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The Intersection Graph of Subgroups of the Dihedral Group of Order $2pq$

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Abstract

For a finite group G, the intersection graph Γ_G of G is the graph whose vertex set is the set of all proper non-trivial subgroups of G, where two distinct vertices are adjacent if their intersection is a non-trivial subgroup of G. In this article, we investigate the detour index, eccentric connectivity, and total eccentricity polynomials of the intersection graph Γ_G of subgroups of the dihedral group $G = D_{2pq}$ for distinct primes $p < q$. We also find the mean distance of the graph Γ_G.

Keywords: dihedral group, intersection graph of subgroups, detour distance, mean distance.
Mathematics Subject Classification: 05C25, 20F16, 05C10.

البيان التقاطعى للزمر الجزئية من زمرة التناضرات من مرتبة $2pq$

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الخلاصة

للزمرة المنتهية G، البيان التقاطعى ل G هو البيان الذى مجموعة رؤسة عبارة عن جميع الزمر الجزئية الفعلية غير التافهة من G بحيث ان رأسين من البيان يرتبطان بحافة(متجاوران) اذا كان تقاطعهما الزمرة الجزئية غير التافهة. فى هذا البحث نتحرى عن دليل اقصى المسافة، مركزية الارتباط و تعددة الحدود للمركزية الكلية للبيان التقاطعى للزمر الجزئية من زمرة التناظرات D_{2pq} حيث p و q عددين أولين مختلفين. وكذللك نقوم بايجاد معدل المسافة للبيان

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1. Introduction

The concept of intersection graph of subgroups of a finite group was defined and studied by Csa'ka'ny and Polla'k in 1969 [1]. They found the clique number and degree of vertices of an intersection graph of subgroups of a dihedral group, quaternion group, and quasi-dihedral group.

Let G be a finite non-abelian group. The intersection graph Γ_G of G is an undirected simple (without loops and multiple edges) graph whose vertex-set consists of all nontrivial proper subgroups of G, for which two distinct vertices H and K of Γ_G are adjacent if $H \cap K$ is a non-trivial subgroup of G. This kind of graph has been studied by researchers; we refer the reader to see [2-6].

Let Γ be any graph. The set of vertices and the set of edges of Γ will be denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. If there is an edge between vertices u and v, then we write $uv \in E(\Gamma)$. The cardinality of $V(\Gamma)$, denoted by $|V(\Gamma)|$, is called the order of Γ, while the cardinality of $E(\Gamma)$, denoted by $|E(\Gamma)|$, is called the size of Γ. For any vertex v in Γ, the number of edges incident to v is called the degree of v and denoted by $deg_\Gamma v$ [7]. The chromatic number of a graph Γ is $\chi(\Gamma)$, which is the smallest number of colors for $V(\Gamma)$ such that adjacent vertices have different colors.

A $u-v$ path is a walk with no two vertices repeated, for any two distinct vertices u and v in Γ. The shortest $u-v$ path in Γ is called the distance between u and v, denoted by $d(u,v)$, and the longest $u-v$ path in Γ is called the detour distance between $u \; and \; v$, denoted by $D(u,v)$. The eccentricity of a vertex $v \in V(\Gamma)$, denoted by $ecc(v)$, is the longest distance between v and all other vertices of Γ. The diameter of a graph Γ, denoted by diam(Γ), is defined as $diam(\Gamma) = max\{ecc(v) \mid v \in V(\Gamma)$ [8]. The detour index, eccentric connectivity and total eccentricity polynomials are defined by $D(\Gamma, x) = \sum_{u,v \in V(\Gamma)} x^{D(u,v)}$[9], $\zeta(\Gamma, x) = \sum_{u \in V(\Gamma)} \deg(u)\, x^{ecc(u)}$ and $\theta(\Gamma, x) = \sum_{u \in V(\Gamma)} x^{ecc(u)}$[10], respectively. The detour index $dd(\Gamma)$, the eccentric connectivity index and the total eccentricity $\zeta(\Gamma)$ of a graph Γ are the first derivatives of their corresponding polynomials at $x = 1$, respectively. The transmission of a vertex v in Γ is $\sigma(\Gamma, v) = \sum_{u \in V(\Gamma)} d(u, v)$. The transmission of a graph Γ is $\sigma(\Gamma) = \sum_{u \in V(\Gamma)} \sigma(\Gamma, v)$. The mean (average) distance of graph Γ is $\mu(\Gamma) = \frac{\sigma(\Gamma)}{p(p-1)}$, where p is the order of Γ [3,1,12].

