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# New Results of Normed Approach Space

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**Abstract**
In this work, we introduce a new convergence formula. We also define cluster point , δ-Cauchy sequence, δ-convergent, δ-completeness , and define sequentially contraction in approach space. In addition, we prove the contraction condition is necessary and sufficient to get the function is sequentially contraction as well as we put a new structure for the norm in the approach space which is called approach – Banach space, we discuss the normed approach space with uniform condition is a Hausdorff space. Also, we prove a normed approach space is complete if and only if the metric generated from approach space is complete as well as prove every finite – dimensional approach normed space is δ-complete. We prove several results and properties in this field.

**Keywords**: Approach space, contraction, approach- metric space, approach vector space.

# نتائج جديدة للفضاء المعياري التقاربي

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**الخلاصه**
في الفضاء التقاربي ، نقدم صيغة تقارب جديدة ، نعرف نقطة التجمع ، المتتابعة كوشي – δ، التقارب –δ, الكمالية من النمط δ– ،عرفنا الانكماش التتابعي في الفضاء التقاربي. برهنا أن حالة الانكماش شرط ضروري وكافي لنحصل دالة منكمشة تتابعيا ، وكذلك وضعنا بنية جديدةً في فضاء الاقتراب التي تسمى فضاء بناخ التقاربي ، نناقش الفضاء المعياري التقاربي يكون هاوزدورف بوجود شرط الانتظام .أيضًا ، نثبت أن الفضاء المعياري التقاربي يكون كامل إذا وفقط إذا كان الفضاء المتري المتولد من الفضاء التقاربي يكون كامل وكذلك برهنا كل فضاء معياري تقاربي منتهي البعد يكون كامل . أثبتنا العديد من النتائج والخصائص في هذا المجال.

## 1.INTRODUCATION

In ( 1989), R.Lowen [4] studied a distance between points and sets in a metric space. In topological space one analogously has that the closure operator gives a distance between points and sets. In (1994), R. Baekeland and R. Lowen [7] studied the measures of Lindelof and separability in approach spaces. In (1996), R. Lowen [13] studied the development of the fundamental theory of approximation. In (1999), R.Lowen, Y. Jinlee [2] defined the notions of approach Cauchy structure and ultra-approach Cauchy structure. In (2000) and (2003) R. Lowen and M. Sioen [8,10] introduced definitions of some separation axioms in the approach

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spaces and found the relationship between them. In (2000), R. Lowen and B. Windels [14] defined an approach groups spaces, semi-group spaces, and uniformly convergent. In(2003) R. Lowen, M. Sion and D. Vaughan [3] defined a complete theory for all approach spaces with an underlying topology that agrees with the usual metric completion theory for metric spaces. In (2004), R. Lowen and S. Verwuwlgen [5] studied approach vector spaces. In (2004), R. Lowen, C. Van Olmen, and T. Vroegrijk [9] found the relationship between Functional ideas and Topological Theories. In (2006), G. C. L. Brümmer, M, Sion [16]developed abicompletion theory for the category of approach spaces in sense of Lowen [20]which extends the completion theory obtained in [14]. In(2009) J.Martnez-Moreno, A. Roldan and C. Roldan [17] defined the notion of Fuzzy approach spaces generalization of Fuzzy metric spaces and proved some properties of Fuzzy approach spaces. In (2009), R. Lowen and C.Van Olmen [11] discussed some notions and relations in approach Theory.In(2013) G. Gutierres. D. Hofmann [12] studied the notion of cocompleteness for approach spaces and proved some properties in cocompleteness approach space.In (2013) K.Van Opdenbosch [18] gave new isomorphic characterizations of approach spaces,pre-approach spaces, convergence approach spaces, uniform gauge spaces, topological spaces and convergence spaces, topological spaces, metric spaces, and uniform spaces.In (2014) R.Lowen, S.Sagiroglu [22] studied in this paper the possibility to weak the concept of approach spaces to incorporate not only topological and metric spaces but also closure spaces.In (2015)R.Lowen[6] in this book approach theory completely solves this by introducing precisely those two new types of numerically structured spaces which are required: approach spaces on the local level and uniform gauge spaces on the uniform level. In (2016) . R. Malčeski, A.Ibrahimi[21]In this paper is proven several generalizations of known theorems of fixed point, and theorems for common fixed points of mapping to 2-Banach space. In (2017) E.Colebunders, M. Sion[1] prove some important consequences on real-valued contractions. In (2017)and (2019), M. Baran and M. Qasim [20,22] characterized Local to distance-approach spaces, Approach spaces, and gauge-approach spaces and compared them with usual approach spaces. In (2018), W. Li, Dexue Zhang [21] introduced the Smyth complete.

We start from a normed approach space, so Banach approach space structure on X is introduced and its properties are investigated . Quantitative results are obtained, which imply their classical qualitative counterparts. These results provide an introduction to approach Banach spaces, which are complete normed approach vector spaces. All vector spaces are assumed to be over the real numbers. This paper is also introduced the concept of an approach to Banach space, and studied its category-theoretic properties. The extension Banach space by complete approach-normed space is also introduced. This leads to expand the space of the norm though. In addition, an additional condition on the norm structure is made, that is $\delta_{\|.\|}$(x,A):= $sup_{x\in X}\, inf_{a\in A}\|x-a\|$ . This means the distance generated by norm function between a point in approach space and a subset of power set. In this case, conditions and the function have been fulfilled, we have to find any Cauchy sequence convergent in approach space and the space has become approach Banach space. Some properties of Banach space are studied. The main goal of this paper is to find and to prove new results in convergent sequences in Approach spaces. We prove approach space is $\delta-$complete if and only if (X,$d_\delta$) is complete, we also define sequentially contraction and prove that it is equivalent contraction. Normed Approach Space is defined, and we prove many results such that every uniform app-normed space (X, $\|.\|, \delta_{\|.\|}$) is a Hausdorff, app-metric of weak approach distance $\delta_{X^*}$ is $d_{\|.\|}$ ,and if a sequence in normed approach $\delta-$convergent sequence in $X$ implies the sequence is bounded. Also, some new results in normed approach Space and Banach approach space are discussed. Futhermore, in this work, a new definition of convergent of cluster point in approach space, $\delta-$Cauch sequence, $\delta-$convergent and $\delta$ −complete are introduced. We prove that A function $f:(X,\delta) \to (Y,\ \delta')$ between approach spaces is a contraction if and only if $f$ sequentily contraction, we also discussed every finite –dimensional app-normed space is $\delta$ −complete and consequent app-Banach space. Also, the metric app-space (X, $d_\delta$) is not to be approach-normed space. A normed Approach space (X, $\|.\|, \delta_{\|.\|}$) is complete if and only if a metric approach space (E,$d_{\|.\|}$) is complete by means every uniform app-normed space (X, $\|.\|, \delta_{\|.\|}$) is a Hausdorff space. If (X, $\|.\|, \delta_{\|.\|}$) is normed Approach space, and $\{x_n\}$ is a $\delta$ −convergent sequence in $X$, then the sequence $\{x_n\}$ in X is norm bounded. New results in normed approach space and convergent are given.

