathcal{S}\,[u(x,t)] = e^{-t} + xe^{-t} + u^v\mathcal{S}[(u(x,t))_{tt} + (u(x,t))_t + u(x,t)] \quad (20)$$

So, according to SDM we can obtain the solution result $u(x,t)$ as:

$u(x,t) = \sum_{n=0}^{\infty} u_n(x,\mathrm{t})$

Now, substituting Eq.(14) into Eq.(20) gives

$$\mathcal{S}[\sum_{n=0}^{\infty} u_n(x,t)]=e^{-t}+xe^{-t}+u^v\mathcal{S}[\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{tt}+\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_t+\sum_{n=0}^{\infty} u_n(x,t)] \quad (21)$$

From Eq.(21) we can define all the coefficients of $u_{n+1}(x,t)$

So we get the zero coefficients $u_0(x,t)$ as:

$$\mathcal{S}[u_0(x,t)] = e^{-t}+xe^{-t} \quad (22)$$

The first component $u_1(x,t)$ as:

$$\mathcal{S}[u_1(x,t)] = u^v\mathcal{S}[(u_0(x,t))_{tt}+(u_0(x,t))_t+u_0(x,t)] \quad (23)$$

Finally the remaining coefficients of $u_n(\mathrm{x},\mathrm{t})$ can be find in a way like each coefficients is found by using the coming before components.

$$\mathcal{S}[u_{n+1}(x,t)] = u^v\mathcal{S}[(u_n(x,t))_t+(u_n(x,t))_t+u_n(x,t)] \quad (24)$$

So, we can use the SI in Eq.(22) we get

$$u_0(x,t) = \mathcal{S}^{-1}[e^{-t}+xe^{-t}] = e^{-t}(1+x) \quad (25)$$

Also,

$$\mathcal{S}[u_1(x,t)] = e^{-t}(u^v+u^{v+1}) \quad (26)$$

Also, by using SI to Eq.(26) we have:

$$u_1(x,t) = e^{-t}\left(\frac{x^v}{\Gamma(v+1)}+\frac{x^{v+1}}{\Gamma(v+2)}\right)$$

Similarly,

$$u_2(x,t) = e^{-t}\left(\frac{x^{2v}}{\Gamma(2v+1)}+\frac{x^{2v+1}}{\Gamma(2v+2)}\right)$$

$$\vdots$$

$$u_{n+1}(x,t) = e^{-t}\left(\frac{x^{nv}}{\Gamma(nv+1)}+\frac{x^{nv+1}}{\Gamma(nv+2)}\right)$$

Therefore, the AS by STM is shown as:

$$\mathrm{u}_n(x,t) = e^{-t}\left(1+x+\frac{x^v}{\Gamma(v+1)}+\frac{x^{v+1}}{\Gamma(v+2)}+\frac{x^{2v}}{\Gamma(2v+1)}+\frac{x^{2v+1}}{\Gamma(2v+2)}+\cdots\right) \quad (27)$$

If we put $v = 2$ in Eq.(27), we can conclude the ES [22].

$$u(x,t) = e^{-t}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right) = e^{-t+x} \quad (28)$$

The Absolute Error (AE), $|u(x,t)-\mathrm{u}_n(x,t)|$ between the ES $u(x,t)$ and the AS $\mathrm{u}_n(x,t)$ by using 3-term of the STM when $v = 1.75$, 1.9 by fixing $t = 1$ and different values of x, are given in Tables-(1, 2). Also Figures-(1, 2) shows the AS using 3-term of the STM when $v = 1.75$ and 1.9.

