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On Quasi-Small Prime Submodules

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Abstract

Let R be a commutative ring with identity, and W be a unitary (left) R -module. A proper submodule H of W is said to be quasi-small prime submodule, if whenever $r, a \in R, w \in W$ with $\langle w \rangle \ll W$, and $raw \in H$, then either $aw \in H$ or $rw \in H$. In this paper, we give a comprehensive study of quasi-small prime submodules.

Keywords: Prime submodules, quasi-prime submodules, small prime submodules, quasi-small prime submodules

المقاسات الجزئية الاولية الصغيره الضاهريه

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الخلاصه

لتكن R حلقة ابدالية ذات محايد, وليكن W مقاساً احادياً (أيسر) على الحلقة R . يدعى المقاس الجزئي H من المقاس W مقاس جزئي اولي صغير ظاهري اذا كان $raw \in H$ حيث $r, a \in R, w \in W$ بحيث $w \ll W$, يؤدي الى اما $aw \in H$ او $rw \in H$. في هذا البحث سوف نعطي دراسة شاملة على المقاسات الجزئية الاولية الصغيره الظاهريه.

1. Introduction:

Throughout this research, we consider R is a commutative ring with identity and W is a unity R - module. A proper submodule H of W is said to be prime if whenever $r \in R, a \in W; ra \in H$ implies either $a \in H$ or $r \in [H:W]$, where $[H:W] = \{r \in R: rW \subseteq H\}$, see[1]. As a generalization of the concept of prime modules and prime submodules, Mahmood, L.S. in 2012, [2] introduced the concepts of small prime modules and small prime submodules. In fact, the concepts of small semiprime modules and small semiprime submodules are also introduced in 2021 by Haider A. Ramadhan, Nuhad S. Al. Mothafar, [3]. A submodule H of W is called small (notational, $H \ll W$) if $H + A = W$ for all submodules A of W implies $A = W$ [4].

A quasi-prime submodules is introduced and studied in 1999 by Abdul-Razak, M. H. [5] as a generalization of a prime submodules. The proper submodule H of an R -module W is called a quasi-prime if whenever $raw \in H$, where $r, a \in R, w \in W$ implies that either $aw \in H$ or $rw \in H$. As a generalization of these concepts Wisam, A. Ali. and Al-Mothafar, N.S. [6] were introduced the concept of quasi-small prime modules in 2021. They call W a quasi-small prime module if $ann_R H$ is prime ideal for all non-zero small submodule H of W .

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The main purpose of this paper is to study the basic properties of quasi-small prime submodules.

2. Quasi-Small Prime Submodules

In this section we introduce and study the concept of a quasi-small prime submodule.

Definition (2.1): A proper submodule H of an R -module W is called a quasi-small prime submodule if whenever $abw \in H$ for $a, b \in R$ and $w \in W$ such that $\langle w \rangle \ll W$, then either $aw \in H$ or $bw \in H$. Equivalently, a proper submodule H of an R -module W is a quasi-small prime if and only if $[H_R \langle w \rangle]$ is a prime ideal of R for all $\langle w \rangle \ll W$.

Remarks and Examples (2.2):

1. It is clear that every quasi-prime submodule is a quasi-small prime submodule.

The following example shows that the converse is not true in general.

Example: The submodule $H = \langle \bar{6} \rangle$ in Z_{12} as a Z -module is a quasi-small prime submodule of Z_{12} , since $\langle \bar{0} \rangle, \langle \bar{6} \rangle$ are small submodules of Z_{12} and for all $a, b \in R$ if $a.b.\bar{6} \in H$, implies either $a.\bar{6} \in H$ or $b.\bar{6} \in H$, but $H = \langle \bar{6} \rangle$ is not a quasi-prime submodule, since $2 \times 3 \times \bar{1} \in H$, but $2 \times \bar{1} \notin H$ and $3 \times \bar{1} \in H$.

2. Every small prime submodule is a quasi-small prime submodule.

Proof: Let H be a small prime submodule of an R -module W . Then from [2], we have $[H_W \langle w \rangle]$ is a prime ideal of W for all $\langle w \rangle \ll W$, this implies that H is a quasi-small prime submodule by definition(2.1).

Next example indicates that the converse of (2) is not true in general.

