The Intersection Graph of Subgroups of the Dihedral Group of Order $2pq$

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Abstract
For a finite group $G$, the intersection graph $\Gamma_G$ of $G$ is the graph whose vertex set is the set of all proper non-trivial subgroups of $G$, where two distinct vertices are adjacent if their intersection is a non-trivial subgroup of $G$. In this article, we investigate the detour index, eccentric connectivity, and total eccentricity polynomials of the intersection graph $\Gamma_G$ of subgroups of the dihedral group $G = D_{2pq}$ for distinct primes $p < q$. We also find the mean distance of the graph $\Gamma_G$.

Keywords: dihedral group, intersection graph of subgroups, detour distance, mean distance.
Mathematics Subject Classification: 05C25, 20F16, 05C10.

البيان التقاطعى للزمرة الجزئية من زمرة التناظرات من مرتبة $2pq$

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الخلاصة
للزمرة المنتهية $G$, البيان التقاطعى ل$G$ هو البيان الذي مجموعة رؤية عبارة عن جميع الزمر الجزئية المنطية غير النافعة من $G$ حيث أن رأسين من البيان يرتبطان بحافة(متجاوران) إذا كان تقابلهما الزمرة الجزئية غير النافعة. في هذا البحث نجري عن دليل اقصى المسافة، مركزية الارتباط و تعدد العدد للمركزيات الكلية للبيان التقاطعى للزمرة الجزئية من زمرة التناظرات $D_{2pq}$ حيث $p$ و $q$ عددين أوليين مختلفين، وكذلك نقوم بإيجاد معدل المسافة للبيان

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1. Introduction

The concept of intersection graph of subgroups of a finite group was defined and studied by Csákány and Pollák in 1969 [1]. They found the clique number and degree of vertices of an intersection graph of subgroups of a dihedral group, quaternion group, and quasi-dihedral group.

Let $G$ be a finite non-abelian group. The intersection graph $\Gamma_G$ of $G$ is an undirected simple (without loops and multiple edges) graph whose vertex-set consists of all nontrivial proper subgroups of $G$, for which two distinct vertices $H$ and $K$ of $\Gamma_G$ are adjacent if $H \cap K$ is a nontrivial subgroup of $G$. This kind of graph has been studied by researchers; we refer the reader to see [2-6].

Let $\Gamma$ be any graph. The set of vertices and the set of edges of $\Gamma$ will be denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. If there is an edge between vertices $u$ and $v$, then we write $uv \in E(\Gamma)$. The cardinality of $V(\Gamma)$, denoted by $|V(\Gamma)|$, is called the order of $\Gamma$, while the cardinality of $E(\Gamma)$, denoted by $|E(\Gamma)|$, is called the size of $\Gamma$. For any vertex $v$ in $\Gamma$, the number of edges incident to $v$ is called the degree of $v$ and denoted by $\deg(v)$ [7]. The chromatic number of a graph $\Gamma$ is $\chi(\Gamma)$, which is the smallest number of colors for $V(\Gamma)$ such that adjacent vertices have different colors.

A $u-v$ path is a walk with no two vertices repeated, for any two distinct vertices $u$ and $v$ in $\Gamma$. The shortest $u-v$ path in $\Gamma$ is called the distance between $u$ and $v$, denoted by $d(u,v)$, and the longest $u-v$ path in $\Gamma$ is called the detour distance between $u$ and $v$, denoted by $D(u,v)$. The eccentricity of a vertex $v \in V(\Gamma)$, denoted by $ecc(v)$, is the longest distance between $v$ and all other vertices of $\Gamma$. The diameter of a graph $\Gamma$, denoted by $\text{diam}(\Gamma)$, is defined as $\text{diam}(\Gamma) = \max\{ecc(v) \mid v \in V(\Gamma)\}$ [8]. The detour index, eccentric connectivity and total eccentricity polynomials are defined by $D(\Gamma,x) = \sum_{u,v \in V(\Gamma)} x^{D(u,v)}$ [9], $\zeta(\Gamma,x) = \sum_{u \in V(\Gamma)} \deg(u)x^{ecc(u)}$ and $\theta(\Gamma,x) = \sum_{u \in V(\Gamma)} x^{ecc(u)}$ [10], respectively. The detour index $dd(\Gamma)$, the eccentric connectivity index and the total eccentricity $\zeta(\Gamma)$ of a graph $\Gamma$ are the first derivatives of their corresponding polynomials at $x = 1$, respectively. The transmission of a vertex $v$ in $\Gamma$ is $\tau(v) = \sum_{u \in V(\Gamma), u \neq v} d(u,v)$. The transmission of a graph $\Gamma$ is $\tau(\Gamma) = \sum_{u \in V(\Gamma)} \tau(u,v)$. The mean (average) distance of graph $\Gamma$ is $\mu(\Gamma) = \frac{\sigma(\Gamma)}{p^{(p-1)/2}}$, where $p$ is the order of $\Gamma$ [3,1,12].