Table1-The AE for t =1 and x=0.1:0.1:1.0 when $v = 1.75$.

x	$u(x,t)$	$\mathrm{u}_n(x,t)$	**AE**
0.100	**0.4065697**	**0.5010931**	**9.4523405e-02**
0.200	**0.4493290**	**0.5530201**	**1.0369115e-01**
0.300	**0.4965853**	**0.6124642**	**1.1587893e-01**
0.400	**0.5488116**	**0.6788092**	**1.2999758e-01**
0.500	**0.6065307**	**0.7516633**	**1.4513259e-01**
0.600	**0.6703200**	**0.8307437**	**1.6042362e-01**
0.700	**0.7408182**	**0.9158318**	**1.7501357e-01**
0.800	**0.8187308**	**1.0067508**	**1.8802005e-01**
0.900	**0.9048374**	**1.1033531**	**1.9851570e-01**
1.000	**1.0000000**	**1.2055127**	**2.0551274e-01**

Table 2-The AE for t =1 and x=0.1:0.1:1.0 when $v = 1.9$.

x	$u(x,t)$	$u_n(x,t)$	AE
0.100	**0.4065697**	**0.4822658**	**7.5696157e-02**
0.200	**0.4493290**	**0.5297864**	**8.0457410e-02**
0.300	**0.4965853**	**0.5837051**	**8.7119841e-02**
0.400	**0.5488116**	**0.6437983**	**9.4986649e-02**
0.500	**0.6065307**	**0.7099152**	**1.03386649e-01**
0.600	**0.6703200**	**0.7819428**	**1.1162278e-01**
0.700	**0.7408182**	**0.8597909**	**1.1897270e-01**
0.800	**0.8187308**	**0.9433846**	**1.2465385e-01**
0.900	**0.9048374**	**1.0326600**	**1.2782260e-01**
1.000	**1.0000000**	**1.1275616**	**1.2756158e-01**

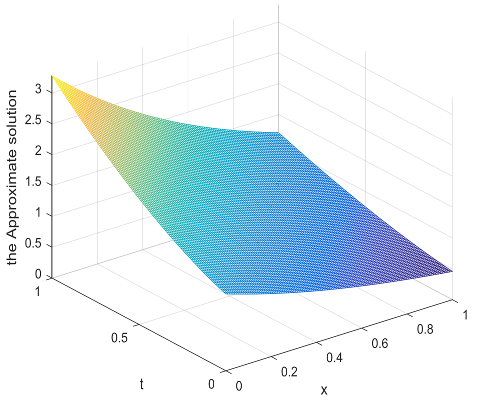


Figure 1-The surface shows the AS for $v = 1.9$ of Example 1.

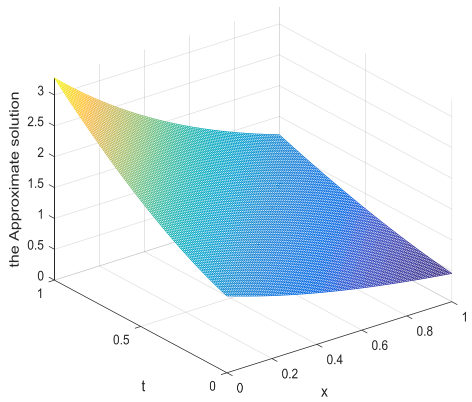


Figure 2-The surface show AS for $v = 1.75$ of Example 1

Example (2)[22]: consider the following nonhomogeneous TESFO

$$D_x^v u(x,t) = u_{tt} + u_t + u - x^2 - t + 1, \; t \geq 0, \quad 0 < x \leq 1, \; 0 < v \leq 2 \,, \tag{29}$$

With the I.C.

$$\begin{cases} u(0,t) = t \;, & t \geq 0 \\ u_x(0,t) = 0, & t \geq 0 \end{cases} \tag{30}$$

Now applying the ST with Eq.(8) into Eqs.(29-30) we get:

$$\mathcal{S}\,[u(x\,,t)] \;= t + u^v \mathcal{S}[(u(x\,,t))_{tt} + (u(x\,,t))_t + u(x,t) - x^2 - t + 1]. \tag{31}$$