Khasraw [13] studied the intersection graph of subgroups of the group D_{2n}, where $n = p^2$, p is a prime. He found some topological indices of the graph $\Gamma_{D_{2p^2}}$ as well as its metric dimension and resolving polynomial.

In this paper, we consider the graph $\Gamma_{D_{2pq}}$of the dihedral group D_{2pq} where p and q are distinct primes. Some properties of the connected graph $\Gamma_{D_{2pq}}$ will be presented. The dihedral group D_{2pq} of order 2pq is defined by $D_{2pq} = \langle r, s \colon r^{pq} = s^2 = 1, srs = r^{-1} \rangle$ for prime numbers $p < q$.

2. Some properties of the intersection graph of D_{2pq} for prime numbers $p < q$

In order to determine the vertex set of the graph $\Gamma_{D_{2pq}}$, it is required to list all non-trivial proper subgroups of the dihedral group D_{2pq} for distinct primes $p < q$. In [6], the set of all non-trivial proper subgroups of the group D_{2n} are classified for all $n \geq 3$. Here, we only consider the case when $n = pq$ for distinct primes $p < q$.

Lemma 2.1[6]. The non-trivial proper subgroups of the dihedral group D_{2pq} for distinct primes $p < q$ are:

1- cyclic groups $G_i = < sr^i >$ of order 2, where $i = 1,2, \dots, pq$.

2- dihedral groups $H_i^p = < r^p, sr^i >$ of order $2p$, where $i = 1,2, \dots, p$ and $H_i^q = < r^q, sr^i >$ of order $2q$, where $i = 1,2, \dots, q$.

3- cyclic groups $I_p =< r^p >$ of order q, $I =< r >$ of order pq, and $I_q =< r^q >$ of order p.
According to the above classification of subgroups of the group D_{2pq} for primes $p < q$, as given in Lemma 2.1, we can determine the structure of the set of vertices of the graph $\Gamma_{D_{2pq}}$ as the non-trivial proper subgroups by $V\left(\Gamma_{D_{2pq}}\right) = A \cup B \cup C$, where $A = \{G_1, G_2, \dots, G_{pq}\}$, $B = \{ H_i^p, H_j^q ; 1 \le i \le p ; 1 \le j \le q\}$, and $C = \{ I_p, I, I_q\}$. So, we can distinguish subgraphs Γ_A as complement of the complete graph K_{pq}, $\Gamma_{B\cup\{I\}}$ as the complete graph K_{p+q+1}, and $\Gamma_{C-\{I\}}$ as the complement of the complete graph K_2. Through this article, we fixed the sets A, B, and C.

In this section, some basic properties of the intersection graph of D_{2pq} are investigated, such as the order and chromatic number of the graph $\Gamma_{D_{2pq}}$.

Theorem 2.2. The order of the graph $\Gamma_{D_{2pq}}$ is $\left|V\left(\Gamma_{D_{2pq}}\right)\right| = pq + p + q + 3$.

Proof: Since the set of vertices of $\Gamma_{D_{2pq}}$ are the non-trivial subgroups of D_{2pq} which are classified in the sets A, B and C, and since $|A| = pq$, $|B| = p + q$, and $|C| = 3$, then
$\left|V\left(\Gamma_{D_{2pq}}\right)\right| = |A| + |B| + |C| = pq + p + q + 3$.

Theorem 2.3. The size of the graph $\Gamma_{D_{2pq}}$ is $\left|E\left(\Gamma_{D_{2pq}}\right)\right| = \frac{(p+q)^2+4(pq+1)+3(p+q)}{2}$.