This paper is divided into six sections: Section one introduces the introduction of the research. In section two, preliminaries with basic definitions are given. In section three, new results in convergent sequences in approach spaces are proved. We also explain the relationship complete and $\delta$ −complete in approach space. In section four, we introduce the definition of normed approach space and prove some results in normed approach Space. In section five, we introduce some results of normed approach space. Section six, we discuss the important conclusions of the research.

## 2. PRELIMINARIES

The metric space $(X,d)$ is a distance between pairs of a given points, and the distance between points and sets is given by the following formula:

$\delta(x,A) := \inf_{a\in A} d(x,a)$ for all $x \in X$ ,for all $A \in 2^X$.

For any subset $A$ of $X$ and any $\varepsilon \in [0,\infty]$ , $A^\varepsilon$is defined by $A^\varepsilon := \{x \in X \mid \delta(x,A) \le \varepsilon\}$.

**Definition 2.1[13]** Let $X$ be a non-empty set. A function $\delta: X \times 2^X \to [0,\infty]$ is called distance on $X$ if the following properties are satisfied:

(D1) $\forall\, x \in X: \delta(x,\{x\}) = 0$,

$(D2)$ $\forall\, x \in X: \delta(x,\emptyset) = \infty$,

$(D3)$ $\forall\, x \in X:\ \forall\, A,B \in 2^X : \delta\ (x, A\cup B) = \min\{\delta(x,A), \delta(x,B)\}$,

$(D4)$ $\forall x \in X: \forall\, A \in 2^X ,\forall \varepsilon \in [0,\infty]: \delta(x,A) \le \delta\big(x, A^{(\varepsilon)}\big) + \varepsilon$.

A pair (X, δ) is called an approach space and denoted by app-spaces, where δ is a distance.

Instead of (D4′) for all x$\in X$ *and* $A,B \in 2^X$, $\delta(x,A) \le \delta(x,B) +\ sup_{b\in B}\ \delta(b,A)$

$(D4')$ is equivalent to $(D4)$.

**Definition 2.2 [4]** Let$(\,X,\delta\,)$ and $(X',\delta')$ are App- spaces**.** A function $f: X \to Y$ is called contraction if for all $x \in X$, for all $A \in 2^X$: $\delta'(f\big((x), f(A)\big) \le \delta(x,A)$.

**Definition 2.3 [14]** A triple $(X,\delta,+)$ is called an approach semi-group if and only if

**1.** $(X,\delta)$ is an approach space **.**

**2.** $(X,*)$ is a semi-group.

**3.** $*: X \otimes X \to X : (x,y) \mapsto x * y$ is a contraction.

**Definition 2.4 [14]** A triple (X, $\delta,+$) is called an approach group if it satisfies the following:

(a) (X,$\delta$) is an approach space.

(b) (X,+) is agroup.

(c) $+: X \oplus X \to X: (x,y) \mapsto x + y$ is contraction.

(d) $\cdot$ :X→ $X$ : X→ $-X$ is contraction .

## 3. Some Properties of Convergent Sequences in Approach spaces.

**Definition 3.1.** If (X, d) is a metric space, then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be a left Cauchy sequence if for all $\varepsilon > 0, there\ exists\ k \in Z^+$such that $d(x_m,x_n) < \varepsilon, for\ all\ m,n \ge N, m \le n.$ Right Cuachy sequence if for all $\varepsilon > 0, there\ exists\ k \in Z^+$such that $d(x_n,x_m) < \varepsilon, for\ all\ m,n \ge N, n \le m.$ If a sequence is left and right Cauchy, then it is called Cauchy sequence.

**Definition 3.2.** A set $A \in P(X)$ is said to be cluster point in an approach space (X, $\delta$) if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\{x_n\}_{n=1}^{\infty} \to A$: where $\{x_n\}_{n=1}^{\infty} \to A$ if and only if $inf_{x\in A}\,\delta(x_n,A)=0$, we denoted the set of all cluster point in approach space by $\Gamma(X)$.

**Definition 3.3.** A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be Cauchy sequence in app-space or Cauchy distance or $\delta$ −Caushy if:for every cluster point $A:\ \underset{n\to\infty}{lim}\ \inf_{x_m\in A}\delta(x_n,A)=0$,

A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be $\delta$ −convergent sequence in app-space if: There exists$x\in X,\forall A\in\ \Gamma(X),:\ \delta(x_n,A)=0$.

**Proposition 3.4.** Let $(X,\delta)$ be an approach space. Then the following statements are equivalent:

1- $\delta$ −Convergent sequence in app-space.

2- $\lim_{n\to\infty} inf_{x\in A}\,\delta(x_n,A)$=0 and $\lim_{n\to\infty} sup_{x\in A}\,\delta(x_n,A)=0$.

$x\in X$

Such that for all $A\in\ \Gamma(X)$, we have : $\delta(x_n,A)=0$.

$\forall A\in\ \Gamma(X),:\ inf_{x\in A}\,\delta(x_n,A)=0$ and $Sup_{x\in A}\,\delta(x_n,A)=0$

$\forall A\in\ \Gamma(X),\ \lim_{n\to\infty} inf_{x\in A}\,\delta(x_n,A)$=0 and $\lim_{n\to\infty} sup_{x\in A}\,\delta(x_n,A)$=0.

