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Convergence and Stability of Iterative Scheme for a Monotone Total Asymptotically Non-expansive Mapping

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Abstract

In this work, we introduce Fibonacci– Halpern iterative scheme (FH scheme) in partial ordered Banach space (POB space) for monotone total asymptotically non-expansive mapping (, MTAN mapping) that defined on weakly compact convex subset. We also discuss the results of weak and strong convergence for this scheme.

Throughout this work, compactness condition of *m*-th iterate of the mapping for some natural m is necessary to ensure strong convergence, while Opial's condition has been employed to show weak convergence. Stability of FH scheme is also studied. A numerical comparison is provided by an example to show that FH scheme is faster than Mann and Halpern iterative schemes.

Keywords: Banach Space, Monotone Mappings,Total Asymptotically Non-expansive Mapping, Fixed Points.

تقارب واستقرارية المخطط التكراري لتطبيقات رتيبة لامتمددة مُقّارِبة كليا

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الخلاصة

في هذا البحث تم تعريف مخطط فيبوناشي–هلبرن التكرارية في فضاء بناخ المرتب جزئيا لتطبيق رتيب لامتمدد مُقّارِب كليا معرف على مجموعة جزئية محدبة ومتراصة بضعف. كذلك تم مناقشة النتائج للتقارب القوي والضعيف لهذا المخطط. خلال هذا البحث ا شرط التراص للتكرار ***m*** للتطبيق لبعض الاعداد الطبيعية ***m*** ضروري لاثبات التقارب القوي بينما وُظِف شرط أوبل لبيان التقارب الضعيف. كذلك تم دراسة استقرارية مخطط فيبوناشي–هلبرن. تم توفير مقارنة عددية بواسطة مثال لنبين ان مخطط فيبوناشي اسرع من مخططي مان و هلبرن.

1.Introduction

As more general classes of asymptotically non-expansive mapping. In [1], Alber introduced total asymptotically non-expansive mappings (TAN). He also studied the iterative method to determine their fixed points. Let A be a Banach space with norm $\|.\|$, a mapping $G: A \to A$ is called TAN if for $r, e \in A$ and $\{B_n\}, \{g_n\}$ ⊂ R^+(R^+ is the set of nonnegative reals),$n \geq 1$.

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$$\|G^n(r) - G^n(e)\| \leq \|r - e\| + B_n\varphi(\|r - e\|) + g_n, \quad (1)$$

where $\varphi: R^+ \to R^+$ is increasing continuous function with $\varphi(0) = 0$.
If $\varphi(t) = t$, then equation (1) reduces to

$$\|G^n(r) - G^n(e)\| \leq \|r - e\| + B_n\|r - e\| + g_n, \text{for } r, e \in A, \forall\, n \geq 1. \quad (2)$$

In addition, if $g_n = 0$, $k_n = 1 + B_n$, $\forall\, n \geq 1$, then G correspond to the asymptotically non-expansive mapping. If $g_n = B_n = 0$, then equation (1) reduces to the non-expansive mapping. See [2-3] for more details.
Let $G : D \to D$. In [4], Halpern studied the following iteration for a non-expansive mapping

$$y_1 \in D, y_{n+1} = \lambda_n x_0 + (1 - \lambda_n)G, \lambda_n \in (0,1) \quad (3)$$

where D is a closed convex subset of Hilbert space.
In this article, we consider a modification of Halpern iteration to be suitable for MTAN mappings. So that equation(3) will be modified by employing Fibonacci sequence of numbers $\{f_i\}$or$\{f(i)\}$, i=1,1,2,3, … ,which is

$$f(i + 1) = f(i) + f(i - 1), i \geq 1.$$

The new iterative scheme is defined by

$$r_0 \in D \text{ and } \{h_n\} ⊂(0,1), \; r_{n+1} = h_n r_n + (1 - h_n)G^{f(n)}(r_n) \quad (4)$$