Proof: It is clear that each vertex of A is adjacent with only two vertices of B. The vertices in the set A are non-adjacent. Also, each vertex of $B \cup\{I\}$ is adjacent with all other vertices of $B \cup\{I\}$; that is, $B \cup\{I\}$ is a complete graph. Moreover, the vertex $I_p \in C$ is adjacent with p vertices of B, which are $H_i^p; i = 1,2, \dots, p$, and $I_q \in C$ is adjacent with q vertices of B which are H_j^q ; $j = 1,2, \dots, q$. Finally, the vertex $I \in C$ is adjacent with I_p and I_q. Thus
$\left|E\left(\Gamma_{D_{2pq}}\right)\right| = 2pq + \frac{(p+q+1)(p+q)}{2} + p + q + 2$.

Theorem 2.4. The chromatic number of the graph $\Gamma_{D_{2pq}}$ is $\chi\left(\Gamma_{D_{2pq}}\right) = p + q + 1$.

Proof: From Theorem 2.2, $cl\left(\Gamma_{D_{2pq}}\right) = p + q + 1$. This means that the graph $\Gamma_{D_{2pq}}$ is at least $p + q + 1$ colorable graph. The vertices $G_1, G_2, \dots, G_{pq}$ can be colored with the same color as the vertex I, the vertices I_p and H_i^q can share the same color, and the vertices I_q and H_i^p can share the same color. Thus, the minimum number of colors that can be used to color the graph $\Gamma_{D_{2pq}}$ is $p + q + 1$.

Therefore, $\chi\left(\Gamma_{D_{2pq}}\right) = p + q + 1$.

Theorem 2.5. Let $\Gamma = \Gamma_{D_{2pq}}$ be the graph of the dihedral group D_{2pq}. Then $diam(\Gamma) = 3$.

Proof: Let u and v be two distinct vertices in Γ. If u and v are joint by an edge, then $d(u, v) = 1$. Otherwise, $u \cap v = \{e\}$. There are five cases to consider.

Case1. If $u = G_i$ and $v = G_j$, where $i \equiv j\ (mod p)$ or $i \equiv j\ (mod q)$, then there exists $v' \in B$ such that $v' = H_k^p$ or $v' = H_{k'}^q$, for some k and k'. If $i \equiv k\ (mod p)$ or $j \equiv k'\ (mod q)$, then $uv', v'v \in E(\Gamma)$ and so $d(u, v) = 2$. Otherwise, if $i \not\equiv j\ (mod p)$ and $i \not\equiv j\ (mod q)$, take $v' = H_k^p$, then there exists $w \in B$, where $w = H_l^p$ such that $k \not\equiv l\ (mod p)$ and $k \not\equiv l\ (mod q)$. Thus, $uv', v'w, wv \in E(\Gamma)$ and then $d(u, v) = 3$.

Case2. If $u = G_j$ and $v = H_i^p$ or $v = H_k^q$, $i = 1, \dots, p;\ k = 1, \dots, q$, where $i \not\equiv j\ (mod p)$ and $k \not\equiv j\ (mod q)$, then there exists $v' \in B$ such that $v' = H_l^p$ or $v' = H_l^q$, where $j \equiv l\ (mod p)$ or $j \equiv l\ (mod q)$, so $uv', v'v \in E(\Gamma)$ and $d(u, v) = 2$.

Case3. If $u = I_p$ and $v = I_q$, then we take $v' = I$ so that $uv', v'v \in E(\Gamma)$ and $d(u, v) = 2$.

Case4. If $u = I_p$ and $v \in \{H_i^q \mid i = 1, \dots, q\}$ (or $u = I_q$ and $v \in \{H_i^p \mid i = 1, \dots, p\}$), then we take $w = I$, which implies that $uw, wv \in E(\Gamma)$ and so $d(u, v) = 2$.