Khasraw [13] studied the intersection graph of subgroups of the group $D_{2n}$, where $n = p^2$, $p$ is a prime. He found some topological indices of the graph $\Gamma_{D_{2p^2}}$ as well as its metric dimension and resolving polynomial.

In this paper, we consider the graph $\Gamma_{D_{2pq}}$ of the dihedral group $D_{2pq}$ where $p$ and $q$ are distinct primes. Some properties of the connected graph $\Gamma_{D_{2pq}}$ will be presented. The dihedral group $D_{2pq}$ of order $2pq$ is defined by $D_{2pq} = \langle r, s \mid r^{pq} = s^2 = 1, srs = r^{-1} \rangle$ for prime numbers $p < q$.

2. Some properties of the intersection graph of $D_{2pq}$ for prime numbers $p < q$

In order to determine the vertex set of the graph $\Gamma_{D_{2pq}}$, it is required to list all non-trivial proper subgroups of the dihedral group $D_{2pq}$ for distinct primes $p < q$. In [6], the set of all non-trivial proper subgroups of the group $D_{2n}$ are classified for all $n \geq 3$. Here, we only consider the case when $n = pq$ for distinct primes $p < q$.

Lemma 2.1[6]. The non-trivial proper subgroups of the dihedral group $D_{2pq}$ for distinct primes $p < q$ are:

1- cyclic groups $G_i = \langle s^i \rangle$ of order $2$, where $i = 1, 2, ..., pq$.
2- dihedral groups $H_i^p = \langle r^p, s^i \rangle$ of order $2p$, where $i = 1, 2, ..., p$ and $H_i^q = \langle r^q, s^i \rangle$ of order $2q$, where $i = 1, 2, ..., q$. 

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3- cyclic groups $I_p = \langle r^p \rangle$ of order $q$, $I = \langle r \rangle$ of order $pq$, and $I_q = \langle r^q \rangle$ of order $p$.

According to the above classification of subgroups of the group $D_{2pq}$ for primes $p < q$, as given in Lemma 2.1, we can determine the structure of the set of vertices of the graph $\Gamma_{D_{2pq}}$ as the non-trivial proper subgroups by $V(\Gamma_{D_{2pq}}) = A \cup B \cup C$, where $A = \{G_1, G_2, \ldots, G_{pq}\}$, $B = \{H_i^p, H_i^q \mid 1 \leq i \leq p; 1 \leq j \leq q\}$, and $C = \{I_p, I, I_q\}$. So, we can distinguish subgraphs $\Gamma_A$ as complement of the complete graph $K_{pq}$, $\Gamma_{B \cup \{l\}}$ as the complete graph $K_{p+q+1}$, and $\Gamma_{C \setminus \{l\}}$ as the complement of the complete graph $K_2$. Through this article, we fixed the sets $A$, $B$, and $C$.

In this section, some basic properties of the intersection graph of $D_{2pq}$ are investigated, such as the order and chromatic number of the graph $\Gamma_{D_{2pq}}$.

