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s-Compressible and s-Prime Modules

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Abstract
Let *R* be a ring with identity and *A* a left *R*-module. In this article, we introduce new generalizations of compressible and prime modules, namely s-compressible module and s-prime module. An *R*-module *A* is s-compressible if for any nonzero submodule *B* of *A* there exists a small *f* in $\text{Hom}_R(A, B)$. An *R*-module *A* is s-prime if for any submodule *B* of *A*, $\text{ann}_R(B)A$ is small in *A*. These concepts and related concepts are studied in as well as many results consist properties and characterizations are obtained.

Keywords: critically s-compressible module, retractable module s-compressible module**,** s-rime module, small submodule.

الموديولات المضغوطة من النمط sوالاولية من النمطs

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الخلاصة
لتكن *R* حلقة مع محايد و*A* مقاس ايسر على الحلقة *R*. في هذا البحث ، تم تقديم تعميمات جديدة للمقاسات القابلة للانضغاط والمقاسات الأولية هي المقاس القابل للانضغاط من النمط s والمقاس الاولي من النمط s. حيث يكون المقاس *A* قابل للانضغاط من النمط s إذا كان لأي مقاس جزئي غير صفري *B* يوجد هومومورفزم صغير *f* من *A* الى*B*. ويكون المقاس *A* اولي من النمط s إذا كان لأي مقاس جزئي *B* ،فأن *A* (ann_R (*B*) يكون صغيرا في *A*. .كذلك تمت دراسة هذه المفاهيم والمفاهيم ذات الصلة وتم الحصول على العديد من النتائج من الخصائص والتوصيفات.

1. Introduction

Compressible module was introduced by Zelmanowitz [1] simultaneous with introducing the concept of weakly primitive ring in the way of generalizing the Jacobson density theorem. He also introduced critically compressible module. In[2], the author studied those concepts in details. A left *R*-module is compressible if it can be embedded in any of its nonzero submodule[1]. A compressible module *A* is critically compressible if it cannot be embedding in any factor *A*/*B*, where *B* is a nonzero submodule of *A* . In[1], Zelmanowitz defined a ring to be weakly primitive if it possesses a faithful critically compressible module. In[3]–[6], authors have been extensively studied compressible, critically compressible and prime modules. By using small submodules one direction of generalizations of compressible and prime modules e appeared in [7]–[9]. A small compressible module is defined as a module that can be embedded in its small submodules, as well as small prime module is defined as a

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module A in which $ann_R B=ann_R A$ for each small submodule B of A. Note that a module A is prime, if $ann_R B=ann_R A$ for each nonzero submodule B of A [7].

Throughout this work, we use the notion of small submodule . Different generalizations are given. We recall that, an R-homomorphism in Hom(A, B) is said to be small if its kernel is small in A[10]. In the new generalization the zero kernel will be replaced by small kernel. An R-module A is said to be s-compressible if for each nonzero submodule B of A there exists a small element f in Hom(A, B), that is ker$f$ is small in A. Note that this definition is also appeared in [11] with different abbreviation, sk-compressible.

An s-compressible module A is critically s-compressible if Hom(A, A/B) has no small element for any non-small submodule B of A. A module A is s-prime if ($ann_R B$) A is small in A for any nonzero submodule B of A. These concepts are studied, and their relationships among them and with other related concepts are discussed. Some properties and characterizations are obtained. Firstly, it is shown that s-compressible with small compressible modules are independent, as well as the s-prime and small prime modules are also independent. The class of compressible modules contains both classes of s- compressible and small compressible modules. As well as the class of prime modules contains both classes of s- prime and small prime modules.