Case5. If $u = G_j$ and $v \in C$, then there are three possibilities for v If $v = I_p$, then there exists $v' \in \{H_i^p \mid i = 1, \dots, p\}$ such that $uv', v'v \in E(\Gamma)$ if $i \equiv j\ (mod p)$. If $v = I_q$, then there exists $v' \in \{H_i^q \mid i = 1, \dots, q\}$ such that $uv', v'v \in E(\Gamma)$ if $i \equiv j\ (mod q)$. Finally, if $v = I$, then there exists $v' \in B$ such that $uv', v'v \in E(\Gamma)$. In all possibilities, $d(u, v) = 2$.
As a consequence from the above theorem, we state the following.
Corollary 2.6. Let $\Gamma = \Gamma_{D_{2pq}}$ be the graph of the dihedral group D_{2pq}. Then

$$d(u,v) = \begin{cases} 1 & \text{if } u = G_i, v = H_j^p \wedge i \equiv j \mod p, \quad 1 \le i \le pq, 1 \le j \le p, \\ & \text{or } u = G_i, v = H_j^q \wedge i \equiv j \mod q, 1 \le i \le pq, 1 \le j \le q, \\ 2 & \text{if } u = G_i, v = G_j, (i \equiv j \mod p \text{ or } q)\ 1 \le i,j \le pq \wedge i \ne j, \\ & \text{or } u = G_i, v = H_j^p \wedge i \not\equiv j \mod p, 1 \le i \le pq, 1 \le j \le p, \\ & \text{or } u = G_i, v = H_j^q \wedge i \not\equiv j \mod q, 1 \le i \le pq, 1 \le j \le q, \\ 3 & \text{if } u = G_i, v = G_j, (i \not\equiv j \mod p \wedge i \not\equiv j \mod q)\ 1 \le i,j \le pq. \end{cases}$$

Lemma 2.7. Let $\Gamma = \Gamma_{D_{2pq}}$ be the intersection graph of subgroups of the dihedral group D_{2pq} with distinct primes p and q. Then

$$\deg_\Gamma(v) = \begin{cases} 2 & \text{if } v = G_i, \text{ for } 1 \le i \le p, \\ p+1 & \text{if } v = I_p, \\ q+1 & \text{if } v = I_q, \\ p+q+2 & \text{if } v = I, \\ p+2q+1 & \text{if } v = H_i^p, \text{ for } 1 \le i \le p, \\ 2p+q+1 & \text{if } v = H_i^q, \text{ for } 1 \le j \le q. \end{cases}$$

Proof: see [7].

3. Detour index, eccentric connectivity, and tot al eccentricity polynomials of the graph $\Gamma_{D_{2pq}}$

In this section, we find detour index, eccentric connectivity, and total eccentricity polynomials of the intersection graph $\Gamma_{D_{2pq}}$ of D_{2pq}.
Theorem 3.1. Let $\Gamma_{D_{2pq}}$ be the intersection graph of D_{2pq} with primes $p < q$. Then

$$D(u,v) = \begin{cases} 3p+q-1 & \text{if } u = H_i^p, v = H_j^p, 1 \le i,j \le p \wedge i \ne j, \\ 3p+q & \text{if } u = H_i^p, v \in \{I, I_q, H_j^q; 1 \le j \le q\}, 1 \le i \le p, \\ 3p+q+1 & \text{if } u = H_i^p, v \in \{I_p, G_j; 1 \le j \le pq\}, 1 \le i \le p, \\ & \text{or } u = H_i^q, v \in \{I_q, I\}, 1 \le i \le q, \\ & \text{or } u = I, v \in \{I_p, I_q\}, \\ & \text{or } u = G_i, v = H_j^q, 1 \le i \le pq, 1 \le j \le q \\ & \quad \wedge\ uv \in E(\Gamma), \\ 3p+q+2 & \text{if } u = G_i, v \in \{I, I_q\}, 1 \le i \le pq, \\ & \text{or } u = I_p, v \in \{I_q, H_i^q; 1 \le i \le q\}, \\ & \text{or } u = G_i, v = H_j^q, 1 \le i \le pq, 1 \le j \le q \\ & \quad \wedge\ uv \notin E(\Gamma), \\ 3p+q+3 & \text{if } u = G_i, v \in \{I_p, G_j\}, 1 \le i,j \le pq \wedge i \ne j. \end{cases}$$