Conversely, suppose that $\{x_n\}_{n=1}^{\infty}$ is convergent sequence

$\lim_{n\to\infty} inf_{x\in A}\,\delta(x_n,A)$=0 and $\lim_{n\to\infty} sup_{x\in A}\,\delta(x_n,A)$=0.

Then, A is cluster set, that is $inf_{x\in A}\,\delta(x_n,A)=0.\ \exists x\in X,\forall A\in\ \Gamma(X):\ \delta(x_n,A)=0$.

Thus, $\{x_n\}_{n=1}^{\infty}$ is $\delta$ −convergent sequence in app-space.

**Remark 3.5.** Every $\delta$ −convergent sequence is Cauchy approach space ($\delta$ −Cauchy).

**Proposition 3.6** If $X$ is an approach space the properties are equivalent;

(1) $\{x_n\}_{n=1}^{\infty}$is a $\delta$ −convergent sequence in app-space.

(2)$sup_{A\in\Gamma(X)}\,inf_{x\in A}\,d_{\delta}(x_n,x)=0$.

**Proof**: It follows that from definition of $\delta$ −convergent sequence

**Proposition 3.7** If (X, $\delta_d$) is a app-metric space then $\delta$ is a Cauchy distance if and only if it is a Cauchy distance in (X, d).

**Proof:** Let $\{x_n\}_{n=1}^{\infty}$ be a $\delta-C$aushy sequence in (X,$d_{\delta}$) so that we have that $\inf_{x_m\in A}\delta(x_n,A)=0$

This implies that $\inf_{x_n\in A}\delta(x_n,\{x_m\})=\inf_{x_n\in A}\ \underset{x_m\in A}{inf}\ d(x_n,x_m)=0$

, that is $d(x_n,x_m)=0$, then $\{x_n\}_{n=1}^{\infty}$ is left Cuachy sequence.

Also, $\inf_{x_n\in A}\delta(x_m,\{x_n\})=\inf_{x_m\in A}\ \underset{x_n\in A}{inf}\ d(x_m,x_n)=0$,that is $d(x_m,x_n)$ =0, $\{x_n\}_{n=1}^{\infty}$ is right Cuachy sequence.

Thus, $\{x_n\}_{n=1}^{\infty}$ is Cuachy sequencein $(X,d)$.

Conversely, if $\{x_n\}_{n=1}^{\infty}$ is Cauchy distance in $(X,d)$. Then, it is left and right Cauchy sequence,

for all $\varepsilon>0, there\ exists\ k\in Z^{+}$such that $d(x_m,x_n)<\varepsilon, for\ all\ m,n\ \geq N, m\leq n$, and for all $\varepsilon>0, there\ exists\ k\in Z^{+}$such that $d(x_n,x_m)<\varepsilon, for\ all\ m,n\ \geq N, n\leq m$.

$\inf_{x_m\in A}\delta(x_n,A)=\inf_{x_m\in A}\ \underset{x_n\in A}{inf}\ d(x_m,x_n)=0$. Hence$\{x_n\}_{n=1}^{\infty}$ is $\delta$ −Cuachy sequence in approach space.

**Theorem 3.8.** A function $f:(X,\delta)\to(Y,\ \delta')$ between approach spaces is a contraction map if and only if for every $\delta$ −convergent sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ and $x\in X$ if $\boldsymbol{x_n}\to x$ then $f(x_n)\to f(x)$.

**Proof:** If this map is a contraction then

$$\delta'\big(f(x_n),f(A)\big)\leq\delta(x_n,A).$$

To prove the condition if $x_n\to x$ ,then $f(x_n)\to f(x)$

Suppose that $\{x_n\}$ is a sequence in approach space (X, $\delta$) such that

$$\delta'\big(f(x_n), f(A)\big) \leq \delta(x_n, A).$$

Let $x_n \to x \Rightarrow \lim_{n\to\infty} \inf_{x\in A} \delta(x_n, A) = 0 \; and \; \delta(x_n, A) = 0$

Then $\lim_{n\to\infty} \delta(x_n, A) = 0.\ \delta'(f\big(x_n, f(A)\big) \leq \delta(x_n, A) = 0$ , $we\ have$

$\lim_{n\to\infty} \sup_{x\in A} \delta'(f\big(x_n, f(A)\big) \leq \lim_{n\to\infty} \sup_{x\in A} \delta(x_n, A)$=0. Thus f($x_n$) $\to f(x)$ , $n \to \infty$

Conversely, suppose that f is not contraction

Then $\delta'\big(f(x_n), f(A)\big) > \delta(x_n, A), \quad n \in Z$

Since f($x_n$) $\to f(x)$ , $n \to \infty$, $\lim_{n\to\infty} \inf_{x\in A} \delta'(f\big(x_n), f(A)\big) = 0$ and $\lim_{n\to\infty} \sup_{x\in A} \delta'(f\big(x_n), f(A)\big) =$ 0

Since $x_n \to x$, $\lim_{n\to\infty} \inf_{x\in A} \delta'\big(f(x_n), f(A)\big) = \lim_{n\to\infty} \sup_{x\in A} \delta'\big(f(x_n), f(A)\big)$ =0

Whenever

$\lim_{n\to\infty} \inf_{x\in A} \delta(x_n, A)$ =0 and $\lim_{n\to\infty} \sup_{x\in A} \delta(x_n, A)$ =0 that contradiction.

Some important properties of Convergent Approach space are given in the following theorem:

**Theorem 3.9.** Let $(X, \delta)$be an app-space. $\{x_n\}, and\ \{y_n\}$ is an app-converge sequence to x, y respectively. Then :

1. $\{x_n + y_n\}$ app- convergence to x + y .
2. $\{\lambda x_n\}$ app- convergence to $\lambda x$.
3. $\{x_n . y_n\}$ app- convergence to $x . y$ .

**Proof :**

**1-** Since $\{x_n\}, \{y_n\}$ $are$app-convergence to $x, y$.

Thus $\lim_{n\to\infty} inf_{x\in A}\, \delta(x_n, A) = 0$ and $\lim_{n\to\infty} sup_{x\in A}\, \delta(x_n, A) = 0$ .