is called Fibonacci– Halpern scheme and denoted by FH scheme.
.**Definition (1.1):** [7] A Banach space $(A, \|.\|)$ is called uniformly convex if $\forall\, \epsilon > 0$ $\exists \delta > 0$ and for $r, e \in A$ if $\|r\| \leq 1$, $\|e\| \leq 1$ and $\|r - e\| \geq \epsilon$ then $\|r + e\| \leq 2(1 - \delta)$.
Definition (1.2): [6] A Banach space $(A, \|.\|)$ is called strictly convex if $\|r\| = \|e\| = 1$ and $r \neq e$, then $\|tr + (1 - t)e\| < 1$ for all $t \in [0,1]$.
Definition (1.3): [7]. The function $\delta_A: [0,2] \to [0,1]$ is called modulus of convexity of A if defined by $\delta_A(\epsilon) = \inf\left\{1 - \frac{\|r+e\|}{2}, \|r\| = \|e\| \leq 1, \|r - e\| \geq \epsilon, \forall\, \epsilon \in [0,2]\right\}$
Definition (1.4): [5] Let $(A, \|.\|, \lesssim)$ be POB space and $D \subset A$. A mapping $G: D \to D$ is said to be monotone if $r \lesssim e \Rightarrow G(r) \lesssim G(e)$ $\quad \forall\, r, e \in D$.
Definition (1.5): [5] A mapping $G: D \to D$ is called monotone Lipschizian if G is monotone and there exist $q \geq 0$ such that $\|G(r) - G(e)\| \leq q\|r - e\|$ for any r, e in D and $r \lesssim e$. While G is said to be MTAN if it is monotone and satisfy (1) for any $r, e \in D$ such that $r \lesssim e$.
Note that, every TAN mapping is MTAN but the opposite is not true, see the following example.
Example (1.6): Let A = $[0, +\infty)$be a Banach space with usual norm $\|r - e\| = |r - e|$, r, e∈ A. Consider the order relation $r \lesssim e$ as $r, e \in [0,1]$ and $r \leq e$ or $r, e \in (i, i + 1]$ for some $i = 1,2, \ldots$ and $r \leq e$
Define G as the following G (0) = 0, $r \in (0,1]$ $i = 0 \Rightarrow G(r) = \frac{0}{2} + \frac{r}{2} = \frac{r}{2}$
$r \in (1,2]$, $i = 1 \Rightarrow G(r) = \frac{1}{2} + \frac{r}{2} = \frac{1+r}{2}$. So, if $r \in (i, i + 1]$, then, $G(r) = \frac{i}{2} + \frac{r}{2}$
G is discontinuous since, for any i, $G(i + 1^-) = \frac{i}{2} + \frac{i+1}{2} = \frac{2i+1}{2} = i + \frac{1}{2} \neq \frac{i+1}{2} + \frac{i+1}{2} = \frac{2(i+1)}{2} = i + 1 = G(i + 1^+)$. So that G is not non-expansive mapping.
Now, if e$\lesssim$ r, then $r, e \in [0,1]$or $r, e \in (i, i + 1]$for some $i = 1,2, \ldots$, and

$$\|G(r) - G(e)\| = \left\|\left(\frac{i}{2} + \frac{r}{2}\right) - \left(\frac{i}{2} + \frac{e}{2}\right)\right\| = |r - e| \leq \|r - e\|$$

So that G is monotone non-expansive mapping but not a non-expansive mapping, therefore G is not TAN mapping.
We will use symbols $\to$ and$\overset{w}{\to}$ denote strong convergence and weak convergence respectively in the following.
Definition (1.7): [5] A Banach space A has the weak-Opial's property if for any sequence $\{r_n\}$ in A, $r_n \overset{w}{\to} r$ implies that $\lim_{n\to\infty} \sup\|r_n - r\| < \lim_{n\to\infty} \sup\|r_n - e\|$, $\forall\, e \in A \;\ni\; r \neq e$.

Therefore, A has monotone weak-Opial's property if whenever any monotone increasing (or, decreasing) sequence $\{r_n\}$ in A, $r_n \overset{w}{\to} r$ implies that

$$\lim_{n\to\infty} \sup\|r_n - r\| \leq \lim_{n\to\infty} \sup\|r_n - e\|, \quad \forall e \in A \text{ such that } r \precsim e \text{ or } e \precsim r$$

Remark (1.8): [13]. Every weak-Opial's property is monotone weak –Opial. It is known that a Hilbert space and Banach spaces ℓ_p, $1 < p < \infty$ are weak Opial property, while Banach spaces $Lp([0,1])$, $1 < p < \infty$, fail to have weak Opial property, however it is monotone weak-Opial.

Proposition (1.9): [9] Let A be reflexive Banach space, and $\{r_n\}$ is bounded monotone increasing (or decreasing) sequence in A. Then $\{r_n\}$ is weakly convergent.

Also, for nonempty compact subset D of A, if $\lim_{n\to\infty} d(r_n, D) = 0$ then $\{r_n\}$ is convergent strongly

Definition (1.10): [9].The norm function $\|.\|$ on A is monotone if $c \precsim u \precsim v \Rightarrow \max\{\|u - c\|, \|u - v\|\} \leq \|v - c\|$, for any $c, u, v \in A$

Here, the subsets $\{r: \alpha \leq r\}, \{r: r \leq \beta\}$ and $\{r: \alpha \leq r \leq \beta\}$ for any $\alpha, \beta \in A$ are denoted to order intervals.

Proposition (1.11): [9] If A is uniformly convex POB space with monotone norm$\|.\|$ for which order intervals are closed and convex then A satisfies the monotone weak-Opial's property.

Lemma (1.12): [5] Let $(A, \|.\|, \precsim)$ be uniformly convex POB space and $\emptyset \neq D \subset A$, D is closed convex .Let $\omega: D \to [0, +\infty)$ be a type function, i.e., there exists a bounded sequence $\{r_n\} \in A$ such that $\omega(r) = \lim_{n\to\infty} \sup\|r_n - r\|, \forall r \in D$.

Then ω has a unique minimum point $s \in D$ such that $\omega(s) = \inf\{\omega(r); r \in D\} = \omega_0$

Moreover, if $\{s_n\}$ is minimizing sequence in D, i.e.,$\lim_{n\to\infty} \omega(s_n) = \omega_0$, then $\{s_n\}$ converges strongly to s .