Proof: For $D(u, v) = 3p + q - 1$, the longest path from H_i^p to H_j^p where $1 \le i, j \le p$ and $i \ne j$ is the path that starts from H_i^p, passing alternatively through $2p - 3$ elements of

A, $p+q-1$ elements of B, and I_p, I and I_q vertices of B, and ending at H_j^p. So, the path has length
$(2p-3)+(p+q-1)+3=3p+q-1$. Hence $D\left(H_i^p, H_j^p\right)=3p+q-1$.
For $D(u,v)=3p+q$, there are two cases. Case1, the longest path, that starts from H_i^p for some $1 \le i \le p$ to $S \in \{I, I_q\}$, is the path passing alternatively through $2p-1$ of vertices of A, $p+q-3$ elements of B, and $\{I, I_q\}$ vertices of B, and ending at $S \in \{I, I_q\}$. So, the length of this path is

$$[1+(2p-1)+(p+q-3)+(1+2)]=3p+q.$$

Thus, $D\left(H_i^p, X\right)=3p+q$, for all $1 \le i \le p$ and $X \in \{I, I_q\}$.
Case2, the longest path, that starts from H_i^p for some $1 \le i \le p$ to H_j^q, for some $1 \le j \le q$, is the path passing alternatively through $2p-1$ vertices of A, $p+q-3$ elements of B, and I_q and I element of B, and ending at H_j^q, for some $1 \le j \le q$. So, the length of this path is

$$[1+(2p-1)+(p+q-3)+(2+1)+1]-1=3p+q.$$

Thus, $D\left(H_i^p, H_j^q\right)=3p+q$, for all $1 \le i \le p$ and $1 \le j \le q$.
For $D(u,v)=3p+q+1$, the longest path, that starts from G_i to H_j^q for some $1 \le j \le p$ for some $1 \le i \le pq$, is the path passing alternatively through $p+q-1$ vertices of B, $2p-1$ vertices of A, and I and I_p vertices of C , and ending at H_j^q for some $1 \le j \le p$. So, the length of the path is

$$[1+(p+q-1)+(2p-1)+2+1]-1=3p+q+1.$$

Thus, $D\left(G_i, H_j^p\right)=3p+q+1$, for all $1 \le i \le pq$ and $1 \le j \le p$.
The longest path, that starts from H_i^p to I_p for some $1 \le i \le p$, is the path passing alternatively through $2p-1$ vertices of A and $p+q-3$ elements of B with I, and ending at I_p. So the length of the path is

$$[1+(2p-1)+(p+q-3)+(2+2)+1]-1=3p+q+1.$$

Hence, $D\left(H_i^p, I_p\right)=3p+q+1$, for all $1 \le i \le p$.
The longest path, that starts from the vertex H_i^q to I for some $1 \le i \le q$, is the path passing alternatively through $2p$ vertices of A, $p+q-1$ vertices of B, and the vertex I_q of C, and ending at I. So the length of the path is

$$[1+(2p)+(p+q-1)+1+1]-1=3p+q+1.$$

Hence, $D\left(H_i^q, I\right)=3p+q+1$, for all $1 \le i \le q$.
In a similar way, we can prove the detour distance between all other vertices in the graph $\Gamma_{D_{2pq}}$.

Theorem 3.2. Let $\Gamma_{D_{2pq}}$ be the intersection graph of D_{2pq} with distinct primes $p<q$. Then

$$D\left(\Gamma_{D_{2pq}}, x\right)= \frac{(pq-1)(pq-2)}{2}x^{3p+q+3}+[p^2(q-1)+q+1]x^{3p+q+2}+\frac{4q(p+1)+2(p+2)+q(q-1)}{2}x^{3p+q+1}+ p(q+2)x^{3p+q}+\frac{p(p-1)}{2}x^{3p+q-1}.$$