So $\lim_{n\to\infty} inf_{y\in A}\, \delta(y_n, A) = 0$ and $\lim_{n\to\infty} sup_{y\in A}\, \delta(y_n, B) = 0$ . Then,

$\lim_{n\to\infty} \delta(y_n, \{y\}) = 0$, that is $\lim_{n\to\infty} inf d(y_n, y) = 0$ , $\lim_{n\to\infty} inf d(x_n, x) = 0$

$\lim_{n\to\infty} inf_{x,y\in A}\, \delta(x_n + y_n, A) =$

$\lim_{n\to\infty} inf_{x,y\in A}\, \delta(x_n + y_n, \{x\} \cup \{y\}) = \lim_{n\to\infty} inf_{x,y\in A} min\{\delta(x_n, \{x\}), \delta(y_n, \{y\}) = 0$

$\lim_{n\to\infty} inf_{x,y\in A}\, \delta(x_n + y_n, A) = \lim_{n\to\infty} inf_{x\in A}\, \delta(x_n, A) + \lim_{n\to\infty} inf_{y\in A}\, \delta(y_n, A) =$ 0, and

$\lim_{n\to\infty} sup_{x,y\in A}\, \delta(x_n + y_n, A) = \lim_{n\to\infty} sup_{x\in A}\, \delta(X_n, A) + \lim_{n\to\infty} sup_{y\in A}\, \delta(y_n, A) = 0$

$\lim_{n\to\infty} inf_{x,y\in A}\, \delta(x_n + y_n, A) = 0$, and $\lim_{n\to\infty} sup_{x,y\in A}\, \delta(x_n + y_n, A) = 0$

$\lim_{n\to\infty} inf_{x,y\in A}\, \delta\big((x_n + y_n), A\big) = 0 \quad and\ \lim_{n\to\infty} sup_{x,y\in A}\, \delta\big((x_n + y_n), A\big) = 0.$

Then, $\{x_n + y_n\}$ is app -convergence sequence to x +y.

**2-**Since$\{x_n\}$ $is$ app-converge sequence to x $\lambda \in F$ .

So that $\lim_{n\to\infty} \inf_{x\in A} \delta(x_n, A) = 0\ and\ \lim_{n\to\infty} \sup_{x\in A}(x_n, A) = 0$. If $\lambda \in F\ and$

$\lambda. \lim_{n\to\infty} \inf_{x\in A} \delta(x_n, A) = 0$ and $\lambda. \lim_{n\to\infty} sup_{x\in A} \delta(x_n, A) = 0$. Then

$\lim_{n\to\infty} \inf_{x\in\lambda A} \delta(\lambda x_n, \lambda A) = \lim_{n\to\infty} \delta(\lambda x_n, \lambda\{x\}) = \lim_{n\to\infty} inf d(\lambda x_n, \lambda x) = \lambda. \lim_{n\to\infty} inf d(x_n, x) =$

$\lambda. \lim_{n\to\infty} \inf_{x\in A} \delta(x_n, A) = 0$, and $\lim_{n\to\infty} \sup_{x\in\lambda A} \delta(\lambda x_n, \lambda A) = \lambda \lim_{n\to\infty} \sup_{x\in A} \delta(x_n, \{x\}) = \lambda. 0 = 0$

Then $\{\lambda x_n\}$ is app-convergence sequence to $\lambda x$.

**3:** Let$\{x_n\}$ , $\{y_n\}$ $be$app-convergent sequence to x,y.

Thus $\lim_{n\to\infty} \inf_{x\in A} \delta(x_n, A) = 0$, $\lim_{n\to\infty} \inf_{y\in A} \delta(y_n, A) = 0$ ,

$\lim_{n\to\infty} \sup_{x\in A} \delta(x_n, A) = 0$, $\lim_{n\to\infty} sup_{x\in A} \delta(y_n, A) = 0$ ,

$\lim_{n\to\infty} inf d(x_n, x) = 0$, and $\lim_{n\to\infty} inf d(y_n, y) = 0$.

Then $\lim_{n\to\infty} inf\, d(x_n y_n, xy) = 0$.
$\lim_{n\to\infty} \inf_{x,y\in A}\big(\delta(x_n, A).\,\delta(y_n, A)\big) = 0$ *and* $\lim_{n\to\infty} \sup_{x,y\in A}\big(\delta(x_n, A).\,\delta(y_n, A)\big) = 0$
$\lim_{n\to\infty} \inf_{x,y\in A} \delta((x_n \cdot y_n), A) = 0$ *and* $\lim_{n\to\infty} \sup_{x,y\in A} \delta((x_n \cdot y_n), A) = 0$.
Then $\{x_n . y_n\}$ is app-convergence sequence to x, y .

**Definition 3.10.** An approach space is called $\delta-$complete if every $\delta-$Cauchy is $\delta-$convergent in $(X, \delta)$

**Theorem 3.11.** Approach space $(X, \delta)$ is $\delta-$complete if and only if (X,$d_\delta$) is complete .

**Proof:** Let $\{x_n\}_{n=1}^{\infty}$ be a Caushy sequence in (X,$d$), then it is $\delta-$Cauchy sequence in $(X, \delta)$
**Since** $(X, \delta)$ is complete, there exists $x \in A$, for all $\Gamma(X),:\ \delta(x_n, A) = 0$.
$sup_{A\in\Gamma(X)}\, inf_{x\in A}\, d_\delta(x_n, x) = 0$. Then$d_\delta(x_n, x) = 0$, that is $(X, d_\delta)$ is complete.
Conversely, Let $\{x_n\}_{n=1}^{\infty}$ be a $\delta-$Caushy sequence in $(X, d)$ so that it is Cauchy sequence in (X,$d_\delta$). The sequence $\{x_n\}$ is left and right sequence in (X,$d_\delta$). (X,d) is complete, that is
$\lim_{n\to\infty} d_\delta(x_n, x) = 0$, that is $\lim_{n\to\infty} \inf_{x\in A} d_\delta(x_n, x) = 0,$ *and* $\lim_{n\to\infty} \sup_{x\in A} d_\delta(x_n, x) = 0$
$\delta(x_n, A) = sup_{A\in\Gamma(X)}\, inf_{x\in A}\, d_\delta(x_n, x) = 0,$
$\lim_{n\to\infty} inf_{x\in X}\, \delta(, A) = \lim_{n\to\infty} inf_{x\in X}\, sup_{A\in\Gamma(X)}\, inf_{x\in A}\, d_\delta(x_n, x) = 0,$ and
$\lim_{n\to\infty} sup_{x\in X}\, \delta(x_n, A) = \lim_{n\to\infty} sup_{x\in X}\, sup_{A\in\Gamma(X)}\, inf_{x\in A}\, d_\delta(x_n, x) = 0$
that is $\exists x \in X, \forall A \in\ \Gamma(X),:\ \delta(x_n, A) = 0$.
Thus, $\{x_n\}_{n=1}^{\infty}$ is convergent in an approach space $(X, \delta)$.