Proposition (1.13): [9] Let D be a convex and bounded nonempty subset of A . Assume that the map $G: D \to D$ is monotone. Let $r_0 \in D$ be such that $r_0 \precsim G(r_0)$(or $, G(r_0) \precsim r_0$) and $\{h_n\}$ ⊂ (0,1) and consider the sequence $\{r_n\}$ generated by (4). Let s be a fixed point of G such that $r_0 \precsim s$ (or, $s \precsim r_0$) (or, $G^{n+1}(r_0) \precsim G^n(r_0)$) then

i. $G^n(r_0) \precsim G^{n+1}(r_0)$

ii. $r_0 \precsim r_n \precsim s$ (or, $s \precsim r_n \precsim r_0$)

iii. $G^{f(n)}(r_0) \precsim G^{f(n)}(r_n) \precsim s$ (or, $s \precsim G^{f(n)}(r_n) \precsim G^{f(n)}(r_0)$)

iv. $r_n \precsim r_{n+1} \precsim G^{f(n)}(r_n)$ (or, $G^{f(n)}(r_n) \precsim r_{n+1} \precsim r_n$)

for any $n \in N$

Lemma (1.14) [10] Let $\{a_n\}, \{b_n\}$, and $\{t_n\}$ be three nonnegative sequences and

$$a_{n+1} \leq (1 + b_n)a_n + t_n, \forall n \geq n_0,$$

for some nonnegative integer n_0. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

1. Convergence Theorems

Throughout this results, we assume that $(A, \|.\|, \precsim)$is POB space such that order intervals are closed and convex and G is TAN mapping whenever $s \in F(G)$ and $\exists M^* > 0$, then $\varphi(\|r - s\|) \leq M^* \|r - s\|$ $\forall r \in A$. We start with the following fixed point result.

Theorem (2.1) LetA be a uniformly convex, and D such that $\emptyset \neq D \subset A$ is closed convex,which has more than one point . Let $G: D \to D$ be a continuous MTAN mapping where$\exists r_0 \in D \ni r_0 \precsim G(r_0)$ then G has a fixed point s such that $r_0 \precsim s$.

Proof: Let $r_0 \in D$ such that $r_0 \precsim G(r_0)$, by monotonicity of G, $G^n(r_0) \precsim G^{n+1}(r_0)$, $\forall n \in N$.

By reflexivity of A and closeness and convexity of order intervals, we get that

$D_\infty = \cap \{r \in D,\ G^n(r_0) \precsim r\} \neq \emptyset$.

Let $r \in D_\infty \Rightarrow G^n(r_0) \lesssim r$, since G is monotone, $G^n(r_0) \lesssim G(G^n(r_0)) = G^{n+1}(r_0) \lesssim G(r), \forall\, n \geq 0$, i.e., $G(D_\infty) \subset D_\infty$. Let $\omega: D \to [0, +\infty)$ be a type function which is generated by $\{G^n(r_0)\}$, $\omega(r) = \lim_{n\to\infty} \sup\|G^n(r_0) - r\|$, by Lemma(1.12)there exists a unique $s \in D_\infty$ such that $\omega(s) = \inf\{\omega(r); r \in D_\infty\} = \omega_0$. Fix $s \in D_\infty \Rightarrow G^m(s) \in D_\infty$
$\forall m \in N$, which implies to

$$\omega(G^m(s)) = \lim_{n\to\infty} \sup\|G^n(r_0) - G^m(s)\|$$
$$\leq \lim_{n\to\infty} \sup[\|G^n(r_0) - s\| + B_m\omega(\|G^n(r_0) - s\|) + g_m]$$

such that $\lim_{m\to\infty} B_m = \lim_{m\to\infty} g_m = 0$, for any $m \in N$ $\omega_0 \leq \omega(G^m(s)) \leq \omega_0$
$\Rightarrow \lim_{m\to\infty} \omega(G^m(s)) = \omega_0$, this means $\{G^m(s)\}$ minimizing sequence of ω. By Lemma (1.12) $\{G^m(s)\}$ converges strongly to s. Note that G is continuous, then $\lim_{m\to\infty} G(G^m(s)) = \lim_{m\to\infty} G^{m+1}(s) = G(s) = s$, so s is fixed point of G.

Note that, by a similar steps in proof of Theorem (2.1), a corresponding result can be satisfied if $G(r_0) \lesssim r_0$ then G has a fixed point s such that $s \lesssim r_0$.

Proposition (2.2): Let D a nonempty closed and convex subset of A and $G: D \to D$ be a MTAN mapping with $r_0 \in D$ such that $r_0 \lesssim G(r_0)$ (or, $G(r_0) \lesssim r_0$)$\{h_n\}$ ⊂ (0,1) . We also consider the sequence $\{r_n\}$ which is generated by equation(4). Let s be a fixed point of G such that $r_0 \lesssim s$ (or, $s \lesssim r_0$). Then $\lim_{n\to\infty}\|r_n - s\|$ exists.