Throughout this article some definitions and notations are given. A module is a left unitary module over a ring R with identity. A submodule B of a module A will be abbreviated by $B \leq A$. A submodule B of a module A is said to be small in A(abbreviated by $B << A$) if it is proper and its sum with any other proper submodule of A is again proper, "in other word if $B + C= A$, where $C\leq A$, then $C= A$ [10]. A is said to be hollow if all its proper submodules are small. Hom_R(D, E) denotes the set of all R-homomorphisms from D into E. If $f\in$Hom(D, E), then ker$f=\{d\in D|f(d)=0\}$, f is a monomorphism if ker$f$=0 and it is small if ker$f<<D$[10].
If $B \subseteq A$, then $ann_R B=\{r \in R|\ rb=0$ for all $b\in B\}$ which is called the annihilator of B in R and it is a left ideal of R if $b\in B$, then $ann_R b= ann_R\{b\}$. If $B \leq A$, then $[B:_R A]= \{ r \in R\ |rA\subseteq B\}$ is a left ideal of R. An R-module A is multiplication if for any submodule B of A there exists an ideal I of R such that $B = I A$, in this case $I=[B:_R A]$ [12]. An R-module A is retractable if $Hom_R (A, B)\neq 0$ for any nonzero submodule B of A [13].

In Section 2 s-compressible and critically s-compressible modules are introduced and investigated. The notion s-compressible is appeared in [11]. It is abbreviated by sk-compressible. In this work this notion is studied in details and more results are given. Section 3 devotes to introduce s-prime module and study the relationships between the present notions and old related notions.

2. s-Compressible and Critically s-Compressible Modules

Definition (2.1): A nonzero R-module A is called s-compressible if for any nonzero submodule B of A there exists a small R-homomorphism from A into B.
Remark (2.2): Any compressible module is s-compressible, however the converse is not true.
Remark (2.3): Any simple module is s-compressible.
Example (2.4): Consider the $\mathbb{Z}$- module $\mathbb{Z}_n$, if $n=mp^k$ where p is a prime which is not dividing m, thus if $s\mathbb{Z}_n$ is a small submodule of $\mathbb{Z}_n$, then $s=pt$ for some t.
Note that, in a R -module A, the submodule Ra is small in A if and only if a belongs to all maximal submodules of A [10].
Now, if f: $\mathbb{Z}_n \rightarrow p^k\ \mathbb{Z}_n$ is a $\mathbb{Z}$ -homomorphism such that ker f = s $\mathbb{Z}_n$ small in $\mathbb{Z}_n$, then |ker f|= n/s, so that $|\ \mathbb{Z}_n/\ker f|=s=pt$, while $|\ p^k\ \mathbb{Z}_n|= m$, this gives a contradiction with the fact that $\mathbb{Z}_n/\ker f$ is isomorphic to a submodule of $p^k\ \mathbb{Z}_n$. Therefore, there is no small $\mathbb{Z}$ -homomorphism from $\mathbb{Z}_n$ into $p^k\ \mathbb{Z}_n$, that is, $\mathbb{Z}_n$ is not s-compressible if $n=mp^k$ and p is a prime which is not dividing m.

On the other hand $n=p^k$, the $\mathbb{Z}$-module $\mathbb{Z}_n$ is hollow, all its proper submodules are small. it is easy to see that it is s-compressible. Therefore the $\mathbb{Z}$- module $\mathbb{Z}_n$ is s-compressible if and only if $n= p^k$ where p is prime.

We note that the two notions small compressible and s-compressible are independent. For example $\mathbb{Z}_6$ is small compressible $\mathbb{Z}$- module which is not s-compressible, while $\mathbb{Z}_4$ is s-compressible that is not small compressible $\mathbb{Z}$- module[8]. Both of two $\mathbb{Z}$- modules are not compressible. The two classes of small and s-compressible modules contain the class of compressible modules.

Remark 2.5: It is clear that any s-compressible module is retractable. However the converse is not true to see that $\mathbb{Z}_6$ as a $\mathbb{Z}$-module is retractable but not s-compressible.

Next proposition gives However, a condition can be added to a retractable module to get s-compressible module, see the following.

Proposition 2.6: Any hollow retractable module is s-compressible.

*Proof***:** Assume that A is hollow retractable module, and B is a nonzero submodule of A, then there exists $0\neq f\in\text{Hom}(A, B)$ such that $\ker f$ is a proper submodule of A, hence small in A. Therefore A is s-compressible.

This proposition can be applied to example 2.4 so that $\mathbb{Z}_{p^k}$ is s-compressible.