**Example. 3.12. Let $E = \mathbb{R}$** be a set of all real numbers. Define $\delta_E: \mathbb{R} \times 2^{\mathbb{R}} \longrightarrow [0,\infty]$

$$\delta_E(x, A) := \begin{cases} 0 & A\ unbounded \\ \infty & A\ bounded \\ \inf_{a\in A}|x - a| & x < \infty \end{cases}$$

This function is distance on [ 0, ∞] and we will prove that (E, $\delta$) is $\delta-$complete approach space.

**Proof:** For $n \in Z^+$, $\delta_E(x_n, A) := \begin{cases} 0 & A\ unbounded \\ \infty & A\ bounded \\ \inf_{a\in A}|x_n - a| & x_n < \infty \end{cases}$

Let $\{x_n\}_{n=1}^{\infty}$ *be a $\delta -$ Cuachy sequence* so that $\lim_{n\to\infty} \inf_{x_m\in A} \delta(x_n, A) = 0$,
That is there exist many cases:
First : If $A\subset\mathbb{R}$ is unbounded, therefore $\delta_E$ $(x_n, A)=0$, $\lim_{n\to\infty} \inf_{x\in A} \delta(x_n, A) = 0$, and $\lim_{n\to\infty} \sup_{x\in A} \delta(x_n, A) = 0$.
If $x_n < \infty$ ,then $\lim_{n\to\infty} \inf_{x_m\in A} \delta(x_n, A) = \inf_{x_m\in A}|x_n - x_m| = 0$ , there exists k$\in Z^+$ such that $|x_n - x_m| = 0$ for all m, n > k, that is $\{x_n\}_{n=1}^{\infty}$is Cuachy sequence in $(E, d)$.
Since $\mathbb{R}$ is complete, then $\{x_n\}_{n=1}^{\infty}$convergent sequence in $\mathbb{R}$.
There exist x$\in A$, for all $A \in P(X), |x_n - x| = 0$,
then $\lim_{n\Rightarrow\infty} \inf_{A\in\Gamma(\mathbb{R})} \inf_{x\in A}|(x_n - x)| = 0$
and $\lim_{n\to\infty} \sup_{A\in\Gamma(\mathbb{R})} \inf_{x\in A}|X_n - a| = 0$
Then $\delta_E$ is convergent on App-space.

## 4. New structure for Normed Approach Space.

**Definition 4.1** A triple (X,$\delta, +$) is called an approach semi-group if and only if the following statements are satisfied

**1.** $(X, \delta)$ is an approach space .
**2.** $(X,*)$ is a semi-group.

**3.** $*: X \otimes X \rightarrow X : (x, y) \mapsto x * y$ is contraction map.

**Definition 4.2** A triple (X, $\delta, +$) is called an approach group if it satisfies the following:

a) (X,$\delta$) $is\ an\ approach\ space$ .

b) (X,$+$) is a group.

c) $+: X \oplus X \rightarrow X: (x, y) \mapsto x + y\ is\ contraction$

d) $\cdot: X \rightarrow X$ : X→ $-X$ is contraction map.

**Definition 4.3** A quadruple $(X, \delta, +, \cdot)$ is said to be Approach vector space if it satisfies the following:

(1) $(X, \delta, +)$ is approach group.

(2) $(X, \delta, \cdot)$ is approach semi-group.

(3) $\lambda(a + b) = \lambda a + \lambda b \quad , \forall \lambda \in F, \forall a, b \in X.$

(4) $(a + b)\lambda = a.\lambda + b.\lambda \quad , \forall \lambda \in F, \forall a, b \in X.$

(5) $1. X = X$ for all $x \in X$.

**Definition 4.4:** Let $X$ be app-vector space. A triple (X,$\|.\|, \delta_{\|.\|}$) said to be normed approach space if it satisfies the following :

(1)$\|x\|$=0 if and only if $x = 0$, for all $x \in X$.

(2)$\|\lambda. x\|$=$|\lambda|. \|x\|$ $\forall \lambda \in F, x \in X$

(3)$\|x + y\| \leq \|x\| + \|y\|$ $\forall x, y \in X$.

(4)$\|x\| \geq 0$ $, \forall x \in X$

(5) $\delta_{\|.\|}(x, A) = sup_{x \in X}\ inf_{a \in A} \|x - a\|$ ,A$\in 2^x$.

**Remark 4.5.** every normed space is not necessary to be normed approach space. The following example shows that: Let C [-1, 1] be a set of all continuous functional on [-1, 1], a vector space C [-1, 1] is normed space under the norm define:

By $\|f\| = sup_{x \in [a,b]}\{|f(x)|\}$, when $f(x) = x - 1$ for all $x \in X$

. However it is not normed app-vector space because:

Since first condition: for $A = \{-1, 0, 1\}$

$$d_{\delta_{\|.\|}}(x, y) = \sup_{x \in X} \inf_{a \in A} \sup_{x \in [0,\infty]} \{|f(x) - f(a)|\} = \sup_{x \in X} \inf_{a \in A} \|f(x) - f(a)\| = 1$$

But $d_{\|.\|}$(x, y)=$\|x - a\| = \sup_{x \in [a,b]} \{|f(x) - f(a)| = 2.$

**Definition 4.6.** Every Banach approach space is complete normed approach space.

**Proposition 4.7.** Every finite –dimensional app-normed space is $\delta -$complete and consequent app-Banach space.

**Proof:** Assume that $dim\,(X) = n > 0$, $\{e_1, e_2, \dots, e_n\}$ is app-basis of $X$. Let$\{x_n\}_{n=1}^{\infty}$ be a $\delta -$Cauchy sequence in $X$, $\lim_{n \to \infty} \inf_{x_m \in A} \delta(x_m, A) = 0$. For $x_m = \sum_{i=1}^{n} \alpha_{im} e_i$ , $x_l = \sum_{i=1}^{n} \alpha_{il} e_i$