Example: Consider $W = Z_{12}$ as a Z -module and $H = \langle \bar{4} \rangle$. H is a quasi-small prime submodule of W , since $[\langle \bar{4} \rangle_W \langle \bar{6} \rangle] = 2Z$ is a prime ideal of Z , but H is not a small prime submodule, since $\langle \bar{6} \rangle \ll Z_{12}$ and $2 \times \bar{6} \in \langle \bar{4} \rangle$, but $\bar{6} \notin \langle \bar{4} \rangle$ and $2 \notin [\langle \bar{4} \rangle_W Z_{12}] = 4Z$.

3. It is clear that every prime submodule is quasi-small prime.

Next example shows that the converse of (3) is not true in general.

Example: Consider $W = Z_{24}$ as a Z -module, $H = \langle \bar{6} \rangle$. H is a quasi-small prime submodule W , since H is a small prime submodule [2], hence it is a quasi-small prime submodule by (2), which is not prime, since $2 \times \bar{3} \in \langle \bar{6} \rangle$, but $\bar{3} \notin \langle \bar{6} \rangle$ and $2 \notin [\langle \bar{6} \rangle_W Z_{24}] = 6Z$.

4. Consider $W = Z$ as a Z -module, let n is a positive integer. If n is a prime number, then the submodule $H = nZ$ is a quasi-small prime submodule, since if n is prime number, then nZ is a quasi-prime submodule,[5]. Hence by (1), $H = nZ$ is a quasi-small prime submodule.

5. Every quasi-small prime submodule is a small semiprime submodule, where a proper submodule H of a module W is called a small semiprime submodule of W if and only if whenever $a \in R, w \in W$ with $\langle w \rangle \ll W$ and $a^2w \in H$, implies $aw \in H$, [3].

Proof: Let H be a quasi-small prime submodule of an R -module W . If $a^2w \in H$ for $a \in R$ and $w \in W$ with $\langle w \rangle \ll W$, then $aw \in H$ by definition (2.1). Hence, H is a small semiprime submodule of W . We give the following example to show that the converse of (5) is not true in general.

Example: $6Z$ is a small semiprime submodule of Z , [3], but $6Z$ is not a quasi-small prime submodule of Z by (4).

6. If W is an R -module in which every cyclic submodule is small, then every quasi-small prime submodule of W is a small prime submodule.

7. If W is a hollow R -module, then every quasi-small prime submodule of W is a quasi-prime submodule, where an R -module W is called a hollow module if every proper submodule of W is small, [4].

8. The submodule Z of the Z -module Q is not a quasi-small prime submodule, since $[Z_Z \langle \frac{1}{6} \rangle] = 6Z$, which is not a prime ideal of Z , where $\langle \frac{1}{6} \rangle \ll Q$, [4,p.108].

9. Since $\langle 0 \rangle$ is the only small submodule of Z as a Z -module, then every proper submodule of Z as a Z -module is a quasi-small prime submodule.

10. If H is a quasi-small prime submodule of an R -module W , then it is not necessary that:

1. $\text{ann}_R H$ is a prime ideal of R .

2. $[H;_R W]$ is a prime ideal of R .

For example: Take $W = Z_{24}$ as a Z -module and $=\langle \bar{6} \rangle$. Then $\text{ann}_Z \langle \bar{6} \rangle = 4Z$ is not a prime ideal of Z and $[H;_Z W] = 6Z$ is also not a prime ideal of Z .

Remark (2.3): A submodule of a quasi-small prime submodule need not be a quasi-small prime submodule of a module.

For example: Let $H = \langle \bar{2} \rangle$ be a quasi-small prime submodule of the Z -module Z_{24} by remarks and examples (2.2, 3) and $K = \langle \bar{8} \rangle \leq \langle \bar{2} \rangle$, K is not a quasi-small prime submodule of Z_{24} , since $[K;_Z \langle \bar{6} \rangle] = 4Z$ which is not a prime ideal of R , where $\langle \bar{6} \rangle \ll Z_{24}$.

Corollary (2.4): A direct summand of a quasi-small prime submodule is not a quasi-small prime submodule.

For example: Let $H = \langle \bar{2} \rangle$ in the Z -module Z_{24} . H is a quasi-small prime submodule of Z_{24} and $H = \langle \bar{6} \rangle \oplus \langle \bar{8} \rangle$, while $\langle \bar{8} \rangle$ is not a quasi-small prime submodule of H , since $[\langle \bar{8} \rangle;_Z \langle \bar{6} \rangle] = 4Z$ is not a prime ideal of R , where $\langle \bar{6} \rangle \ll Z_{24}$.