So, according to SDM we can obtain the solution result $u(x\,,t)$ as:

$$u(x\,,t) = \sum_{n=0}^{\infty} u_n\,(x,\mathrm{t})$$

Now, substituting Eq.(14) into Eq.(31) gives

$$\mathcal{S}[\sum_{n=0}^{\infty} u_n(x,t)] = t + u^v \mathcal{S}[\left(\sum_{n=0}^{\infty} u_n\,(x,t)\right)_{tt} + \left(\sum_{n=0}^{\infty} u_n\,(x,t)\right)_t + \sum_{n=0}^{\infty} u_n(x,t) - \; x^2 - t + 1] \tag{32}$$

From Eq.(32) we can define all the coefficients of $u_{n+1}(x\,,t)$

So we get the zero coefficients $u_0(x\,,t)$ as:

$$\mathcal{S}[u_0(x\,,t)] \;= t - 2u^{v+2} - tu^v + u^v \tag{33}$$

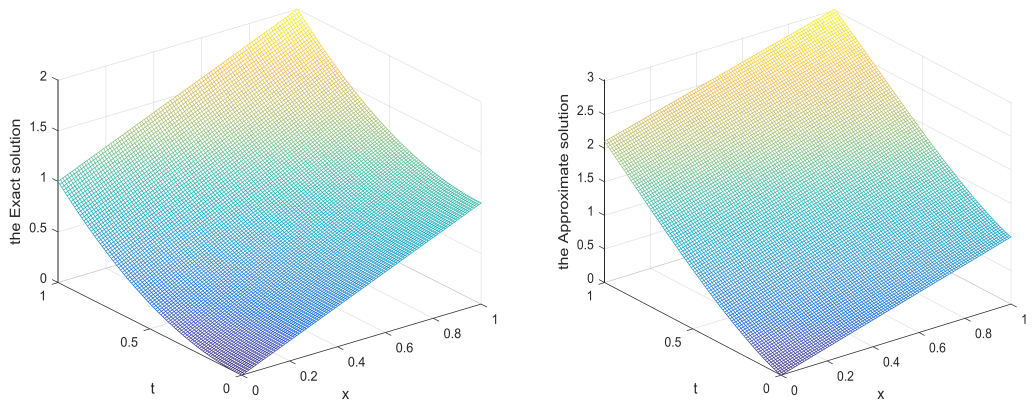
The first component $u_1(x\,,t)$ as:

$$\mathcal{S}[u_1(x\,,t)] = \; u^v \mathcal{S}[(u_0\,(x\,,t))_{tt} + (u_0\,(x\,,t))_t + u_0(x,t)] \tag{34}$$

Finally the remaining coefficients of $u_n(\mathrm{x}\,,\mathrm{t})$ can be find in a way like each coefficients is found by using the coming before components.

$$\mathcal{S}[u_{n+1}(x,t)] = u^{v}\mathcal{S}[(u_n(x,t))_{tt} + (u_n(x,t))_t + u_n(x,t)] \tag{35}$$

Now, by using the SI to Eq.(33) we have:



(a) **(b)**

Figure 3-The comparison between (a) the ES and (b) the AS using 3-term of the STM for v = 1.2 of Example 2.

$$u_0(x,t) = t - \frac{2x^{v+2}}{\Gamma(v+3)} - \frac{tx^{v}}{\Gamma(v+1)} + \frac{x^{v}}{\Gamma(v+1)} \tag{36}$$

Also,

$$\mathcal{S}[u_1(x,t)] = u^{v} - u^{2v} + tu^{v} - 2u^{2v+2} - tu^{2v} + u^{v} \tag{37}$$

Also, by using SI of Eq.(37) we have:

$$u_1(x,t) = \frac{x^{v}}{\Gamma(v+1)} - \frac{x^{2v}}{\Gamma(2v+1)} + \frac{tx^{v}}{\Gamma(v+1)} - \frac{2x^{2v+2}}{\Gamma(2v+3)} - \frac{tx^{2v}}{\Gamma(2v+1)} + \frac{x^{2v}}{\Gamma(2v+1)}$$

Similarly,

$$u_2(x,t) = \frac{2x^{2v}}{\Gamma(2v+1)} - \frac{x^{3v}}{\Gamma(3v+1)} + \frac{tx^{2v}}{\Gamma(2v+1)} - \frac{2x^{3v+2}}{\Gamma(3v+3)} - \frac{tx^{3v}}{\Gamma(3v+1)}$$

And so on.