We note that the $\mathbb{Z}$-module $\mathbb{Z}$ is s-compressible but not hollow, and this proves that the converse of proposition 2.6 is not true.

Proposition 2.7: If B is a submodule of an s- compressible module A such that $J(B)=J(A)$, then B is s-compressible.

Proof: Assume that B is a submodule of an s- compressible module A and $J(B)=J(A)$. If $K\leq B$, then $K\leq A$, hence there exists $f\in\text{Hom}(A, K)$ with $\ker f << A$. Now if $g=f|_B$ then $g\in\text{Hom}(B, K)$, and $\ker g= B\cap\ker f \subseteq B\cap J(A)= B\cap J(B) \leq J(B)$, so that $\ker g<< B$. Therefore B is s-compressible. □

Example 2.8:

(i) Consider $A = \mathbb{Q}\oplus\mathbb{Z}_p$, where p is prime, as a $\mathbb{Z}$-module and $B = \mathbb{Q}\oplus 0$,

then $B \leq A$ and $J(B)=J(A)= \mathbb{Q}\oplus 0$.

(ii) Let $A = \mathbb{Z}$ as a $\mathbb{Z}$-module and $B= n\,\mathbb{Z}$, then $J(B)=J(A)= 0$.

Corollary 2.9: If $J(A)=0$ and A is an s-compressible module, then any submodule of A is s-compressible.

Proposition 2.10: If A is an s-compressible module and B is a nonzero submodule of A, then $(\text{ann } B)\, A << A$.

Proof: Since A is an s-compressible, then there exists $f\in\text{Hom}(A, B)$ with $\ker f << A$. Let $r\in \text{ann } B$, so that for each $m\in A$, $f(rm)=rf(m)=0$, then $rm\in \ker f<< A$, this implies that $(\text{ann } B) A\subseteq \ker f<< A$. Therefore $(\text{ann } B)\, A << A$. □

The converse of Proposition 2.10 is not true, for example if A is a torsion free R-module, then $\text{ann } B =0$ for any non zero submodule B of A, hence $(\text{ann } B)\, A =0<< A$. While there are many torsion free modules not s-compressible, e.g. the $\mathbb{Z}$-module $\mathbb{Q}$.

Proposition 2.11: If A is an R- module with $J(A)=0$, then A is s- compressible if and only if it is compressible.

Proof: The sufficiency is clear. Conversely, $J(A)=0$ implies that A has no nonzero small submodule, so, if A is s-compressible, there exists $f\in\text{Hom}(A, B)$ with $\ker f$ small in A which implies $\ker f=0$ and f is a monomorphism. □

It is well known that a nonzero submodule of a compressible module is compressible. In the following this property will be discussed under certain condition for s-compressibility.

Recall that, an R- module A is said to be fully stable, if for each submodule B of A and for each $f\in \text{Hom}(B, A)$, it follows $f(B) \subseteq B$ [12], In fact A is fully stable if and only if $\text{Hom}(B, A)=\text{End}(B)$ for each submodule B of A ,and more details about fully stable modules can be found in [12]. For completeness a proof will be given.

Lemma 2.12: If A is a fully stable module, $B = B_1 \oplus B_2$ and K are submodules of A, then $K \cap B = (K \cap B_1) \oplus (K \cap B_2)$.

Proof: The natural projections of B onto B_1 and B_2, respectively π_1, π_2 are elements of Hom(B, B)= End(B), in fact, $\pi_1 \in$Hom(B, B_1) and $\pi_2 \in$ Hom(B, B_2). On the other hand $\pi_1 + \pi_2 = 1_B$, so, $K \cap B = \pi_1(K \cap B) + \pi_2(K \cap B)$. Since A is fully stable, $\pi_i(K \cap B_i) \subseteq K \cap B_i$, ($i$=1, 2) but $\pi_i(K \cap B) \subseteq B_i$ so $\pi_i(K \cap B) \subseteq K \cap B_i$. Hence $K \cap B \subseteq (K \cap B_1) \oplus (K \cap B_2) \subseteq K \cap B$. □

It is known that any small submodule of a module is contained in its Jacobson radical,while a submodule that contained in the Jacobson radical of the module is small if it is finitely generated [10].