$0 = \lim_{n \to \infty} \inf_{\sum_{i=1}^{n} \alpha_{il} e_i \in A} \delta(\sum_{i=1}^{n} \alpha_{im} e_i\ , A) = \lim_{n \to \infty} \inf_{\sum_{i=1}^{n} \alpha_{il} e_i \in A} inf_{\ y \in A} \mathrm{f}\, d_{\delta_{\|.\|}}(\sum_{i=1}^{n} \alpha_{im} e_i\ , y)$

$= \lim_{n \to \infty} \inf_{\sum_{i=1}^{n} \alpha_{il} e_i \in A} inf_{\ y \in A} \mathrm{f}\, d_{\delta_{\|.\|}}(\sum_{i=1}^{n} \alpha_{im} e_i\ , \sum_{i=1}^{n} \alpha_{il} e_i) = \lim_{n \to \infty}$ $\inf_{\sum_{i=1}^{n} \alpha_{il} e_i \in A} inf_{\ y \in A} \inf ||\sum_{i=1}^{n} \alpha_{im} e_i - \sum_{i=1}^{n} \alpha_{il} e_i||$, that is $\sum_{i=1}^{n} ||\alpha_{im} - \alpha_{il}||$=0. Then $\{\alpha_{im}\}$ is Cauchy sequence in real field $\mathbb{R}$ or complex field $\mathbb{C}$, since real field $\mathbb{R}$ or complex field $\mathbb{C}$ are complete, therefore for all I there exists $\alpha_i \in F$ such that $\lim_{n \to \infty} \alpha_{im} = \alpha_i$, put $x = \sum_{i=1}^{n} \alpha_i e_i$ There exists $x \in A, for\ all\ A \in P(X)$, $\lim_{n \to \infty} \inf_{\sum_{i=1}^{n} \alpha_i e_i \in A} \delta(\sum_{i=1}^{n} \alpha_{im} e_i\ , A) = 0$

Thus $X$ is $-$complete

This follows , because of both $\mathbb{R}\ and\ \mathbb{C}$ are complete and every finite –dimentional is isomorphism to $\mathbb{R}^n or\ \mathbb{C}^n$ for some N.

**Remark 4.8.**The metric app-space $(X, d_\delta)$ is not app-normed space. Let X be a set of all complex sequence $\{x_i\}$ , and let $\delta: X \times 2^X \rightarrow [0, \infty]$ defined by

$$\delta(x,A)=\begin{cases}0 & if\ x_i=y_i, A\ bounded\\ \infty & if\ x_i\neq y_i, A\ unbounded\\ \inf_{y_i\in A}\sum_{i=1}^{n}\frac{1}{2i}\ (\frac{|x_i-y_i|}{1+|x_i-y_i|}) & if\ \ A\ bounded\end{cases}$$

$d_{\delta}(x,y)$=$\delta(x,\{y\})=\inf_{y_i\in A}\sum_{i=1}^{n}\frac{1}{2i}\ (\frac{|x_i-y_i|}{1+|x_i-y_i|})$

for all $x\in X, y\in A\ and\ \ A\subset 2^X\ \ , i=1,\ldots,n$

(X,$d_\delta$) is a metric app-space but it is not normed app- space .

**5. Main Result**

In this section, we introduce some important results ,and we also give the proof between App-normed spaces and other spaces.

**Proposition 5.1.** If $E_1$ and $E_2$ are normed approach spaces, and $f\colon E_1\to E_2$ is a surjective linear function, then the following properties are equivalent:

**(1)** $f$: $(E_1, \|.\|_1, \delta_1)\to(E_2, \|.\|_2, \delta_2)$ is contraction.

**(2)** $E_1$ Banach app- space if and only if $(E_2, \delta_{\|.\|})$ is complete.

**Proof:** If f: $E_1\ \to E_2$ is contraction. Then for every x$\in E_1$ and each subset A$\subset E_1$ .

$$\delta(f(x),f(A))\leq\delta(x,A)$$

If $(E_1, \|.\|_1)$ is Banach app-space.

Then A normed approach space is complete .

Let $\{y_n\}$ be a $\delta$ −caushy sequence in $E_2$ , then there exists $\{x_n\}$ such that $f(x_n)=y_n$

$\lim_{n\to\infty}\inf_{x_m\in A}\delta(y_n,A)=0$, $\lim_{n\to\infty}\inf_{x_m\in B}\delta(f(x_n),f(B))=0$, Since f is contraction map , we have $\lim_{n\to\infty}\inf_{x_m\in B}\delta(f(x_n),f(B))\leq\lim_{n\to\infty}\inf_{x_m\in A}\delta(x_n,A)$ .Hence $\lim_{n\to\infty}\inf_{x_m\in A}\delta(x_n,A)$=0.

That is $\{x_n\}$ is $\delta-C$aushy sequence in $E_1$ , $E_1$ is complete app-space. There exists $x\in B,$ for all $B\in f(E_1)$ such that $\lim_{n\to\infty}\inf_{x\in A}\delta(x_n,B)$=0

$\delta'(f(x_n),f(B))\leq\delta(x_n,B)=0.$ Therefore $\delta'(f(x_n),f(B))=0$

There exists $y=f(x)\in A=f(B), for\ all\ A\in f(E_2)$such that $\delta'(y_n,A)$= $\delta'(f(x_n),f(B))=0,$ and hence $\{y_n\}$ be a $\delta$ −convergent sequence in $E_2$ .

There is $\{x_n\}$ $\delta$ −Cauchy sequence in $E_1$ such that:

,$\delta_1(x_n,A)=0$ ,$x_n\to x$ ,$\forall x\in E_1$

Since f contraction map, then we get

$\delta_2(f(x_n),f(A))\leq\ \delta_1(x_n,A)$

$\delta_2(f(x_n),f(A))$ ≤0, thus f($x_n$) $\to f(x), for\ all\ x\in E_2$

Then $\lim_{n\to\infty} inf_{x\in E_2}\ \delta_2\big(f(x_n),f(A)\big)=0\ and\ \lim_{n\to\infty} sup_{x\in E_2}\ \delta_2\big(f(x_n),f(A)\big)=0$

Then $(E_2, \|.\|_2, \delta_2)$ is complete.