Therefore, the AS by STM is given by:

$$u_n(x,t) = t - \frac{2x^{v+2}}{\Gamma(v+3)} - \frac{tx^{v}}{\Gamma(v+1)} + \frac{x^{v}}{\Gamma(v+1)} + \frac{x^{v}}{\Gamma(v+1)} - \frac{x^{2v}}{\Gamma(2v+1)} + \frac{tx^{v}}{\Gamma(v+1)} - \frac{2x^{2v+2}}{\Gamma(2v+3)} - \frac{tx^{2v}}{\Gamma(2v+1)} + \frac{x^{2v}}{\Gamma(2v+1)} + \frac{2x^{2v}}{\Gamma(2v+1)} - \frac{x^{3v}}{\Gamma(3v+1)} + \frac{tx^{2v}}{\Gamma(2v+1)} - \frac{2x^{3v+2}}{\Gamma(3v+3)} - \frac{tx^{3v}}{\Gamma(3v+1)} + \cdots \tag{38}$$

If we put $v = 2$ in Eq.(38), we get the required ES [22].

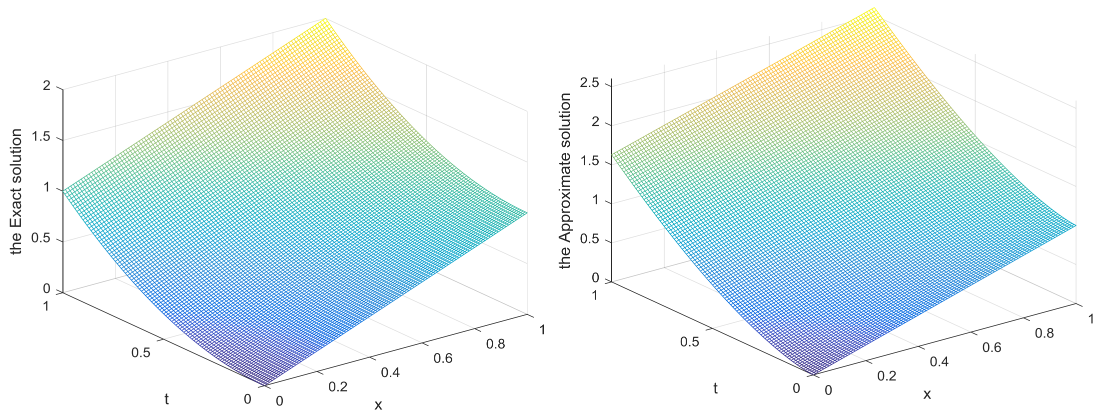
$$u(x,t) = \left(t + \frac{2x^2}{2!} - \frac{2x^4}{4!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \frac{2x^6}{6!} - \frac{2x^8}{8!} + \frac{2x^8}{8!} + \frac{2x^8}{8!} + \cdots\right) = t + x^2 \tag{39}$$

Table-3 show the AE between the ES $u(x,t)$ and the AS $u_n(x,t)$ using 3-term of the STM when $v = 1.9$ by fixing $t = 1$ and different values of x.

Figures-(3, 4 and 5) shows the comparison between the ES $u(x,t)$ and the AS $u_n(x,t)$ using 3-term of the STM when $v = 1.2, 1.5$ and 1.9.