Proposition 2.13: A finitely generated direct summand of a fully stable s-compressible module is s-compressible.

Proof : Assume that $A = A_1 \oplus A_2$ is an s-compressible module and B is a submodule of A_1, then B is a submodule of A, by assumption there exists $f \in$Hom(A, B) with ker$f << A$. Let $g = f|_{A_1}$, then ker$g = A_1 \cap$ ker$f$. It is known that $J(A) = J(A_1) \oplus J(A_2)$. But ker$f \subseteq J(A) = J(A_1) \oplus J(A_2)$ implies $A_1 \cap$ ker$f \subseteq A_1 \cap (J(A_1) \oplus J(A_2)) = J(A_1)$(by full stability) so that ker$g \subseteq J(A_1)$ and ker$g << A_1$. Therefore A_1 is s-compressible. □

Remark 2.14: The converse of Proposition 2.13 is not true to see that let $\mathbb{Z}_6 = \langle\bar{2}\rangle \oplus \langle\bar{3}\rangle$ is fully stable [11] and both $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are s-compressible ,however $\mathbb{Z}_6$ is not s-compressible, as we have seen in Example2.4.

Remark 2.15: It is clear that a homomorphic image of an s-compressible module need not be s-compressible. For instance $\mathbb{Z}$ is an s-compressible $\mathbb{Z}$-module, however $\mathbb{Z}/6\mathbb{Z}$ is not.

Proposition 2.16: If $A_1$and A_2 are two isomorphic modules, then A_1 is s-compressible if and only if A_2 is s-compressible.

Proof: Assume that φ: $A_1 \to A_2$ is an isomorphism and A_1 is s-compressible. Let B be a nonzero submodule of A_2. Then $\varphi^{-1}(B)$ is a nonzero submodule of A_1, by assumption there exists $\alpha: A_1 \to \varphi^{-1}(B)$ with ker $\alpha << A_1$. Let $\delta = j\alpha\, \varphi^{-1}$, where $j = \varphi|\, \varphi^{-1}(B)$, then $\delta \in$Hom(A_2, B) and ker $\delta = \varphi$(ker α) $<< A_2$. Hence A_2 is s-compressible. The proof of the other direction is similar. □

Lemma 1.17: If A is a multiplication module and $A = A_1 \oplus A_2$, then ann$_R$ A_i=[A_j: A], $i \neq j$, i, j=1, 2.

Proof: Let $r \in$ ann$_R$ A_1, then for each $m = m_1 + m_2$, $r(m_1 + m_2) = r\, m_2 \in A_2$, so that $r \in$[A_2: A]. Conversely, The $r \in$[A_2: A] implies that for each $m_1 \in A_1$, if m_2 is any element of A_2, then $m_1 + m_2 \in A$ and $r(m_1 + m_2) \in A_2$, which implies $rm_1 \in A_1 \cap A_2$, hence rm_1=0 and $r \in$ ann$_R$ A_1. This proves ann$_R$ A_1=[A_2: A]. By the same manner the other case can be proved.

Proposition 2.18: If A is a multiplication and s-compressible R-module then it is indecomposable.

*Proof***:** Assume that $A = A_1 \oplus A_2$, since A is multiplication, we have A_1= [A_1: A] A and A_2= [A_2: A] A. By Lemma 1.17, A_1= (ann$_R$ A_2) A and A_2= (ann$_R$ A_1) A, then A = (ann$_R$ A_2) $A \oplus$ (ann$_R$ A_1) A. But by Proposition 2.3 (ann$_R$ A_1) A and (ann$_R$ A_2) A are both small in A, which is a contradiction. Therefore A is indecomposable. □

An R-module A is said to be duo if any submodule of A is full invariant, that is, for each $f \in$ End(A) and for each $B \leq A$, $f(B) \subseteq B$" [14].,and it is said to be torsion free if $rm \neq 0$ whenever $0 \neq r \in R$ and $0 \neq m \in A$, or equivalently $0 \neq m \in A$ and rm=0 implies r=0

Next theorems give a characterization of Duo modules, we will start with the following lemma.