Proof: It follows from Theorem3.1 that

$$D\left(\Gamma_{D_{2pq}}, x\right)=\sum_{u,v\in V(\Gamma)} x^{D(u,v)}=\binom{pq-1}{2}x^{D(G_i,G_j)}+(pq)(p)D^{D(G_i,H_i^p)}+(pq-p)qx^{D(G_i,H_j^q)}+pqx^{D(G_i,H_j^q)}+pqx^{D(G_i,I_p)}+\text{pq}x^{D(G_i,I)}+pqx^{D(G_i,I_q)}+\binom{p}{2}x^{D(H_i^p,H_i^p)}+ pqx^{D(H_i^p,H_j^q)}+px^{D(H_i^p,I_p)}+px^{D(H_j^p,I)}+px^{D(H_i^p,I_q)}+\binom{q}{2}x^{D(H_j^q,H_j^q)}+qx^{D(H_j^q,I_p)}+$$

$px^{D(H_j^q,I)} + qx^{D(H_j^q,I_q)} + x^{D(I_p,I)} + x^{D(I_p,I_q)} + x^{D(I,I_q)}$ where $\binom{pq-1}{2} = \frac{(pq-1)(pq-2)}{2}$, $\binom{p}{2} = \frac{p(p-1)}{2}$ and $\binom{q}{2} = \frac{q(q-1)}{2}$.

Therefore, $D\left(\Gamma_{D_{2pq}}, x\right) = \frac{(pq-1)(pq-2)}{2}x^{3p+q+3} + [p^2(q-1)+q+1]x^{3p+q+2} + \left[2pq + p + \frac{q(q-1)}{2} + 2q + 2\right]x^{3p+q+1} + p(q+2)x^{3p+q} + \frac{p(p-1)}{2}x^{3p+q-1}$

Corollary 3.3. Let $\Gamma_{D_{2pq}}$ be the intersection graph of D_{2pq} with distinct primes $p < q$. Then

$$dd(\Gamma_{D_{2pq}}) = 3p^3q(q+1) + q^3\left(p^2 + \frac{1}{2}\right) - \frac{3}{2}p^3 + \frac{3}{2}p^2q + \frac{3}{2}q^2p + 4p^2q^2 + 5p^2 + 3q^2 + 3pq + \frac{33}{2}p + \frac{17}{2}q + 10.$$

Proof: The result follows directly by taking the first derivative of $D\left(\Gamma_{D_{2pq}}, x\right)$ at $x = 1$.

Theorem 3.4. Let $\Gamma_{D_{2pq}}$ be the intersection graph of D_{2pq} with distinct primes $p < q$. Then
$ecc(v) = \begin{cases} 2 & \text{if } v \in B \cup \{I\}, \\ 3 & \text{if } v \in A \cup C - \{I\}. \end{cases}$

Proof: The proof follows directly from Corollary 2.6.

Theorem 3.5. Let $\Gamma_{D_{2pq}}$ be the intersection graph of D_{2pq} with distinct primes $p < q$. Then $\zeta\left(\Gamma_{D_{2pq}}, x\right) = (2pq+p+q+2)x^3 + [(p+q)^2 + 2(pq+p+q+1)]x^2$.

Proof: It follows from Lemma 2.7 and Theorem 3.4 that $\zeta\left(\Gamma_{D_{2pq}}, x\right) = \sum_{u \in V(\Gamma_{D_{2pq}})} \deg(u)x^{ecc(u)} = 2pqx^3 + (q+2+p+q-1)px^2 + (p+2+p+q-1)qx^2 + (p+1)x^3 + (p+q+2)x^2 + (q+1)x^3$.

Theorem 3.6. Let $\Gamma_{D_{2pq}}$ be the intersection graph of D_{2pq} with distinct primes $p < q$. Then $\theta\left(\Gamma_{D_{2pq}}, x\right) = (pq+2)x^3 + (p+q+1)x^2$.

Proof: It follows from Theorem 3.4 that $\theta\left(\Gamma_{D_{2pq}}, x\right) = \sum_{u \in V(\Gamma_{D_{2pq}})} x^{ecc(u)} = pqx^3 + px^2 + qx^2 + x^3 + x^2 + x^3 = (pq+2)x^3 + (p+q+1)x^2$.