Table 3-The Absolute error for t =1 and x=0.1:0.1:1.0 when $v = 1.9$

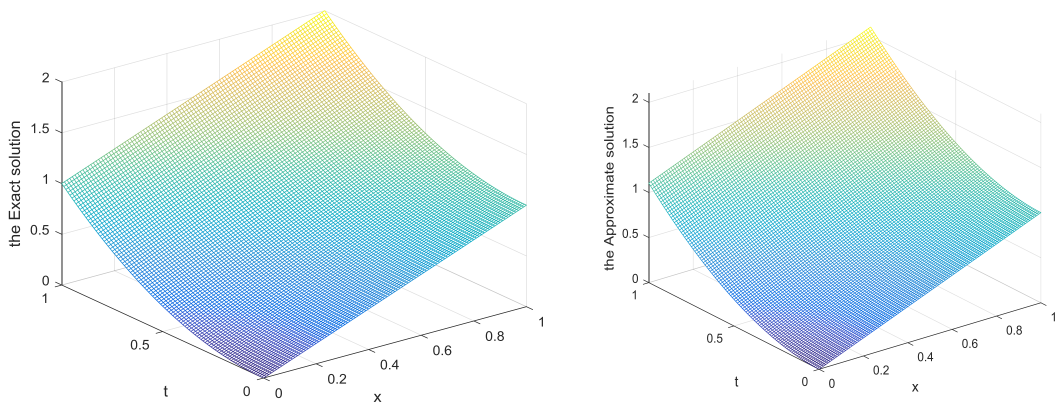
x	$u(x,t)$	$u_n(x,t)$	AE
0.100	**1.0100**	**1.01377**	**3.779642e-3**
0.200	**1.0400**	**1.05145**	**1.14520e-02**
0.300	**1.0900**	**1.11112**	**2.12469e-02**
0.400	**1.1600**	**1.19232**	**3.23258e-02**
0.500	**1.2500**	**1.29413**	**4.41316e-02**
0.600	**1.3600**	**1.41622**	**5.62230e-02**
0.700	**1.4900**	**1.55581**	**6.81921e-02**
0.800	**1.6400**	**1.71960**	**7.96092e-02**
0.900	**1.8100**	**1.89997**	**8.99776e-02**
1.000	**2.0000**	**2.09869**	**9.86922e-02**



(a)

(b)

Figure 4-The comparison between (a) the ES and (b) the AS using 3-term of the STM for $v = \mathbf{1.5}$ of Example 2



(a) **(b)**

Figure 5-The comparison between (a) the ES and (b) the AS using 3-term of the STM for $v = \mathbf{1.9}$ of Example 2.

Example (3)[31]: consider the following homogeneous TESFO

$$D_x^{2v}u(x,t) = u_{tt} + 4u_t + 4u\ , t \geq 0,\ \ 0 < x, \leq 1,\ \ 0 < v \leq 1, \quad (40)$$

With the I.C.

$$\begin{cases} u(0,t) = e^{-2t} + 1 , & t \geq 0 \\ u_x(0,t) = 2, & t \geq 0 \end{cases} \quad (41)$$

Now applying the ST with Eq.(8) into Eqs.(40-41) we get:

$$\mathcal{S}\,[u(x\,,t)] \;= e^{-2t} + \;1 + 2u + u^{2v}\mathcal{S}[(u(x\,,t))_{tt} + 4(u(x\,,t))_t + 4u(x,t)] \quad (42)$$

So, according to SDM we can obtain the solution result $u(x\,,t)$ as:

$$u(x\,,t) = \sum_{n=0}^{\infty} u_n\,(x, \mathrm{t})$$

Now, substituting Eq.(14) into Eq.(42) gives

$$\mathcal{S}[\sum_{n=0}^{\infty} u_n(x,t)]{=}t{+}u^{2v}\mathcal{S}[\big(\sum_{n=0}^{\infty} u_n\,(x,t)\big)_{tt} + 4\big(\sum_{n=0}^{\infty} u_n\,(x,t)\big)_t + 4\sum_{n=0}^{\infty} u_n(x,t)] \quad (43)$$