Proof: By definition of $\{r_n\}$

$$\|r_{n+1} - s\| \leq \|(h_n r_n + (1-h_n)G^{f(n)}(r_n) - (h_n s + (1-h_n)s)\|$$
$$\leq h_n\|r_n - s\| + (1-h_n)\|G^{f(n)}(r_n) - G^{f(n)}(s)\|,$$

for any $n \geq 1$. Since G is MTAN mapping,

$$\|r_{n+1} - s\| \leq h_n\|r_n - s\| + (1-h_n)[\|r_n - s\| + B_{f(n)}\varphi(\|r_n - s\|) + g_{f(n)}]$$
$$\|r_{n+1} - s\| \leq h_n\|r_n - s\| + (1-h_n)\|r_n - s\| + (1-h_n)B_{f(n)}M^*\|r_n - s\| + (1-h_n)g_{f(n)}$$
$$\leq (h_n + (1-h_n) + (1-h_n)B_{f(n)}M^*)\|r_n - s\| + (1-h_n)g_{f(n)}$$
$$\leq (1+u_n)\|r_n - s\| + v_n$$

Where $u_n = (1-h_n)B_{f(n)}M^*$ and $\sum_{n=1}^{\infty} B_{f(n)} < \infty$
$v_n = (1-h_n)g_{f(n)}$ and $\sum_{n=1}^{\infty} g_{f(n)} < \infty$
Then by Lemma (1.14) we obtain that $\lim_{n\to\infty}\|r_n - s\|$ exist.

To prove the next results, we need the concept of ultra-power of Banach space. For more details, see [5].

Lemma (2.3) [5]: Let $\mathcal{U}$ be a nontrivial ultra-filter over$\mathbb{N}$, then for any bounded sequence of real numbers $\{\mu_n\}$, $\lim_{n,\mathcal{U}} \mu_n$ exist. Moreover, if A is Banach space then
$\ell_\infty(A) = \{\{r_n\} \subset A : \|\{r_n\}\|_\infty = \sup\|r_n\| < \infty\}$ endowed with norm $\|.\|_\infty$
is Banach space and $A_0 = \left\{\{r_n\} \in \ell_\infty(A): \lim_{n,\mathcal{U}}\|r_n\| = 0\right\}$ is closed subspace of $\ell_\infty(A)$.
The quotient space $(A_\mathcal{U}) = \ell_\infty(A)/A_0$ is called ultra-power of the Banach space A.In particular for every $\tilde{r} \in (A)_\mathcal{U}$, $\|\tilde{r}\|_\mathcal{U} = \lim_{n,\mathcal{U}}\|r_n\|$, where $\{r_n\}$ is any representative of $\tilde{r}$

Proposition (2.4): Let $(A, \|.\|, \lesssim)$ be a uniformly convex POB space and D a nonempty weakly compact convex subset of A. Let $G: D \to D$ be a continuous MTAN mapping as in (1). Let $r_0 \in D$ such that $r_0 \lesssim G(r_0)$(or, $G(r_0) \lesssim r_0$)and $\{r_n\}$ as in (4). Let s be a fixed point of G such that $r_0 \lesssim s$ (or, $s \lesssim r_0$) Then $\lim_{n\to\infty}\|r_n - G^{f(n)}(r_n)\| = 0$.

Proof: Suppose that $r_0 \lesssim G(r_0)$. Theorem (2.1) implies that $\exists\, s \ni G(s) = s$ such that$r_0 \lesssim s$, and Proposition (2.2), $c = \|r_n - s\|$ exists.
If c = 0, the result trivially holds . Suppose that $c > 0$, then

$$\lim_{n\to\infty} \sup\|G^{f(n)}(r_n) - s\| = \lim_{n\to\infty} \sup\|G^{f(n)}(r_n) - G^{f(n)}(s)\|$$