Conversely, suppose that f is not contraction map that means

$\delta_2(f(x),f(B))>\delta_1(x,B)$.

Let $\{x_n\}$ be a $\delta$ −convergent sequence in $E_1$ , that is $\{x_n\}$ is a $\delta-C$auchy sequence in $E_1$ , $\{f(x_n)\}$ is a $\delta-C$auchy sequence in $E_2$ . The condition holds, then there is f($x_n$) $in\ E_2$ such that f($x_n$) $\to f(x)$. There exists $y=f(x)\in f(B)=A\in f(E_2)$ such that $\delta_2\big(f(x_n),f(B)\big)=0, that\ is\ \delta_1(x_n,B)<0,$ this is impossible

Since $E_1$ Banach app- space there is $\{x_n\}$ in $E_1$ such that $\{x_n\}\ \to x$ and

$\delta_1(x_n,A)$=0, that is contradiction.

Then f is contraction map.

**Proposition 5.2.** A normed Approach space (X, $\|.\|,\delta_{\|.\|}$)is complete if and only if a metric approach space (E,$d_{\|.\|}$) is complete.

**Proof:** let E be a normed app- space and that $\delta$ is generated by the $\|.\|$. let $\langle x_n\rangle$ be Cauchy sequence in (E,$d_{\|.\|}$) so that , we have $d_{\|.\|}(x_n,x_m)=0$, for all $n\in Z^+$

this implies that $\delta_{\|.\|}(x_n, A) = sup_{x_n \in X}\, inf_{x_m \in A}\, d_{\|.\|}\,(x_n, x_m) = 0$
$\inf_{x_m \in A} \delta_{\|.\|}(x_n A) = 0,$ $\{x_n\}$ is $\delta$ −Cauchy in (E,$\delta_{\|.\|}$, $\|.\|$) by proposition 2.(2).
Since E is $\delta$ −complete this implies that there exists x$\in X$ for all $A \in P(X)$, such that $\delta_{\|.\|}(x_n, A) = 0$,for all $n \in z^+$, $d(x_n, x) = \delta(x_n, \{x\}) = 0$. That is $\{x_n\} \to x$.
Conversely; suppose that (E,$d_{\|.\|}$) is complete and let $\{x_n\}$ be $\delta -$ Cauchy sequence in normed app-space.
Then $\delta_{\|.\|}(x_n, A) = \inf_{x_m \in A} \delta_{\|.\|}(x_n, \{x_m\}) = \sup_{x \in X} inf_{x_m \in A}\|x_n - x_m\| = 0.$
$d_{\|.\|}(x_n, x_m) = inf_{x_m \in A}\, \delta_{\|.\|}(x_n, \{x_m\}) = 0$ . That is $\langle x_n \rangle$ is Cauchy sequence in $(X, d_{\|.\|})$ . $(X,$ $d_{\|.\|})$ is complete, therefore $\{x_n\}$ is convergent sequence , there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.
$d_{\|.\|}(x_n, x) = inf_{x_m \in A}\, \delta_{\|.\|}(x_n, \{x\}) = 0$ . There exists $x \in A$, for all $A \in P(X)$ such that $\delta_{\|.\|}(x_n, A) = sup_{x_n \in X}\, inf_{x \in A}\, d_{\|.\|}\,(x_n, x) = 0$. Hence (E, $\delta_{\|.\|}$, $\|.\|$) is $\delta$ −complete.

**Corollary 5.3.** A normed app-vector space is Banach app-space if and only if (X, $d_\delta$) is Banach space.

**Proof:** It follows from proposition (4.1) and proposition (4.2)

**Proposition 5.4.** If (X, $\|.\|, \delta_{\|.\|}$) is a normed app-vector space, then the following are equivalent:
**(1)** (X, $\|.\|, \delta_{\|.\|}$)is a Banach app-space.
**(2)** (X,$\delta$) is complete.

**Proof:** That is clear by previous corollary.

**Proposition 5.5.** If (X, $\|.\|, \delta_{\|.\|}$)is app-normed space ,then we have
**(1)**- The function f:(x,y )→ $x + y$ is contraction.
**(2)**- The function f:(x,y)→ $\alpha x$ is contraction.

**Proof:**
**(1)** Let $\{(x_n, y_n)\}$ be a convergent sequence in $X$, there exists $x, y \in X, for\ all\ A, B \in \Gamma(X)\ (respectively) such\ that\ \delta(x_n, A) = 0, \delta(y_n, B) = 0$
since $\delta_{\|.\|}(x_n, A) = sup_{x \in X}\, inf_{A \subset X}\|x_n - x\| = \sup_{x \in X} \inf_{A \subset X} d_\delta(x_n, x) = 0$
$\delta_{\|.\|}(y_n, \boldsymbol{A}) = sup_{x \in X}\, inf_{A \subset X}\|y_n - y\| = \sup_{x \in X} \inf_{A \subset X} d_\delta(y_n, y) = 0$
$\delta'_{\|.\|}($ f$(x_n, y_n)$,f(A, B)) = $\delta'_{\|.\|}(x_n + y_n,$ A + B) $= sup_{x,y \in X}\, inf_{A,B \subset X}\|x_n + y_n - (x + y)\| \leq sup_{x,y \in X}\, inf_{A,B \subset X}\, d_\delta(x_n + y_n, x + y) \leq sup_{x \in X}\, inf_{A \subset X}\big||x_n - x|\big| + \sup_{y \in X} \inf_{B \subset X}\big||y_n - y|\big| = 0$
Then $f$ is sequentialy contraction and therefore $f$ is contraction.
**(2)** Let $\{(\alpha_n, x_n)\}$ be a convergent sequence in $F \times X$, then Let $x \in X$, for all $A \in \Gamma(X)$ such that $\delta(x_n, A) = 0, \delta'_{\|.\|}(f(x_n), f(A)) = \delta'_{\|.\|}(\alpha x_n, \alpha A)$ $= sup_{x \in X}\, inf_{A \subset X}\|\alpha_n x_n - \alpha x\| = sup_{x \in X}\, inf_{A \subset X}\|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\| = 0.$
Thus $f(\alpha, x) = \alpha x$ is sequentially contraction

**Theorem 5.6.** If $M = (X, d_\delta)$ metric app-space. Then $M$ is a Hausdorff space.

**Proof:** Let $x, y \in X : x \neq y$ so that from distinct points in Metric app-space have disjoint open-balls exists open $\epsilon$ −balls $B_\epsilon(x) and\ B_\epsilon(y)$ which are disjoint open sets containing x and y respectively. Hence the result by the definition of Hausdorff space.