From Eq.(43) we can define all the coefficients of $u_{n+1}(x\,,t)$

So we get the zero coefficients $u_0(x\,,t)$ as:

$$\mathcal{S}[u_0(x\,,t)] \;= e^{-2t} + \;1 + 2u \quad (44)$$

The first component $u_1(x\,,t)$ as:

$$\mathcal{S}[u_1(x\,,t)] = \;u^{2v}\mathcal{S}[(u_0(x\,,t))_{tt} + +4(u_0(x\,,t))_t + 4u_0(x,t)] \quad (45)$$

Finally the remaining coefficients of $u_n(\mathrm{x}\,,\mathrm{t})$ can be find in a way like each coefficients is found by using the coming before components.

$$\mathcal{S}[u_{n+1}(x\,,t)] = \;u^{2v}\mathcal{S}[(u_n(x\,,t))_{tt} + 4(u_n(x\,,t))_t + 4u_n(x,t)] \quad (46)$$

So, by using the SI in Eq.(44) we have:

$$u_0(x\,,t) = e^{-2t} + \;1 + 2x \quad (47)$$

Also,

$$\mathcal{S}[u_1(x\,,t)] = \;4u^{2v} + 8u^{2v+1} \quad (48)$$

Also, by using SI of Eq.(48) we have:

$$u_1(x\,,t) = \frac{4x^{2v}}{\Gamma(2v+1)} + \frac{8x^{2v+1}}{\Gamma(2v+2)}$$

Similarly,

$$u_2(x\,,t) = \frac{16x^{4v}}{\Gamma(4v+1)} + \frac{32x^{4v+1}}{\Gamma(4v+2)}$$

And so on.

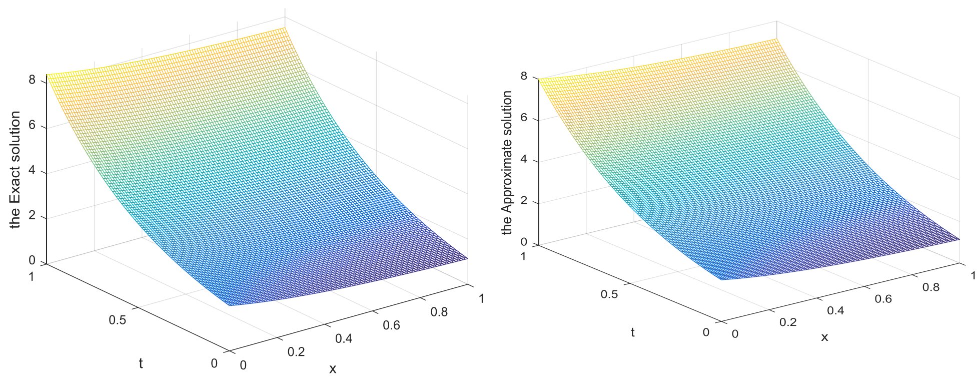
Therefore, the AS by STM is given by:

$$\mathrm{u}_n(x\,,t) = e^{-2t} + 1 + 2x + \frac{4x^{2v}}{\Gamma(2v+1)} + \frac{8x^{2v+1}}{\Gamma(2v+2)} + \frac{16x^{4v}}{\Gamma(4v+1)} + \frac{32x^{4v+1}}{\Gamma(4v+2)} + \cdots \quad (49)$$

If we put $v = 1$ in Eq.(49), we get the required ES [31].