Corollary (2.5): If $K \subseteq H$ are submodules of W and H is a quasi-small prime of W , then it is not necessary that K is a quasi-small prime in H .

For example: Let $K = \langle \bar{8} \rangle$ and $H = \langle \bar{2} \rangle$ in the Z -module Z_{24} . H is a quasi-small prime submodule in Z_{24} , but K is not a quasi-small prime submodule in H , since $[K;_Z \langle \bar{6} \rangle]$ is not a prime ideal of R , where $\langle \bar{6} \rangle \ll H$.

Proposition (2.6): Let H be a proper submodule of an R -module W . Then H is a quasi-small prime submodule of W if and only if whenever $IJL \subseteq H$, where I, J are ideals in R and L is a small submodule of W , this implies that either $IL \subseteq H$ or $JL \subseteq H$.

Proof:

(\Rightarrow) Suppose that $IJL \subseteq H$, where I, J are ideals in R and L is a small submodule of W , with $IL \not\subseteq H$ and $JL \not\subseteq H$. So there exists $a, b \in L$ and $i \in I, j \in J$ such that $ia \notin H$ and $jb \notin H$. Since $\langle a \rangle \leq L \ll W$, then $\langle a \rangle \ll W$ and $\langle b \rangle \leq L \ll W$, then $\langle b \rangle \ll W$. But H is a quasi-small prime submodule of W and $ija \in H$ and $ia \notin H$, implies that $ja \in H$. Also $ijb \in H$ and $jb \notin H$, implies that $ib \in H$. It follows that either $IL \subseteq H$ or $JL \subseteq H$.

(\Leftarrow) Assume that $abw \in H$, where $a, b \in R, w \in W$ with $\langle w \rangle \ll W$, implies that $ab\langle w \rangle \subseteq H$, so either $a\langle w \rangle \subseteq H$ or $b\langle w \rangle \subseteq H$. Thus, either $aw \in H$ or $bw \in H$. Hence H is a quasi-small prime submodule of W .

As direct application of proposition (2.6), we get the following corollaries.

Corollary (2.7): Let H be a proper submodule of an R -module W . Then H is a quasi-small prime submodule of W if and only if whenever $abL \subseteq H$, where $a, b \in R$ and L is a small submodule of W , it implies that either $aL \subseteq H$ or $bL \subseteq H$.

Corollary (2.8): Let H be a proper submodule of an R -module W . Then H is a quasi-small prime submodule of W if and only if whenever $aIw \subseteq H$, where $a \in R, w \in W$ with $\langle w \rangle \ll W$ and I is an ideal of R , which implies that either $aw \in H$ or $Iw \subseteq H$.

Corollary (2.9): Let H be a proper submodule of an R -module W . Then H is a quasi-small prime submodule of W if and only if whenever $IJw \subseteq H$, where I, J are ideals in R , and $w \in W$ with $\langle w \rangle \ll W$, which implies that either $Iw \subseteq H$ or $Jw \subseteq H$.

Proposition (2.10): Let W be an R -module and I be an ideal of R such that $I \subseteq \text{ann}_R W$. Let H be a submodule of W . Then H is a quasi-small prime R -submodule of W if and only if H is a quasi-small prime R/I -submodule of W .

Proof: Let $\bar{a} \in R/I, w \in W$ with $\langle w \rangle \ll W$ and $\bar{a}w \in H$. But $\bar{a}w = aw$, since $I \subseteq \text{ann}_R W$. Hence the result follows easily.

Recall that a proper submodule H of an R -module W is called irreducible if for all submodules K_1 and K_2 of W such that $K_1 \cap K_2 = H$, then either $K_1 = H$ or $K_2 = H$, [7].

Proposition (2.11):

If H be an irreducible submodule of a hollow R -module W , then the following statements are equivalent:

1. H is a small prime submodule.
2. H is a quasi-small prime submodule.
3. H is a small semiprime submodule.
4. H is a semiprime submodule.

Proof:

(1) \Rightarrow (2), by remarks and examples (2.2, (2)).

(2) \Rightarrow (3), by remarks and examples (2.2, (5)).

(3) \Rightarrow (4), it is clear by [3].