**Proposition 5.7.** Every uniform app-normed space (X, $\|.\|, \delta_{\|.\|}$)is a Hausdorff space.

**Proof:** suppose that $X^*$ be a topological duall of X. that is
$X^* = \left\{f : (X, T_{d_{\|.\|}}) \to (R, T_E)\right|$ f is linear and continuous functionals },
Let $\Im_X^*$ is the set of all non − negative closed unit ball in $X^*$ , so $\Im_X^*$={$f \epsilon X^*$: $\|f\| \leq 1$}.

and the norm on duall is defined by $\|f\|_{*} = \inf_{x \subset \Im^{*}_{X}} \|f(x)\|$

it is clear $that\ (X^{*}, \|f\|_{*})$ is Banach space. The dual of $(X^{*}, \|f\|_{*})$ is called biduall of $X$which is denoted by $X^{**}$

Let $\Psi$ be a non-empty subset of $X^{*}$, for each the functional $\|x\|_{\Psi}: X \to \mathbb{R}$ as followes:

$\|x\| = \mathrm{Sup}_{f \in \Psi} |\, f(x)|$ is a semi-norm on $X$, we have $\mathcal{N}_{X^{*}} = \{\|x\|_{\Psi} \mid \Psi \subseteq \Im^{*}_{X}\}$ and $\mathcal{D}_{X^{*}} = \{d_{\|x\|_{\Psi}} | \Psi \subseteq \Im^{*}_{X}\}$. Then a basis for the weak topology $\sigma_{(X,X^{*})}$ on $X$ is given by:

$\{\{b \in X \mid \forall f \in \Psi: |f(x-b)| < \varepsilon\}: \emptyset \neq \Psi \subseteq X^{*}, \varepsilon > 0\}$ for $x \in X$.

Define $\delta_{X^{*}} : X \times 2^{X} \to [0, \infty]$ by $\delta_{X^{*}}(x, A) = \sup_{\Psi \subseteq \Im^{*}_{X}} \inf_{a \in A} \|x - a\|_{\Psi}$ . It is clear that $\delta_{X^{*}}$ satisfies the condition of approach distance is said to be weak distance or weak approach distance. Since $\delta_{X^{*}}$is the uniform app-space. Generated by $\mathcal{D}_{X^{*}}$ , an app-basis for the $\tau_{\delta_{X^{*}}}$ is $\mathcal{N}_{X^{*}} = \{\|x\|_{\Psi} \mid \Psi \subseteq \Im^{*}_{X}\}$ which equals a basis for a weak topology $\sigma_{(X,X^{*})}$ which is given as: $\{\{b \in X \mid \forall f \in \Psi: |f(x-b)| < \varepsilon\} | \emptyset \neq \Psi \subseteq X^{*}, \varepsilon > 0\}$ that is equally a basis for the weak the weak topology $\sigma_{(X,X^{*})}$ is Housdorff that is the normed app-space is Housdorff space.

**Proposition 5.8 .** If (X, $\|.\|, \delta_{\|.\|}$) is normed app-space, then the app-metric of weak approach distance $\delta_{X^{*}}$ is $d_{\|.\|}$.

Proof: For all a,b $\in X, d_{\delta_{X^{*}}}(a, b) = \max\{\delta_{X^{*}}(a, \{b\}), \delta_{X^{*}}(b, \{a\})\}$

$= \mathop{\mathrm{Sup}}_{\Psi \subseteq \Im^{*}_{X}} \mathop{\mathrm{Sup}}_{f \in \Psi} |\, f(a-b)|$

$= \mathop{\mathrm{Sup}}_{\Psi \subseteq \Im^{*}_{X}} |f(a-b)| = \|a - b\|$

**Theorem 5.9.** If (X, $\|.\|, \delta_{\|.\|}$) be normed App- space, and$\{x_n\}$ a $\delta$ −convergent sequence in $X$. Then a sequence $\{x_n\}$ in X is norm bounded.

**Proof**: Suppose that M:$= sup_{f \in \Psi}\, lim\, sup_{n} |f(a - x_n)| < \infty$

For some $a \in X$.then we have that

$\forall f \in \Psi \exists n_f, \forall n > n_f: |f(a) - f(x_n)| \leq M + 1,$

We have for every f $\in X^{*}$ and every $|f(x_n)| \leq$ ($\|f\|$+1).($\left|\frac{f}{\|f\|+1}(x)\right| + M + 1$. Which shows that (f($x_n$))$_n$ is bounded sequence for every f $\in X^{*}$. Applying a well-known consequence of the Banach –stenin haus theorem (see e.g.Brezis (2011). Now, this yields that $(x_n)_n$ is norm bounded ,

Conversely, Let 3$\Rightarrow$ 1. Note that for each x$\in X$.

$\delta(x_n, A) = \delta_{\|.\|}(x_n, A)$=lim sup$\|x - x_n\| \leq \|x\| + sup_n \|x_n\|$.

## 6. CONCLUSIONS

In this paper, we proved that every uniform app-normed space (X, $\|.\|, \delta_{\|.\|}$) is a Hausdorff space and app-metric of weak approach distance $\delta_{X^{*}}$ is $d_{\|.\|}$. Also, we show that normed app-space (X, $\|.\|, \delta_{\|.\|}$) is complete if and only if a metric approach space (E,$d_{\|.\|}$) is complete. An example is given to show that the metric app-space (X, $d_{\delta}$) is not app-normed space. We proved every finite –dimensional app-normed space is $\delta$ −complete.Also, the necessary and sufficient conditions are found to prove (X, $d_{\delta}$) is Banach space. Some other results that relate to the convergent in approach space and normed approach space are proved.

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