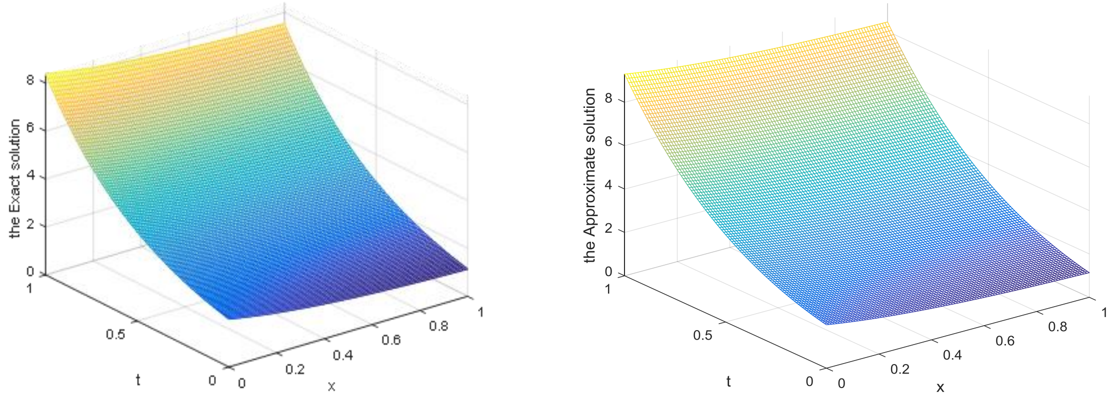
$$u(x\,,t\,) = e^{-2t} + \left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \cdots + \frac{(2x)^n}{n!} + \cdots\right)$$
$$= e^{-2t} + e^{2x} \quad (50)$$

The graph of ES $u(x\,,t\,)$ and the AS $\mathrm{u}_n(x\,,t)$ using 3-term of the STM when $v = 0.8,\; 0.9$ and 1, are shown in Figures-(6, 7 and 8).



(a) **(b)**

Figure 6: The comparison between (a) the ES and (b) the AS using 3-term of the STM for $v = \mathbf{1}$ of Example 3.



(a) **(b)**

Figure 7-The comparison between (a) the ES and (b) the AS using 3-term of the STM for $v = \mathbf{0.9}$ of Example 3.

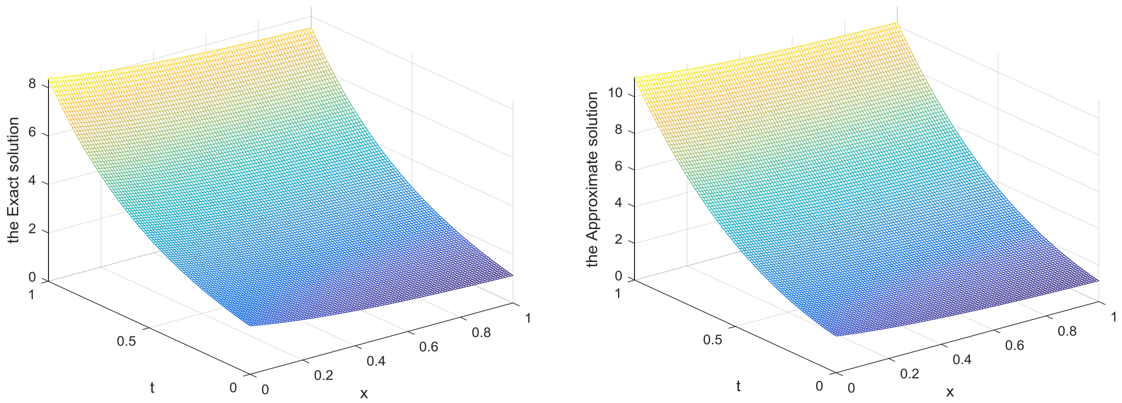


Figure 8-The comparison between (a) the ES and (b) the AS using 3-term of the STM for $v = \mathbf{0.8}$ of Example 3.

Conclusion

The application of STM was extended successfully for solving the TESFO. The STM was clearly very efficient and powerful technique in finding the AS of the proposed equations. In order to check the effectiveness of the introduced procedure, three numerical examples are tested, by comparing the AS with the ES. A critical advantage of the new approach will be about its low computational load.

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