(4) \Rightarrow (1), since H is a semiprime submodule, then H is a prime submodule, [8, prop. 1.10, chapter two], implies H is a small prime submodule, [2].

Theorem (2.12): Let H be a proper submodule of an R -module W . Then, the following are equivalent:

1. H is a quasi-small prime submodule of W .
2. $[H_R^i K]$ is a prime ideal of R for all small submodule K of W where $[H_R^i K] = \{a \in R, aK \subseteq H\}$.
3. $[H_R^i(bw)] = [H_R^i(w)]$ for all $w \in W, \langle w \rangle \ll W, b \in R$ and $b \notin [H_R^i(w)]$.

Proof:

(1) \Rightarrow (2) Let H be a quasi-small prime submodule of W . Let $K \ll W$. Then $[H_R^i(w)]$ is a prime ideal of R for all $w \in W$ such that $\langle w \rangle \ll W$. So $[H_R^i(w)]$ is a prime ideal for all $w \in K$. Then $[H_R^i K]$ is a prime ideal of R , [5, lemma(1.2.5)].

(2) \Rightarrow (3). It is clear that $[H_R^i(w)] \subseteq [H_R^i(bw)]$. Let $a \in [H_R^i(bw)]$ for all $b \notin [H_R^i(w)]$ and $w \in W, \langle w \rangle \ll W$. Hence, $a(bw) \subseteq H$. It follows that $ab \in [H_R^i(w)]$ which is a prime ideal by (2). But $a \notin [H_R^i(w)]$, so $a \in [H_R^i(w)]$. Thus, $[H_R^i(bw)] \subseteq [H_R^i(w)]$. Therefore, $[H_R^i(w)] = [H_R^i(bw)]$.

(3) \Rightarrow (1). Let $w \in W, \langle w \rangle \ll W$ and $a, b \in R$ such that $a \in [H_R^i(w)]$, suppose $b \notin [H_R^i(w)]$, hence by (3), $[H_R^i(bw)] = [H_R^i(w)]$. But $a \in [H_R^i(bw)]$, so $a \in [H_R^i(w)]$ and hence H is a quasi-small prime submodule.

Proposition (2.13): Let W be an R -module, and H, L are a quasi-small prime submodules. Then $H \cap L$ is a quasi-small prime submodule of W .

Proof: Let $a, b \in R, w \in W$ with $\langle w \rangle \ll W$ such that $abw \in H \cap L$, then $abw \in H$ and $abw \in L$. Since both H and L are a quasi-small prime submodules of W , so either $aw \in H$ or $bw \in H$ and either $aw \in L$ or $bw \in L$. Thus we have either $aw \in H \cap L$ or $bw \in H \cap L$. That is $H \cap L$ is a quasi-small prime submodule of W .

Proposition (2.14): Let H and K be two submodules of an R -module W such that H is a quasi-small prime submodule of W and $K \not\subseteq H$. Then $K \cap H$ is a quasi-small prime submodule of K .

Proof: Since $K \not\subseteq H, K \cap H$ is a proper submodule of K . Let $a, b \in R$ and $y \in K, \langle y \rangle \ll K$ such that $aby \in K \cap H$, so that $aby \in K$ and $aby \in H$. But H is a quasi-small prime submodule of W and $\langle y \rangle \ll W$, so either $ay \in H$ or $by \in H$ by definition (2.1). Since $y \in K$, implies that $ay \in K \cap H$ or $by \in K \cap H$. Hence $K \cap H$ is a quasi-small prime submodule of K .

Proposition (2.15): Let H be a submodule of an R -module W . Then, H is a quasi-small prime submodule of W if and only if $[H_W^i I]$ is a quasi-small prime submodule of W for every ideal I of R .

Proof: Suppose that H is a quasi-small prime submodule of W . Let I be an ideal of R , let $a, b \in R$ and $w \in W$ with $\langle w \rangle \ll W$ such that $abw \in [H_W^i I]$, then $Iabw \subseteq H$. Hence, $iabw \in H$ for all $i \in I$. But H is a quasi-small prime submodule of W , so either $iaw \in H$ or $ibw \in H$ for all $i \in I$. Therefore, either $aw \in [H_W^i I]$ or $bw \in [H_W^i I]$, which means that $[H_W^i I]$ is a quasi-small prime submodule of W .