$\leq \lim_{n\to\infty} \sup \left[\|r_n - s\| + B_{f(n)}\varphi(\|r_n - s\|) + g_{f(n)}\right]$
$\leq \lim_{n\to\infty} \sup \left(1 + B_{f(n)}M^*\right)\|r_n - s\| + g_{f(n)}$
Since $\lim_{n\to\infty} B_{f(n)} = \lim_{n\to\infty} g_{f(n)} = 0$
$\leq \lim_{n\to\infty} \sup \|r_n - s\| = c$
Note that $r_n \precsim s$, for any $n \geq 1$. Also, for any ≥ 1 , we obtain that
$\|r_{n+1} - e\| \leq h_n\|r_n - e\| + (1 - h_n)\|G^{f(n)}r_n - e\|$
Let $\mathcal{U}$ be a nontrivial ultera-filter over $\mathbb{N}$ .Then $\lim_{\mathcal{U}} h_n = h \in [a, b]$ where $0 \leq a \leq b \leq 1$
Thus, $c = \lim_{\mathcal{U}}\|r_{n+1} - s\| \leq \lim_{\mathcal{U}}[h_n\|r_n - s\| + (1 - h_n)\|G^{f(n)}r_n - s\|]$
$c \leq hc + (1 - h) \lim_{\mathcal{U}}\|G^{f(n)}r_n - s\|$
$c - hc \leq (1 - h) \lim_{\mathcal{U}}\|G^{f(n)}r_n - s\|$
$(1 - h)c \leq (1 - h) \lim_{\mathcal{U}}\|G^{f(n)}r_n - s\|$, since $c \neq 0$, we have $c \leq \lim_{\mathcal{U}}\|G^{f(n)}r_n - s\|$.
Hence $c \leq \lim_{\mathcal{U}}\|G^{f(n)}r_n - s\| \leq \lim_{n\to\infty} \sup\|G^{f(n)}r_n - s\| \leq c$,
which implies $\lim_{\mathcal{U}}\|G^{f(n)}r_n - s\| = c$
Let $(A)_{\mathcal{U}}$ be ulter-power of A and $\tilde{r} = (\{r_n\})$, $\tilde{s} = (\{s_n\})$, $\tilde{e} = (\{G^{f(n)}r_n\})$ .Then
$\|\tilde{r} - \tilde{s}\|_{\mathcal{U}} = \|\tilde{e} - \tilde{s}\|_{\mathcal{U}} = \|h\tilde{r} + (1 - h)\tilde{e} - \tilde{s}\|_{\mathcal{U}} = c$. Since A is uniformly convex then $(A)_{\mathcal{U}}$ is strictly convex. Note that $h \in (0,1)$,we get $\tilde{r} = \tilde{e}$, that means $\lim_{\mathcal{U}}\|r_n - G^{f(n)}(r_n)\| = 0$.
Since $\mathcal{U}$ is arbitrary nontrivial ultra-filtre then $\lim_{n\to\infty}\|r_n - G^{f(n)}(r_n)\| = 0$.
Next, we employ the concept of compact mapping to get another result

Definition (2.5):[14] Let $G: A \to B$ be a mapping between two Banach spaces A and B, then G is said to be compact if G is continuous and G(D) is relatively compact for any subset D of A.

Theorem (2.6): Let D be a nonempty weakly compact convex subset of a uniformly convex A and $G: D \to D$ be a MTAN mapping as in (1) and $G^m$is compact for some $m \geq 1$. Let $r_0 \in D$ such that $r_0 \precsim G(r_0)$(or, $G(r_0) \precsim r_0$) and $\{r_n\}$ as in (4). Then $\{r_n\}$ converges strongly to a fixed point s of G with $r_0 \precsim s$ (or, $s \precsim r_0$).

Proof: Suppose that $r_0 \precsim G(r_0)$, by Proposition (1.13) $r_0 \precsim r_n \ \forall n \in \mathbb{N}$
By Proposition (2.4), $\lim_{n\to\infty}\|r_n - G^{f(n)}(r_n)\| = 0$
Fix $m \geq 1$, such that $G^m$is compact and set $D_0 = \overline{G^m}(D)$ .Then D_0 is nonempty compact set.
For every $n > m$ and $r \in D$, then $G^n(r) \in D_0$. Note that $f(n) > m$ for $n > m$, so, $G^{f(n)}(r_n) \in D_0$.
Hence, $\lim_{n\to\infty} d(r_n, D_0) \leq \lim_{n\to\infty}\|r_n - G^{f(n)}(r_n)\| = 0$ and $\lim_{n\to\infty} d(G^n(r_0), D_0) = 0$.
The monotonically of $\{r_n\}$,$\{G^n(r_n)\}$ and reflexivity of A makes Proposition (1.9) implies $\{r_n\}$ and$\{G^n(r_n)\}$ are strongly convergent. Thus, $\{G^{f(n)}(r_n)\}$ converges strongly to the same limit of $\{r_n\}$.
Let s be the strongly limit of $\{G^n(r_0)\}$, to prove that s is fixed point of G. Since $\{G^n(r_0)\}$ is monotone increasing, $G^n(r_0) \precsim s$ for every $n \in \mathbb{N}$. Then
$\|G^{n+1}(r_0) - G(s)\| \leq \|G^n(r_0) - s\| + B_1\varphi(\|G^n(r_0) - s\|)$+$g_1$ for every $n \geq 1$. Therefore,
$G^{n+1}(r_0) \to G(s)$ and s that is $G(s) = s$.
By Proposition (1.13) $G^{f(n)}(r_0) \precsim G^{f(n)}(r_n) \precsim s$, for everyn $\in \mathbb{N}$. By closeness of the bounded intervals, s is also the limit of $\{G^{f(n)}(r_n)\}$ and $\{r_n\}$. Then $\{r_n\}$ converges strongly to a fixed point s of G.

The following proposition is needed to prove the next theorem.

Proposition (2.7) [5]: Let $\{r_n\}$ be a bounded increasing (or decreasing) sequence of a uniformly convex space A with its norm $\|.\|$ is monotone. Suppose that $D = \{r; r_n \precsim r$, for any $n \in N\}$ and $\omega(r): D \to [0,\infty)$ be a function such that $\omega(r) = \lim_{n\to\infty}\|r_n - r\|$. If $r_n \xrightarrow{w} s$ then $\omega(s) = \inf\{\omega(r); r \in D\}$ and any minimizing sequence $\{s_n\}$ of ω in D converges strongly to s. Moreover, ω has one minimum point.