**Theorem 2.2.** The order of the graph $\Gamma_{D_{2pq}}$ is $|V(\Gamma_{D_{2pq}})| = p + q + 3$.

**Proof:** Since the set of vertices of $\Gamma_{D_{2pq}}$ are the non-trivial subgroups of $D_{2pq}$ which are classified in the sets $A$, $B$, and $C$, and since $|A| = pq$, $|B| = p + q$, and $|C| = 3$, then $|V(\Gamma_{D_{2pq}})| = |A| + |B| + |C| = pq + p + q + 3$.

**Theorem 2.3.** The size of the graph $\Gamma_{D_{2pq}}$ is $|E(\Gamma_{D_{2pq}})| = \frac{(p+q)^2 + 4(pq+1)+3(p+q)}{2}$.

**Proof:** It is clear that each vertex of $A$ is adjacent with only two vertices of $B$. The vertices in the set $A$ are non-adjacent. Also, each vertex of $B \cup \{l\}$ is adjacent with all other vertices of $B \cup \{l\}$; that is, $B \cup \{l\}$ is a complete graph. Moreover, the vertex $I_p \in C$ is adjacent with $p$ vertices of $B$, which are $H_i^p; i = 1, 2, \ldots, p$, and $I_q \in C$ is adjacent with $q$ vertices of $B$ which are $H_i^q; j = 1, 2, \ldots, q$. Finally, the vertex $I \in C$ is adjacent with $I_p$ and $I_q$. Thus $|E(\Gamma_{D_{2pq}})| = 2pq + \frac{(p+q+1)(p+q)}{2} + p + q + 2$.

**Theorem 2.4.** The chromatic number of the graph $\Gamma_{D_{2pq}}$ is $\chi(\Gamma_{D_{2pq}}) = p + q + 1$.

**Proof:** From Theorem 2.2, $cl(\Gamma_{D_{2pq}}) = p + q + 1$. This means that the graph $\Gamma_{D_{2pq}}$ is at least $p + q + 1$ colorable graph. The vertices $G_1, G_2, \ldots, G_{pq}$ can be colored with the same color as the vertex $I$, the vertices $I_p$ and $H_i^q$ can share the same color, and the vertices $I_q$ and $H_i^p$ can share the same color. Thus, the minimum number of colors that can be used to color the graph $\Gamma_{D_{2pq}}$ is $p + q + 1$.

Therefore, $\chi(\Gamma_{D_{2pq}}) = p + q + 1$.

**Theorem 2.5.** Let $\Gamma = \Gamma_{D_{2pq}}$ be the graph of the dihedral group $D_{2pq}$. Then $diam(\Gamma) = 3$.

**Proof:** Let $u$ and $v$ be two distinct vertices in $\Gamma$. If $u$ and $v$ are joint by an edge, then $d(u, v) = 1$. Otherwise, $u \cap v = \{e\}$. There are five cases to consider.

Case1. If $u = G_i$ and $v = G_j$, where $i \equiv j \pmod{p}$ or $i \equiv j \pmod{q}$, then there exists $v' \in B$ such that $v' = H_k^p$ or $v' = H_k^q$ for some $k$ and $k'$. If $i \equiv k \pmod{p}$ or $j \equiv k' \pmod{q}$, then $uv', v'v \in E(\Gamma)$ and so $d(u, v) = 2$. Otherwise, if $i \not\equiv j \pmod{p}$ and $i \not\equiv j \pmod{q}$, take $v' = H_k^p$, then there exists $w \in B$, where $w = H_l^p$ such that $k \not\equiv l \pmod{p}$ and $k \not\equiv l \pmod{q}$.

Thus, $uv', v'w, wv \in E(\Gamma)$ and then $d(u, v) = 3$.

Case2. If $u = G_i$ and $v = H_i^p$ or $v = H_i^q$, $i = 1, \ldots, p; k = 1, \ldots, q$, where $i \not\equiv j \pmod{p}$ and $k \not\equiv j \pmod{q}$, then there exists $v' \in B$ such that $v' = H_i^p$ or $v' = H_i^q$, where $j \equiv l \pmod{q}$, so $uv', v'v \in E(\Gamma)$ and $d(u, v) = 2$.