The converse follows by taking $I = R$ and $[H_W^i R] = H$, [9, p.13], [10, p.16]. Thus, H is a quasi-small prime submodule of W .

Remark (2.16): Let W be an R -module and I be a maximal ideal of R such that IW a proper submodule of W . Then IW is a quasi-small prime submodule of W .

Proof: Since IW is a prime submodule of W , ([8], Coro. (2.6), chapter one), hence IW is a quasi-small prime submodule of W by remarks and examples (2.2, (3)).

Proposition (2.17): Let W be a cyclic R -module. If I be a small prime ideal of a ring R ($I \neq R$) with $\text{ann}_R W \subseteq I$. Then IW is a quasi-small prime submodule of W .

Proof: It is clear that $IW \neq W$. Let $W = \langle y \rangle$ for $y \in W$ and let $abw \in IW$, where $a, b \in R$ and $w \in W$ such that $\langle w \rangle \ll W$, then $w = sy$ for some $s \in R$, and $absy = rhy$ for some $h \in R$ and $r \in I$, so $(abs - rh)y = 0$, implies that $abs - rh \in \text{ann}_R W \subseteq I$. Hence, $abs \in I$, since $\langle sy \rangle \ll W$ and W is cyclic with $\text{ann}_R W \subseteq I$, then $\langle s \rangle \ll R$, [11]. But I is a small prime ideal of R , so either $a \in I$ or $b \in I$ or $s \in I$. Therefore, either $aw \in IW$ or $bw \in IW$ or $sy \in IW$. But $sy \in IW$, implies that $IW = W$ which is a contradiction. Thus, either $aw \in IW$ or $bw \in IW$ and hence IW is a quasi-small prime submodule of W .

Recall that an R -module W is called multiplication if for all submodule H of W , there exists an ideal I of R such that $IW = H$. Equivalently, for all submodule H of W , $H = [H:W]W$, [10].

Recall that an R -module W is called faithful if $\text{ann}_R W = (0)$, [11].

Proposition (2.18): Let W be a faithful finitely generated multiplication R -module. If I is a quasi-small prime ideal of R , then IW is a quasi-small prime submodule of W .

Proof: Suppose that $abw \in IW$, where $a, b \in R$, $w \in W$ with $\langle w \rangle \ll W$. Then $ab\langle w \rangle \subseteq IW$. Since W is a multiplication, this implies that $\langle w \rangle = JW$ for some ideal J of R . Thus $abJW \subseteq IW$. But W is finitely generated, so $abJ \subseteq I + \text{ann}_R W$, [11]. But W is faithful, it follows that $\text{ann}_R W = (0)$ hence $abJ \subseteq I$, and $JW \ll W$, then $J \ll R$. But I is a quasi-small prime ideal of R , then by corollary (2.7), either $aJ \subseteq I$ or $bJ \subseteq I$, it follows that, either $aJW \subseteq IW$ or $bJW \subseteq IW$. Hence, either $aw \in IW$ or $bw \in IW$. Therefore IW is a quasi-small prime submodule of W .

Proposition (2.19): Let W be a faithful finitely generated multiplication R -module and H be a proper submodule of W . Then H is a quasi-small prime submodule of W if and only if $[H_R^i W]$ is a quasi-small prime ideal of R .

Proof: (\Rightarrow) Let $a, b, y \in R$ with $\langle y \rangle \ll R$ and $aby \in [H_R^i W]$. Then $\langle aby \rangle W \subseteq H$. But $\langle y \rangle \ll R$, implies that $\langle aby \rangle \ll R$, ([12], prop. 1.1.3) and since W is a faithful finitely generated multiplication R -module, then $yW \ll W$ and H is a quasi-small prime submodule of W , then $ayW \subseteq H$ or $byW \subseteq H$. Hence either $ay \in [H_R^i W]$ or $by \in [H_R^i W]$ and hence $[H_R^i W]$ is a quasi-small prime ideal of R .

(\Leftarrow) Let $a, b \in R$, $w \in W$ with $\langle w \rangle \ll W$ such that $abw \in H$, this implies that $ab\langle w \rangle \subseteq H$. Since W be a multiplication, then $\langle w \rangle = IW$ for some ideal I of R . That is $abIW \subseteq H$, it follows that $abI \subseteq [H_R^i W]$, and $IW \ll W$, then $I \ll R$. But $[H_R^i W]$ is a quasi-small prime ideal of R , then by corollary (2.7), either $aI \subseteq [H_R^i W]$ or $bI \subseteq [H_R^i W]$, it follows that either

$aIW \subseteq [H_R^i W]W$ or $bIW \subseteq [H_R^i W]W$. Since W is a multiplication, therefore $aw \in H$ or $bw \in H$. Hence H is a quasi-small prime submodule of W .