Theorem (2.8): Let $(A, \|.\|, \precsim)$ be a uniformly convex space with $\|.\|$ is monotone and D is a nonempty convex weakly compact subset of A. If $G: D \to D$ is a MTAN as in (1), $r_0 \in D$ be such that $r_0 \precsim G(r_0)$ and $\{r_n\}$ as in (4) with $\{h_n\} \subset (0,1)$, then $\{r_n\}$ converges weakly to a fixed point of G which comparable to r_0.

Proof: Suppose that $r_0 \precsim G(r_0)$, so $\{G^n(r_0)\}$ is monotone increasing. Since D is weakly compact, and $\{G^n(r_0)\}$ is weakly convergent to a point s. By Proposition (1.11) A satisfies the monotone weak Opial's condition. Then s is the minimum point of ω, $\omega: D_\infty \to [0,\infty)$,

$\omega(e) = \lim_{n\to\infty}\inf\|G^n(r_0) - e\| = \lim_{n\to\infty}\|G^n(r_0) - e\|$, where

$D_\infty = \{e \in D: G^n(r_0) \leq s$ for any $n \in N\}$.

Since $s \in D_\infty$ where $G^m(s) \in D_\infty$, for every $m \in \mathbb{N}$, which implies

$$\omega(G^m(s)) = \lim_{n\to\infty}\inf\|G^n(r_0) - G^m(s)\| \leq \|G^n(r_0) - s\| + B_m\varphi(\|G^n(r_0) - s\|) + g_m$$

Since $\lim_{m\to\infty} B_m = \lim_{m\to\infty} g_m = 0$, since G is MTAN mapping, then

$\omega(s) \leq \omega(G^m(s)) \leq \omega(s)$ for every $m \in \mathbb{N}$

$\{G^m(s)\}$ is a minimizing sequence of ω. By Proposition (2.7) $G^m(s) \to s$.

Since $G^n(r_0) \precsim s$ we have $G^{n+1}(r_0) \precsim G(s)$ For any $n > 1$

Since $G^m(s) \to s$, and by closeness and convexity of order intervals $s \precsim G(s)$. Since G is monotone then the sequence $\{G^m(s)\}$ is monotone increasing and it converges to s.

We must have $G^m(s) \precsim s \;\forall\, m \geq 1$. By Proposition (1.13) $G^{f(n)}(r_0) \precsim G^{f(n)}(r_n) \precsim s$ $\forall n \in \mathbb{N}$.

Since $\{G^n(r_0)\}$ is monotone increasing, and it converges weakly to s. By closeness and convexity of intervals, we have $\{G^{f(n)}(r_n)\}$ also converges weakly to s. By Proposition (2.4) $\lim_{n\to\infty}\|r_n - G^{f(n)}(r_n)\| = 0$., then $\{r_n\}$ converges weakly to s, a fixed point of G.

Corollary (2.9): Suppose that $(A, \|.\|, \precsim), G, r_0, D$ and $\{r_n\}$ are in Theorem (2.8). If G is continuous then G has a fixed point s and the sequence $\{r_n\}$ is generated by (4) converges weakly to s which is comparable to r_0.

In the following example, we present a compression between the behaviors of FH-scheme and two different iterative schemes [18].

Example (2.10): Let $G: [0,\infty) \to [0,\infty), G(s) = \frac{s+3}{2}$ be a function with fixed point s=3.Consider the following three

$x_1 \in [0,\infty), x_{n+1} = h_n x_n + (1-h_n)G^{f(n)}G(x_n)$, (FH scheme)

$y_1 \in [0,\infty), y_{n+1} = h_n y_n + (1-h_n)G^n(y_n)$, (Modified Mann scheme)

$z_1 \in [0,\infty), z_{n+1} = h_n z_n + (1-h_n)G(z_n)$, (Mann scheme)

Fix $x_1 = y_1 = z_1 = 20$ and $h_n = \frac{1}{\sqrt{n+1}}$

In Table 1,and Figure 1 it is shown that $\{x_n\}$ is faster than $\{y_n\}$ and $\{z_n\}$.

Table 1-

n	x_n	y_n	z_n
1	20.00000000	20.00000000	20.00000000
2	17.51040764	17.51040764	17.51040764
3	14.44399770	12.91079273	14.44399770
..	…	..	..
20	3.00000003	3.00003902	3.00861677
21	3.00000001	3.00001614	3.00524855

22	3.00000000	3.00000043	3.00069495
..	…	…	…
30	3.00000000	3.00000000	3.00005164
…	…..	…	…
42	3.00000000	3.00000000	3.00000008
43	3.00000000	3.00000000	3.00000005
44	3.00000000	3.00000000	3.00000003
45	3.00000000	3.00000000	3.00000001
46	3.00000000	3.00000000	3.00000000
47	3.00000000	3.00000000	3.00000000