Theorem 3.7. Let $\Gamma_{D_{2pq}}$ be the intersection graph of D_{2pq} with distinct primes $p < q$. Then $\xi\left(\Gamma_{D_{2pq}}\right) = 2(p^2+q^2) + 7(2pq+p+q) + 10$.

Proof: From Theorem 3.5, one can see that
$\frac{d}{dx}\zeta\left(\Gamma_{D_{2pq}}, x\right)|_{x=1} = 3(2pq+p+q+2) + 2[(p+q)^2 + 2(pq+p+q+1)]$. The result follows.

4. The mean distance of the intersection graph $\Gamma_{D_{2pq}}$

In this section, we find the mean distance of the intersection graph of subgroups of D_{2pq} for distinct prime numbers p and q.

Theorem 4.1. The transmission of the graph $\Gamma_{D_{2pq}}$ is

$$\sigma\left(\Gamma_{D_{2pq}}\right) = p^2(3q+1)(q+1) + q^2(3p+1) + q(8p+7) + 7p + 8.$$

Proof: From Corollary 2.6, we have

$$\sigma(G_i) = q(2) + \left(pq - (q+1)\right)(3) + 2(1) + (p+q-2)(2) + 2(2) + (1)(3) = 3pq + q + 2p + 2, \text{ for all } i = 1,2,\dots,pq,$$

$$\sigma\left(H_i^p\right) = q(1) + (pq-q)(2) + (p+q-1)(1) + 2(1) + (1)(2) = 2pq + p + 3, \text{ for all } i = 1,2,\dots,p.$$

Also, $\sigma\left(H_i^q\right) = p(1) + (pq-p)(2) + (p+q-1)(1) + 2(1) + (1)(2)$
$= 2pq + q + 3$, for all $i = 1,2,\dots,q$.

Note that the vertices I_p and I_q are non-adjacent but the vertex I is adjacent to both I_p and I_q.
So, $C = \{I_p, I, I_q\}$ induced a path subgraph of $\Gamma_{D_{2pq}}$.
Thus, $(I_p) = pq(2) + p(1) + q(2) + (1)(1) + (1)(2) = 2pq + p + 2q + 3$,
$\sigma(I) = pq(2) + (p + q)(1) + 2(1) = 2pq + p + q + 2$, and
$\sigma(I_q) = pq(2) + q(1) + p(2) + (1)(1) + (1)(2) = 2pq + 2p + q + 3$.
Now, we can find the transmission of the graph $\Gamma_{D_{2pq}}$ as
$\sigma\left(\Gamma_{D_{2pq}}\right) = \sum_{i=1}^{pq} \sigma(G_i) + \sum_{i=1}^{p} \sigma(H_i^p) + \sum_{i=1}^{q} \sigma(H_i^q) + \sigma(I) + \sigma(I_p) + \sigma(I_q)$
$= pq[3pq + 2p + q + 2] + p[2pq + p + 3] + q[2pq + q + 3] + 6pq + 4(p + q) + 8$
$= p^2(3q + 1)(q + 1) + q^2(3p + 1) + q(8p + 7) + 7p + 8$.

Theorem 4.2. The mean distance of the graph $\Gamma_{D_{2pq}}$ is
$\mu(\Gamma_{D_{2pq}}) = \frac{p^2(3q+1)(q+1)+q^2(3p+1)+q(8p+7)+7p+8}{(pq+p+q+3)(pq+p+q+2)}$.
Proof: Since the order of the graph $\Gamma_{D_{2pq}}$ is $pq + p + q + 3$ and the transmission of the graph $\Gamma_{D_{2pq}}$ is given in Theorem 4.1, we can find the mean distance of the graph $\Gamma_{D_{2pq}}$ as
$\mu(\Gamma_{D_{2pq}}) = \frac{p^2(3q+1)(q+1)+q^2(3p+1)+q(8p+7)+7p+8}{(pq+p+q+3)(pq+p+q+2)}$, where $p < q$ are prime numbers.

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