Lemma 2.19: "An R-module A is duo if and only if for each $f \in$ End(A) and for each $m \in A$ there exists $r \in R$ such that $f(m) = rm$"[14]**.**

Theorem 2.20: Let A be a duo torsion free R-module. Then A is compressible if and only if it is retractable.
Proof: (⟹) It is clear so that it is omitted .
(⟸) Assume that A is a duo torsion free R-module and retractable, let $0\neq B\leq A$, then there exists $0\neq f\in\text{Hom}(A, B)$, it can be considered that $f\in\text{End}(A)$. By Lemma2.19, for each $m\in A$ there exists $r\in R$ such that $f(m)=rm$. So $\ker f=\{ m\in A \mid rm=0 \text{ for some } r\in R\}$, as A is torsion free and $0\neq f$, it follows $\ker f=0$, that is A embed in B. Therefore A is compressible. □

A compressible module is said to be critically compressible if it cannot be embedded in any of its proper factors[2]. This notion was generalized in [7] using small submodule this way gives that a small compressible module A is called small critically compressible if A cannot be embedded in any proper quotient module A/B with $0\neq B\ll A$".

Another generalization will be given by using small submodule.

Definition 2.21: An R-module A is called critically s-compressible if it is s-compressible and for any not small submodule B of A, $\text{Hom}(A, A/B)$ contains no small element.

Remark 1.22: The two classes small critically compressible modules, and critically s-compressible modules are different (see Example2.23(ii)), and their intersection contains the class of critically compressible modules.

Example 2.23: (i) The $\mathbb{Z}$- module $\mathbb{Z}_n$ is critically s-compressible if and only if n= p^k where p is a prime.

*Proof***:** In Example 2.4, we proved that $\mathbb{Z}_n$ is s-compressible if and only if n= p^k where p is a prime. Since $\mathbb{Z}_{p^k}$has no proper submodule which is not small, so it is critically s-compressible.

(ii) $\mathbb{Z}_{p^k}$**,** is not small critically compressible module for $k>1$.While $\mathbb{Z}_6$ is small critically compressible $\mathbb{Z}$-module but not critically s-compressible.

(iii) The $\mathbb{Z}$- module $\mathbb{Z}$**,** also is critically s-compressible.

(iv) Any critically compressible module is critically s-compressible. But the converse is not true.

(v) Any simple module is critically s-compressible.

By partial endomorphism of a module A it means an element of $\text{Hom}(B, A)$ where B is a submodule of A.

Proposition 2.24: If A is a critically s-compressible module, then any nonzero partial endomorphism of A has kernel small in A.

Proof: Assume that A is a critically s-compressible module and $0\neq f\in\text{Hom}(B, A)$**,** where $B\leq A$, suppose that $\ker f$ is not small in A. Then $\text{Im}f\neq 0$ and there exists $0\neq g\in\text{Hom}(A, \text{Im}f)$ such that $\ker g<< A$ since A is s-compressible. On the other hand $\text{Im}f\cong N/\ker f\leq A/\ker f$, let h: $\text{Im}f\to B/\ker f$ be an isomorphism and i: $B/\ker f\to/\ker f$ be the inclusion map. Then $ihg\in\text{Hom}(A, A/\ker f)$ and $\ker ihg=\ker g<< A$. This contradicts the assumption that A is critically s-compressible.

To prove the converse of Proposition 2.24, we need a condition this is given in next proposition.

Proposition 2.25: Let A be an s-compressible module such that for any $L\leq A$ and $K << A$, any element of $\text{Hom}(L, A/K)$ has kernel small in A. Then A is critically s-compressible.