Proposition (2.20): Let W be a faithful finitely generated multiplication R -module and H be a proper submodule of W . Then $[H_R^i W]$ is a quasi-small prime ideal of R if and only if $H = IW$ for some quasi-small prime ideal I of R .

Proof: (\Rightarrow) Since $[H_R^i W]$ is a quasi-small prime ideal of R and W is a multiplication, then $H = [H_R^i W]W$, it follows that $H = IW$ and $I = [H_R^i W]$ is a quasi-small prime ideal of R .

(\Leftarrow) Suppose that $H = IW$ for some quasi-small prime ideal I of R . But W is a multiplication, we have $H = [H_R^i W]W = IW$. Thus, since W is a faithful finitely generated multiplication, then $I = [H_R^i W]$, [11]. It follows that $[H_R^i W]$ is a quasi-small prime ideal of R .

Proposition (2.21): Let W be a faithful finitely generated multiplication R -module and H be a proper submodule of W , the following statements are equivalent:

1. H is a quasi-small prime submodule of W .
2. $[H_R^i W]$ is a quasi-small prime ideal of R .
3. $H = IW$ for some quasi-small prime ideal I of R .

Proof:

(1) \Leftrightarrow (2), by proposition (2.19).

(2) \Leftrightarrow (3), by proposition (2.20).

3. More Properties a bout Quasi-Small Prime Submodules.

Recall that an R -module W is called a quasi-small prime modules, if and only if $ann_R H$ is a prime ideal of R for all non-zero small submodule H of W , [6]. We know that an R -module W is a prime (quasi-prime) if and only if $\langle 0 \rangle$ is a prime (quasi-prime) submodule of W , [5, proposition 2.2.1].

Proposition (3.1): A module W is quasi-small prime if and only if $\langle 0 \rangle$ is a quasi-small prime submodule of W .

Proof: (\Rightarrow) Assume that W is a quasi-small prime R -module, then by [6, proposition (2.3)] $ann_R \langle w \rangle$ is a prime ideal of R for all small submodule $\langle w \rangle$ of W . But $ann_R \langle w \rangle = [0_R^i(w)]$ for all small submodule $\langle w \rangle$ of W . Hence $\langle 0 \rangle$ is a quasi-small prime submodule of W by definition (2.1).

(\Leftarrow) It is clear, so that the details are omitted.

Corollary (3.2): A proper submodule of an R -module W , H is a quasi-small prime submodule of W if and only if W/H is a quasi-small prime R -module.

Proof: It is clear since this the zero of W/H .

Corollary (3.3): If W is an R -module. Then the following statements are equivalent:

1. W is a quasi-small prime R -module.
2. $ann_R \langle w \rangle$ is a prime ideal of R for all non-zero small submodule $\langle w \rangle$ of W .
3. $\langle 0 \rangle$ is a quasi-small prime submodule of W .

Proof:

(1) \Rightarrow (2) It is clear by [6, proposition 2.1.7].

(2) \Rightarrow (3) It is clear by definition (2.1).

(3) \Rightarrow (1) By proposition (3.1).

Proposition (3.4): If W is a quasi-small prime R -module, then $ann_R I$ is a quasi-small prime submodule of W for every ideal I of R .

Proof: If W is a quasi-small prime R -module, then $\langle 0 \rangle$ is a quasi-small prime submodule of W by proposition (3.1). But $[0_W^i I] = ann_R I$, [10, p.16]. Thus, $ann_R I$ is a quasi-small prime submodule of W , by proposition (2.15).

Proposition (3.5): Let $W = W_1 \oplus W_2$, where W_1 and W_2 be two R-modules. If $H = H_1 \oplus H_2$ is a quasi-small prime R-submodule of W , then H_1 and H_2 are quasi-small prime R-submodules of W_1 and W_2 , respectively.