Case3. If $u = I_p$ and $v = I_q$, then we take $v' = I$ so that $uv', v'v \in E(\Gamma)$ and $d(u, v) = 2$.

Case4. If $u = I_p$ and $v \in \{H_i^q \mid i = 1, \ldots, q\}$ (or $u = I_q$ and $v \in \{H_i^p \mid i = 1, \ldots, p\}$), then we take $w = I$, which implies that $uw, wv \in E(\Gamma)$ and so $d(u, v) = 2$. 

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Case 5. If \( u = G_j \) and \( v \in C \), then there are three possibilities for \( v \) If \( v = I_p \), then there exists \( v' \in \{ H_i^p \mid i = 1, \ldots, p \} \) such that \( uv', v'v \in E(\Gamma) \) if \( i \equiv j \pmod{p} \). If \( v = I_q \), then there exists \( v' \in \{ H_i^q \mid i = 1, \ldots, q \} \) such that \( uv', v'v \in E(\Gamma) \) if \( i \equiv j \pmod{q} \). Finally, if \( v = I \), then there exists \( v' \in B \) such that \( uv', v'v \in E(\Gamma) \). In all possibilities, \( d(u, v) = 2 \).

As a consequence from the above theorem, we state the following.

**Corollary 2.6.** Let \( \Gamma = \Gamma_{D_{2pq}} \) be the graph of the dihedral group \( D_{2pq} \). Then

\[
d(u, v) =
\begin{cases}
1 & \text{if } u = G_i, v = H_j^p \land i \equiv j \pmod{p}, \ 1 \leq i \leq pq, 1 \leq j \leq p, \\
or \ u = G_i, v = H_j^q \land i \equiv j \pmod{q}, 1 \leq i \leq pq, 1 \leq j \leq q, \\
2 & \text{if } u = G_i, v = G_j, (i \equiv j \pmod{p} \ or \ q) 1 \leq i, j \leq pq \land i \neq j, \\
or \ u = G_i, v = H_j^p \land i \neq j \pmod{p}, 1 \leq i \leq pq, 1 \leq j \leq p, \\
or \ u = G_i, v = H_j^q \land i \neq j \pmod{q}, 1 \leq i \leq pq, 1 \leq j \leq q, \\
3 & \text{if } u = G_i, v = G_j, (i \neq j \pmod{p} \land i \neq j \pmod{q}) 1 \leq i, j \leq pq.
\end{cases}
\]

**Lemma 2.7.** Let \( \Gamma = \Gamma_{D_{2pq}} \) be the intersection graph of subgroups of the dihedral group \( D_{2pq} \) with distinct primes \( p \) and \( q \). Then

\[
deg_{\Gamma}(v) =
\begin{cases}
2 & \text{if } v = G_i, \ for \ 1 \leq i \leq p, \\
p + 1 & \text{if } v = I_p, \\
q + 1 & \text{if } v = I_q, \\
p + q + 2 & \text{if } v = I, \\
p + 2q + 1 & \text{if } v = H_i^p, \ for \ 1 \leq i \leq p, \\
2p + q + 1 & \text{if } v = H_i^q, \ for \ 1 \leq j \leq q.
\end{cases}
\]

Proof: see [7].

### 3. Detour index, eccentric connectivity, and total eccentricity polynomials of the graph \( \Gamma_{D_{2pq}} \)

In this section, we find detour index, eccentric connectivity, and total eccentricity polynomials of the intersection graph \( \Gamma_{D_{2pq}} \) of \( D_{2pq} \).