Proof: Assume that A is an s-compressible module satisfying the above condition. Let B be a submodule of A which is not small and $f\in\text{Hom}(A, A/B)$ and $\ker f<< A$. Then $A/\ker f\cong L/B$, where L is a submodule of A containing B. Let ν: $L\to L/B$ be the natural epimorphism and φ: $L/B\to A/\ker f$ be an isomorphism, then $\varphi\,\nu\in\text{Hom}(L, A/\ker f)$ and $\ker\varphi\,\nu= B$ which is not small in A, a contradiction with the assumed condition. Therefore, the kernel of any element of $\text{Hom}(A, A/B)$ is not small in A, that is, A is critically s-compressible. □

3. s-Prime Modules

Prime modules are defined and investigated in the literatures see [3][4][15]. An R-module A is said to be prime if for any nonzero submodule B of A, $\text{ann}_R B = \text{ann}_R A$. This notion is generalized in[7] using the concept of small submodules in this way an R-module A is a small prime module if $\text{ann}_R A = \text{ann}_R B$ for each non-zero small submodule B of A . We also use small submodules to give a different generalization for prime module, its properties, and characterizations as well as relations with s-compressible is also studied.

Definition 3.1: A nonzero R-module A is called s-prime if for any nonzero submodule B of A, $(\text{ann}_R B) A << A$.

Remark 3.2:

(i) The two notions small prime, and s-prime are independent. For example the $\mathbb{Z}$-module $\mathbb{Z}_4$ is an s-prime but not small prime (can be easily checked), while the $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is small prime , this is also shown in[7], However it is not s-prime since $(ann_{\mathbb{Z}}(\langle\bar{8}\rangle)\ \mathbb{Z}_{24}=3\mathbb{Z}(\mathbb{Z}_{24})=\langle\bar{3}\rangle$ not small in $\mathbb{Z}_{24}$.

(ii) We have seen that any torsion free R -module is s-prime.

(iii) It is clear that any prime module is s-prime. However the converse is not true, for example the $\mathbb{Z}$ -module $\mathbb{Z}_8$ is not prime since $ann_{\mathbb{Z}}(4\,\mathbb{Z}_8)= 2\mathbb{Z}$, while $ann_{\mathbb{Z}}(\mathbb{Z}_8)=8$. But $\mathbb{Z}_8$ is s-prime $\mathbb{Z}$-module (can be easily checked).

Proposition 3.3: An s-compressible module is s-prime.

Proof: See Proposition 2.10. □

It is clear the converse of Proposition 3.3 is not true, the $\mathbb{Z}$-module $\mathbb{Q}$ is s-prime since it is torsionfree (Remark 3.2(ii)) but not s-compressible since $\text{Hom}(\mathbb{Q}, \mathbb{Z})=0$.

Proposition 3.4: A nonzero R-module A is s-prime if and only if $(\text{ann}_R Rx) A << A$ for each $0\neq x\in A$.

Proof: (⟹) It is clear.

(⟸) Assume that $(\text{ann}_R Rx) A << A$ for each $0\neq x\in A$, and let $0\neq B\leq A$, then there exists $0\neq x\in B$ and $(\text{ann}_R B) \subseteq (\text{ann}_R Rx)$ which implies $(\text{ann}_R B) A \subseteq (\text{ann}_R Rx) A << A$, hence $(\text{ann}_R B) A << A$. Therefore A is s-prime.

Proposition 3.5: A nonzero R-module A is s-prime if and only if for each $0\neq B \leq A$ and for each ideal I of R, $I B =0$ implies $I A << A$.

Proof: It is clear that $I B =0$ means $I\subseteq (\text{ann}_R B)$.

Theorem 3.6: Let A be a multiplication retractable R-module, then A is s-compressible if and only if it is s-prime.

Proof: (⟹) See Proposition 3.3.

(⟸)Assume that A is a multiplication retractable R-module and $0\neq B\leq A$.

Then, there exists $0\neq f\in\text{Hom}(A, B)$, since A is retractable, that is $\text{Im}f \neq 0$. As A is s-prime, it follows $(\text{ann}_R \text{Im}f) A << A$.

Now, $\text{ann}_R (\text{Im}f)=\{r\in R \mid rf(m)=0, \forall m\in A\}=\{r\in R \mid f(rm)=0, \forall m\in A\}= \{r\in R \mid rm\in\ker f\ \forall m\in A\}= [\ker f: A]$. Hence $(\text{ann}_R \text{Im}f) A = [\ker f: A] A =\ker f$, since A is multiplication. Therefore, we have $\ker f<< A$, and A is s-compressible. □

In [16] author proved that a faithful multiplication R-module is retractable according to this result Theorem 3.6 can be rewritten as following.