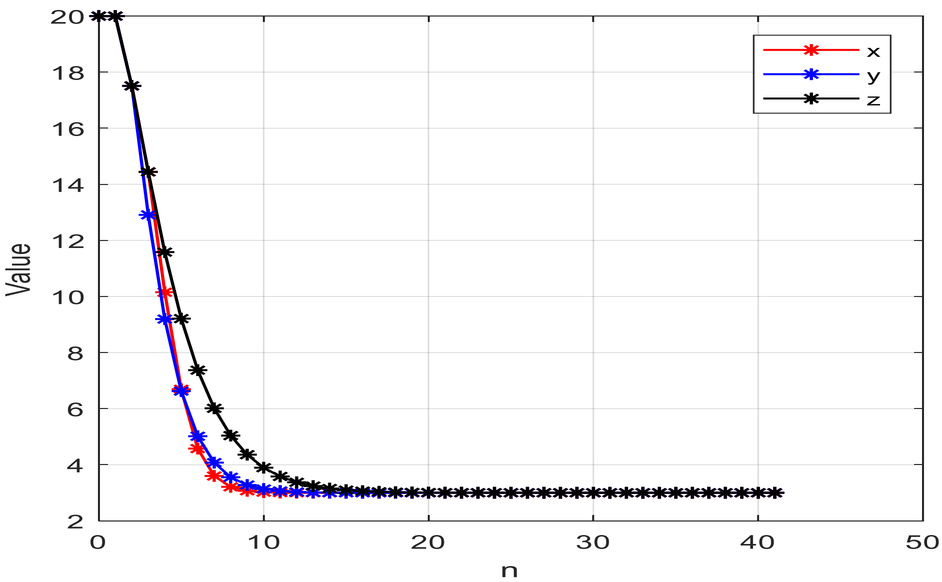

Figure 1-

2. Stability of FH-iterative Scheme

Definition (3.1): [18] Let $(A, \|.\|)$ be a normed space and $G: A \to A$ is a mapping.. $\{r_n\}$ is an iterative scheme converges strongly to $s \in F(G)$, which is produced by G

$$\begin{cases} r_1 \in A \\ r_{n+1} = f(G, r_n) \end{cases}$$

where f is a function. If for an arbitrary sequence $\{e_n\} \subset A$ $\lim_{n\to\infty}\|e_{n+1} - f(G, r_n)\| = 0$ implies $\lim_{n\to\infty} e_n = s$, then $\{r_n\}$ is said to be stable w.r.t G.

Definition (3.2): [16] The sequences $\{r_n\}$ and $\{e_n\}$ are called equivalent if $\lim_{n\to\infty}\|r_n - e_n\| = 0$

Definition (3.3): [15] Let $\{r_n\}$ be an iterative scheme and it converges strongly to $s \in F(G)$. If for any equivalent sequence $\{e_n\} \subset A$ of $\{r_n\}$, $\lim_{n\to\infty}\|e_{n+1} - f(G, r_n)\| = 0 \Rightarrow \lim_{n\to\infty} e_n = s$, then the iteration sequence $\{r_n\}$ is said to be weak-w^2 stable w.r.t G.

Theorem (3.4): Let $(A, \|.\|, \precsim)$ be a uniformly convex space and D is a nonempty convex closed subset of A. Let $G: D \to D$ be MTAN mapping as in (1) with fixed point s. Suppose that $\{r_n\}$ as in (4) with $r_0 \precsim G(r_0)$, $h_n \subset (0,1)$ and $s \precsim r_0$ If $\{e_n\}$ is any equivalent sequence of $\{r_n\}$ with $r_n \precsim e_n$ (or, $e_n \precsim r_n$), then $\{r_n\}$is weak-w^2 stable w.r.t G..

Proof: Consider $\{e_n\}$to be an equivalent sequence of $\{r_n\}$

$$\lim_{n\to\infty}\|e_{n+1} - f(G, r_n)\| = 0 \Rightarrow \lim_{n\to\infty} e_n = s$$

Let $r_n \precsim e_n$ by monotonicity of G $G^{f(n)}(r_n) \precsim G^{f(n)}(e_n)$.
Set $\epsilon_n = \|e_{n+1} - f(G, r_n)\|$, suppose $\epsilon_n \to 0$ as $n \to \infty$
$\|e_{n+1} - s\| \le \|e_{n+1} - f(G, r_n)\| + \|f(G, r_n) - r_{n+1}\| + \|r_{n+1} - s\|$
$\le \epsilon_n + \left\|(h_n e_n + (1-h_n)G^{f(n)}(e_n)) - (h_n r_n + (1-h_n)G^{f(n)}(r_n)\right\| + \|r_{n+1} - s\|$
$\le \epsilon_n + h_n\|e_n - r_n\| + (1-h_n)\left\|G^{f(n)}(e_n) - G^{f(n)}(r_n)\right\| + \|r_{n+1} - s\|$
$\le \epsilon_n + h_n\|e_n - r_n\| + (1-h_n)\left[\|e_n - r_n\| + B_{f(n)}\varphi(\|e_n - r_n\|) + g_{f(n)}\right] + \|r_{n+1} - s\|$
$\le \epsilon_n + h_n\|e_n - r_n\| + (1-h_n)\left[\|e_n - r_n\| + B_{f(n)}\|e_n - r_n\| + g_{f(n)}\right] + \|r_{n+1} - s\|$
Let , then $\lim_{n\to\infty}\|e_{n+1} - s\| = 0$
So, $\{r_n\}$ is weak $- w^2$ stable w.r.t G.