**Theorem 3.1.** Let \( \Gamma_{D_{2pq}} \) be the intersection graph of \( D_{2pq} \) with primes \( p < q \). Then

\[
D(u, v) =
\begin{cases}
3p + q - 1 & \text{if } u = H_i^p, v = H_j^p, 1 \leq i, j \leq p \land i \neq j, \\
3p + q & \text{if } u = H_i^p, v \in \{I, I_q, H_j^q; 1 \leq j \leq q\}, 1 \leq i \leq p, \\
3p + q + 1 & \text{if } u = H_i^p, v \in \{I_p, G_j; 1 \leq j \leq pq\}, 1 \leq i \leq p, \\
or \ u = H_i^q, v \in \{I, I_q, L_j^i, I_q; 1 \leq i \leq q, \\
or \ v = I, v \in \{I_p, I_q\}, \\
or \ u = G_i, v = H_j^q, 1 \leq i \leq pq, 1 \leq j \leq q \\
or \land uv \in E(\Gamma), \\
3p + q + 2 & \text{if } u = G_i, v \in \{I, I_q\}, 1 \leq i \leq pq, \\
or \ u = I_p, v \in \{I_q, H_i^q; 1 \leq i \leq q\}, \\
or \ v = G_i, v = H_j^q, 1 \leq i \leq pq, 1 \leq j \leq q \\
or \land uv \in E(\Gamma), \\
3p + q + 3 & \text{if } u = G_i, v \in \{I_p, G_j\}, 1 \leq i, j \leq pq \land i \neq j.
\end{cases}
\]

Proof: For \( D(u, v) = 3p + q - 1 \), the longest path from \( H_i^p \) to \( H_j^p \) where \( 1 \leq i, j \leq p \) and \( i \neq j \) is the path that starts from \( H_i^p \), passing alternatively through \( 2p - 3 \) elements of...
A. p + q − 1 elements of B, and \(I_p, I\) and \(I_q\) vertices of B, and ending at \(H_i^P\). So, the path has length

\[(2p - 3) + (p + q - 1) + 3 = 3p + q - 1.\]

Hence \(D(H_i^P, H_j^P) = 3p + q - 1\).

For \(D(u, v) = 3p + q\), there are two cases. Case 1, the longest path, that starts from \(H_i^P\) for some \(1 \leq i \leq p\) to \(S \in \{I, I_q\}\), is the path passing alternatively through \(2p - 1\) of vertices of A, \(p + q - 3\) elements of B, and \(\{I, I_q\}\) vertices of B, and ending at \(S \in \{I, I_q\}\). So, the length of this path is

\[1 + (2p - 1) + (p + q - 3) + (1 + 2) = 3p + q.\]

Thus, \(D(H_i^P, X) = 3p + q\), for all \(1 \leq i \leq p\) and \(X \in \{I, I_q\}\).

Case 2, the longest path, that starts from \(H_i^P\) for some \(1 \leq i \leq p\) to \(H_j^q\), for some \(1 \leq j \leq q\), is the path passing alternatively through \(2p - 1\) of vertices of A, \(p + q - 3\) elements of B, and \(I_q\) and \(I\) element of B, and ending at \(H_j^q\), for some \(1 \leq j \leq q\). So, the length of this path is

\[1 + (2p - 1) + (p + q - 3) + (2 + 1) + 1 - 1 = 3p + q.\]

Thus, \(D(H_i^P, H_j^q) = 3p + q\), for all \(1 \leq i \leq p\) and \(1 \leq j \leq q\).

For \(D(u, v) = 3p + q + 1\), the longest path, that starts from \(G_i\) to \(H_j^q\) for some \(1 \leq j \leq p\) for some \(1 \leq i \leq pq\), is the path passing alternatively through \(p + q - 1\) vertices of B, \(2p - 1\) vertices of A, and \(I\) and \(I_p\) vertices of C, and ending at \(H_j^q\) for some \(1 \leq j \leq p\). So, the length of the path is

\[1 + (p + q - 1) + (2p - 1) + 2 + 1 - 1 = 3p + q + 1.\]

Thus, \(D(G_i, H_i^P) = 3p + q + 1\), for all \(1 \leq i \leq p\) and \(1 \leq j \leq p\).

The longest path, that starts from \(H_i^P\) to \(I_p\) for some \(1 \leq i \leq p\), is the path passing alternatively through \(2p - 1\) vertices of A and \(p + q - 3\) elements of B with I, and ending at \(I_p\). So the length of the path is

\[1 + (2p - 1) + (p + q - 3) + (2 + 1) + 1 - 1 = 3p + q + 1.\]

Thus, \(D(H_i^P, I_p) = 3p + q + 1\), for all \(1 \leq i \leq p\).