Corollary 3.7: Let A be a faithful multiplication R-module, then A is s-compressible if and only if it is s-prime.

Recall that a ring R is called left duo if any left ideal is two sided ideal [14].

Proposition 3.8: Let R be a left duo ring. A nonzero R-module A is s-prime if and only if for each $0\neq x\in A$ and for each ideal I of R, $Ix=0$ implies $I A << A$.

Proof: (⟹) Assume that $0\neq x\in A$ and $Ix=0$, then $IRx= RIx=0$ where $0\neq Rx<< A$ and by assumption $I A << A$.

(⟸) Let $0\neq B\leq A$ and $IB=0$, if $0\neq x\in B$, then $Ix=0$, by assumption $IA << A$, therefore A is s-prime.

Remark 3.9: The $\mathbb{Z}$- module $\mathbb{Z}_n$ is s-prime if and only if $n= p^k$ where p is a prime number.

Proof: If $n= p^k$, then $\mathbb{Z}_n$ is s-compressible (see Example 2.4), and by Proposition 3.3, it is s-prime. If $n= mk$ with $(m, k)=1$, then $ann_{\mathbb{Z}}\langle\bar{m}\rangle = k\mathbb{Z}$, $(k\mathbb{Z})\mathbb{Z}_n=\langle\bar{k}\rangle$ which is not small in $\mathbb{Z}_n$ since $\langle\bar{m}\rangle+\langle\bar{k}\rangle= \mathbb{Z}_n$.

Proposition 3.10: Let B be a finitely generated submodule of an R-module A and $J(B)=J(A)$. If A is s- prime then B is also s-prime.

Proof: Assume that A is s-prime and $K\leq B$, then $K\leq A$ and $(ann_R K)A << A$.

Now, $(ann_R K)B \leq (ann_R K)A$ and $(ann_R K)A << A$ implies $(ann_R K)A\subseteq J(A) = J(B)$. Therefore $(ann_R K)B\subseteq J(B)$, that is, $(ann_R K)B << B$. □

It is known that, if R is a commutative ring and $0\neq x\in A$, where A is an R-module, then $ann_R\, x$ is an ideal of R and $ann_R\, Rx= ann_R\, x$. The following lemma is needed to get the next result.

Lemma 3.11: Let R be a commutative ring with identity and A a finitely generated faithful multiplication R-module. If I is any ideal of R, then $I<< R$ if and only if $IA << A$.

Proof: (⟹) Assume that $I<< R$ and $IA + B = A$ where $B \leq A$. Since A is multiplication, $B=JA$ for some ideal J of R, then $IA + JA = A$, hence $(I+ J)A = A$. This implies $I+ J= R$ (see Theorem 3.1,[16]), then $J= R$ (since $I<< R$).Therefore, $B= A$, that is $IA << A$.

(⟸) Assume that $IA << A$ and $I+ J= R$ for some ideal J of R, then $IA + JA = RA = A$, and so, $JA = A$ since $IA << A$. Again, by (Theorem 3.1,[17]) $J= R$, hence $I<< R$.

Theorem 3.12: Let R be a commutative ring with identity and A a finitely generated faithful multiplication R-module. Then, A is s-prime if and only if $(ann_R\, x) << R$ for each $0\neq x\in A$.

Proof:See Proposition 3.4 and Lemma3.11.

Corollary 3.13: Let R be a commutative ring with identity and A a finitely generated faithful multiplication R-module, then, A is s-compressible.

Proof: Let A be a finitely generated faithful multiplication R-module. By (Lemma4.1,[17]), faithful multiplication modules are torsion free, then

$(ann_R\, x)=0$, then $(ann_R\, x) << R$ for all $0\neq x\in A$. By Theorem 3.12, A is s-prime. Then by Corollary 3.7, A is s-compressible.

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