3. Application to Integral Equations

Consider the Hilbert space
$(L^2[0,1], \mathbb{R}) = \left\{f: [0,1] \to \mathbb{R}, \text{f is Lebesgue masurable and } \int_0^1 |f(t)|^2\, dt < \infty\right\}$
with $\|f\| = \sqrt{\int_0^1 f^2\,(t)dt}$. Theorem (2.8) solves the following of integral equation:

$$r(t) = k(t) + \int_0^1 B(t, s, r(s))ds \quad t \in [0,1] \qquad (5)$$

such that
i. $k \in L^2([0,1], \mathbb{R})$
ii. B: $[0,1]^2 \times L^2([0,1], \mathbb{R}) \to \mathbb{R}$ is measurable and satisfies the condition
$0 \le |B(t,s,r) - B(t,s,e)| \le \|r - e\|\ \forall$ t ,s $\in [0,1]$ and $r, e \in L^2([0,1], \mathbb{R}) \ni r \le e$.

It is known that for any $r, e \in L^2([0,1], \mathbb{R})$, we have $r \precsim e \Leftrightarrow r(t) \le e(t)$almost every for $t \in [0,1]$

Suppose that there exist a nonnegative function $g(.,.) \in L^2([0,1] \times [0,1])$ and M$< \frac{1}{2}$ such that
$|B(t,s,r)| \le g(t,s) + M\|r\|$
for every t , s $\in [0,1]$ and $r \in L^2([0,1], \mathbb{R})$
Define H $= \{e \in L^2([0,1], \mathbb{R}) \ni \|e\| \le \sigma\}$, where σ large enough, hence H is closed ball of $L^2([0,1], \mathbb{R})$ with center 0 and radius σ.
Define the operator G: $L^2([0,1], \mathbb{R}) \to L^2([0,1], \mathbb{R})$ by

$$G(e(t)) = k(t) + \int_0^1 B(t, s, e(s))d(s). \qquad (6)$$

To show that G(H)⊂H, let $e(t) \in G(H)$, then by using the Cauchy-Schwarz inequality and condition (ii) with this property ,namely for any reals s, t the inequality $(s+t)^2 \le 2s^2 + 2t^2$holds . Therefore

$$\|G(e)\|^2 = \int_0^1 |e(t)|^2\, dt$$
$$= \int_0^1 \left|k(t) + \int_0^1 B(t,s,e(s))ds\right|^2 dt$$
$$\le 2\int_0^1 |k(t)|^2\, dt + 2\int_0^1\int_0^1 \left|B(t,s,e(s))\right|^2 ds\, dt$$
$$\le 2\int_0^1 |k(t)|^2\, dt + 2\int_0^1\int_0^1 \left|g(t,s) + M|e(s)|\right|^2 ds\, dt$$
$$\le 2\int_0^1 |k(t)|^2\, dt + 4\int_0^1\int_0^1 |g(t,s)|^2\, ds\, dt + 4M\int_0^1\int_0^1 |e(s)|^2\, ds\, dt$$

$$= 2\int_0^1 |k(t)|^2\,dt + 4\int_0^1\int_0^1 |g(t,s)|^2\,ds\,dt + 4M\|e\|^2$$
$$\leq 2\int_0^1 |k(t)|^2\,dt + 4\int_0^1\int_0^1 |g(t,s)|^2\,ds\,dt + 4M\sigma^2$$
By $M < \frac{1}{2}$ choosing σ such that
$$\frac{2}{1-4M^2}\int_0^1 |k(t)|^2\,dt + \frac{2}{1-4M^2}\int_0^1\int_0^1 |g(t,s)|^2\,ds\,dt \leq \sigma^2$$
,we get $G(e) \in H$.

Now, to show that G is MTAN mapping,one canemploy condition (ii) and Theorem 12 in [17].

G is monotone. Thus,

$$\|G(r) - G(e)\|^2 = \int_0^1 (G(r(t)) - G(e(t)))^2 dt$$
$$= \int_0^1 (\int_0^1 B(t,s,r(s)ds) - B(t,s,e(s)ds))^2 dt$$
$$\leq \int_0^1 (\int_0^1 (r(s) - e(s))ds)^2 dt$$
$$\leq \int_0^1 (r(s) - e(s))^2 ds$$
$$= \|r - e\|^2$$

which means that G is non-expansive and then it is MTAN. Since every Hilbert space is uniformly convex POB, A= $L^2([0,1],\mathbb{R})$ and the Theorem (2.8) is satisfied.

Theorem (4.1): Under all pervious hypotheses in this section, the iterative scheme generated by (4) converges weakly to a solution of integral equation (5).

The second application is that possible to employ the results in [20] after we prove similar results to the previous results.

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