The longest path, that starts from the vertex \(H_i^q\) to \(I\) for some \(1 \leq i \leq q\), is the path passing alternatively through \(2p\) vertices of A, \(p + q - 1\) vertices of B, and the vertex \(I_q\) of C, and ending at \(I\). So the length of the path is

\[1 + (2p) + (p + q - 1) + 1 + 1 - 1 = 3p + q + 1.\]

Hence, \(D(H_i^q, I) = 3p + q + 1\), for all \(1 \leq i \leq q\).

In a similar way, we can prove the detour distance between all other vertices in the graph \(\Gamma_{D_{2pq}}\).

Theorem 3.2. Let \(\Gamma_{D_{2pq}}\) be the intersection graph of \(D_{2pq}\) with distinct primes \(p < q\). Then

\[
D\left(\Gamma_{D_{2pq}}\right) = \left(\frac{pq-1}{p-2}\right)^3 + \sum_{u,v} D^{D(u,v)} + \left(\frac{pq}{2}\right)^3 + \left(\frac{pq}{2}\right)^2 \frac{pq-1}{p-2}.
\]

Proof: It follows from Theorem 3.1 that

\[
D\left(\Gamma_{D_{2pq}}\right) = \sum_{u,v \in V} x^{D(u,v)} = \left(\frac{pq}{2}\right)^3 + \left(\frac{pq}{2}\right)^2 \frac{pq-1}{p-2}.
\]
where $(p-1)(p-2)$, $(p)$ = $p(p-1)$ and $(q)$ = $q(q-1)$.

Therefore,

$D \left( I_{D_{pq}}'x \right) = \frac{(p-1)(p-2)}{2} + \frac{q(q-1)}{2} + 2p + 2q$.

### Corollary 3.3.
Let $\Gamma_{D_{pq}}$ be the intersection graph of $D_{pq}$ with distinct primes $p < q$. Then

$$dd(\Gamma_{D_{pq}}) = 3p^3(q + 1) + q^3 \left( p^2 + \frac{1}{2} \right) - \frac{3}{2}p^3 + \frac{3}{2}q^2p + \frac{3}{2}q^2 + 4p^2q^2 + 5p^2 + 3q^2 + 3pq + \frac{33}{2}p + \frac{17}{2}q + 10.$$

**Proof:** The result follows directly by taking the first derivative of $D \left( I_{D_{pq}}'x \right)$ at $x = 1$.

### Theorem 3.4.
Let $\Gamma_{D_{pq}}$ be the intersection graph of $D_{pq}$ with distinct primes $p < q$. Then

$ecc(v) = \begin{cases} 2 & \text{if } v \in B \cup \{i\}, \\ 3 & \text{if } v \in A \cup C \setminus \{i\}. \end{cases}$

**Proof:** The proof follows directly from Corollary 2.6.

### Theorem 3.5.
Let $\Gamma_{D_{pq}}$ be the intersection graph of $D_{pq}$ with distinct primes $p < q$. Then

$$\zeta \left( I_{D_{pq}}'x \right) = (2pq + p + q + 2)x^3 + [(p + q)^2 + 2(pq + p + q + 1)]x^2.$$  

**Proof:** It follows from Lemma 2.7 and Theorem 3.4 that

$$\zeta \left( I_{D_{pq}}'x \right) = \sum_{u \in V(\Gamma_{D_{pq}})} \deg(u)x^{ecc(u)} = 2pqx^3 + (q + 2 + p + q - 1)p + 2 + (p + q - 1)x^2 + (p + q + 2)x^2 + (q + 1)x^3.$$  

### Theorem 3.6.
Let $\Gamma_{D_{pq}}$ be the intersection graph of $D_{pq}$ with distinct primes $p < q$. Then

$$\theta \left( I_{D_{pq}}'x \right) = (pq + 2)x^3 + (p + q + 1)x^2.$$  

**Proof:** It follows from Theorem 3.4 that

$$\theta \left( I_{D_{pq}}'x \right) = \sum_{u \in V(\Gamma_{D_{pq}})} x^{ecc(u)} = pqx^3 + px^2 + qx^2 + x^2 + x^2 + x^2 = (pq + 2)x^3 + (p + q + 1)x^2.$$  

### Theorem 3.7.
Let $\Gamma_{D_{pq}}$ be the intersection graph of $D_{pq}$ with distinct primes $p < q$. Then

$$\zeta \left( I_{D_{pq}}'x \right) = 2(p^2 + q^2) + 7(2pq + p + q) + 10.$$  

**Proof:** From Theorem 3.5, one can see that

$$\frac{d}{dx} \zeta \left( I_{D_{pq}}'x \right) |_{x=1} = 3(2pq + p + q + 2) + 2[(p + q)^2 + 2(pq + p + q + 1)].$$  

The result follows.

### 4. The mean distance of the intersection graph $\Gamma_{D_{pq}}$

In this section, we find the mean distance of the intersection graph of subgroups of $D_{pq}$ for distinct prime numbers $p$ and $q$.

### Theorem 4.1.
The transmission of the graph $\Gamma_{D_{pq}}$ is

$$\sigma \left( I_{D_{pq}}' \right) = p^2(3q + 1)(q + 1) + q^2(3p + 1) + q(8p + 7) + 7p + 8.$$  

**Proof:** From Corollary 2.6, we have

$$\sigma(G_i) = q(2) + (pq - (q + 1))(3) + 2(1) + (p + q - 2)(2) + 2(2) + (1)(3) = 3pq + q + 2p + 2,$$

for all $i = 1,2,...,pq$,

$$\sigma(H_i^p) = q(1) + (pq - q)(2) + (p + q - 1)(1) + 2(1) + (1)(2) = 2pq + p + 3,$$

for all $i = 1,2,...,p$.

Also,

$$\sigma(H_i^q) = p(1) + (pq - p)(2) + (p + q - 1)(1) + 2(1) + (1)(2) = 2pq + q + 3,$$

for all $i = 1,2,...,q$.  

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Note that the vertices $I_p$ and $I_q$ are non-adjacent but the vertex $I$ is adjacent to both $I_p$ and $I_q$. So, $C = \{I_p, I, I_q\}$ induced a path subgraph of $\Gamma_{D_{2pq}}$.

Thus, $\sigma(I_p) = pq(2) + p(1) + q(2) + (1)(1) + (1)(2) = 2pq + p + 2q + 3$, 
$\sigma(I) = pq(2) + (p + q)(1) + 2(1) = 2pq + p + q + 2$, and 
$\sigma(I_q) = pq(2) + q(1) + p(2) + (1)(1) + (1)(2) = 2pq + 2p + q + 3$.

Now, we can find the transmission of the graph $\Gamma_{D_{2pq}}$ as 
\[
\sigma(\Gamma_{D_{2pq}}) = \sum_{i=1}^{p} \sigma(G_i) + \sum_{i=1}^{p} \sigma(H_i) + \sum_{i=1}^{q} \sigma(H_i) + \sigma(I) + \sigma(I_p) + \sigma(I_q) 
= p^2(3q + 1)(q + 1) + q^2(3p + 1) + q(8p + 7) + 7p + 8.
\]

**Theorem 4.2.** The mean distance of the graph $\Gamma_{D_{2pq}}$ is 
\[
\mu(\Gamma_{D_{2pq}}) = \frac{p^2(3q+1)(q+1)+q^2(3p+1)+q(8p+7)+7p+8}{(pq+p+q+3)(pq+p+q+2)}.
\]

Proof: Since the order of the graph $\Gamma_{D_{2pq}}$ is $pq + p + q + 3$ and the transmission of the graph $\Gamma_{D_{2pq}}$ is given in Theorem 4.1, we can find the mean distance of the graph $\Gamma_{D_{2pq}}$ as 
\[
\mu(\Gamma_{D_{2pq}}) = \frac{p^2(3q+1)(q+1)+q^2(3p+1)+q(8p+7)+7p+8}{(pq+p+q+3)(pq+p+q+2)}, \text{ where } p < q \text{ are prime numbers.}
\]

**References**