Proof: Let $ab\langle w_1 \rangle \subseteq H_1$, where $a, b \in R$, $\langle w_1 \rangle \ll W$, then $ab\langle w_1 \rangle \oplus 0 \subseteq H_1 \oplus H_2$, hence $ab(\langle w_1 \rangle \oplus 0) \subseteq H_1 \oplus H_2$, this implies that $ab \in [H_R(\langle w_1 \rangle \oplus 0)]$. $\langle w_1 \rangle \ll W$, then $\langle w_1 \rangle \oplus 0 \ll W$, [12, Proposition 1.14]. But H is a quasi-small prime R-submodule, so $[H_R(\langle w_1 \rangle \oplus 0)]$ is a prime ideal of R . Thus, $a \in [H_R(\langle w_1 \rangle \oplus 0)]$ or $b \in [H_R(\langle w_1 \rangle \oplus 0)]$, and then $a(\langle w_1 \rangle \oplus 0) \subseteq H_1 \oplus H_2$ or $b(\langle w_1 \rangle \oplus 0) \subseteq H_1 \oplus H_2$, hence $(aw_1, 0) \in H_1 \oplus H_2$ or $(bw_1, 0) \in H_1 \oplus H_2$, which implies $aw_1 \in H_1$ or $bw_1 \in H_1$, so $a \in [H_{1R}(\langle w_1 \rangle)]$ or $b \in [H_{1R}(\langle w_1 \rangle)]$. Therefore, $[H_{1R}(\langle w_1 \rangle)]$ is a prime ideal of R and hence H_1 is a quasi-small prime submodule of W_1 .

By similar proof, H_2 is a quasi-small prime submodule of W_2 .

Using the mathematical induction, we obtain the following corollary:

Corollary (3.6): Let $W = W_1 \oplus W_2 \oplus \dots \oplus W_n$. Where W_1, W_2, \dots, W_n are a finite collection of R-modules. If $H = H_1 \oplus H_2 \oplus \dots \oplus H_n$ is a quasi-small prime submodule of W , then H_1, H_2, \dots, H_n are quasi-small prime submodules of W_1, W_2, \dots, W_n , respectively.

Proof: By induction the result follows.

Remark (3.7): Let $W = W_1 \oplus W_2$, where W_1 and W_2 be two R-modules. If H_1 and H_2 are quasi-small prime submodules of W_1 and W_2 respectively, then $H = H_1 \oplus H_2$ is not necessary quasi-small prime submodule.

For example: Consider the Z-module $W = Z \oplus Z$ and let $H = 2Z \oplus 3Z$, so by (2.2,(4)), we have $2Z$ and $3Z$ are quasi-small prime submodules of Z . But $[H_R(0)] = 6Z$, which is not a prime ideal of Z , where (0) is the only small submodule of W . Thus, H is not a quasi-small prime submodule of W .

Proposition (3.8): If H is a quasi-small prime submodule of an R-module W , then $H^2 = H \oplus H$ is a quasi-small prime submodule of $W^2 = W \oplus W$.

Proof: Let $a, b \in R$ and $w = (w_1, w_2) \in W^2$ with $\langle w \rangle = \langle w_1 \rangle \oplus \langle w_2 \rangle \ll W^2$, then $\langle w_1 \rangle \ll W$ and $\langle w_2 \rangle \ll W$, [12, Proposition 1.14] such that $abw \in H^2$. Hence, $abw_1 \in H$ and $abw_2 \in H$. It follows that $aw_1 \in H$ or $bw_1 \in H$ and $aw_2 \in H$ or $bw_2 \in H$, which implies that either $aw \in H^2$ or $bw \in H^2$. Therefore, H^2 is a quasi-small prime R-submodule.

Proposition (3.9): Let $W = W_1 \oplus W_2$ where W_1, W_2 be R-modules. Then

- H_1 is a quasi-small prime submodule of W_1 , if and only if $H_1 \oplus W_2$ is a quasi-small prime submodule of W .
- H_2 is a quasi-small prime submodule of W_2 , if and only if $W_1 \oplus H_2$ is a quasi-small prime submodule of W .

Proof (1): (\Rightarrow) Let $ab \in [H_1 + W_{2R}(\langle w \rangle)]$, where $a, b \in R, w \in W$ with $\langle w \rangle = \langle w_1 \rangle \oplus \langle w_2 \rangle \ll W$, such that $w_1 \in W_1$ and $w_2 \in W_2$. Since $w \in W$, then $w_1 + w_2 = w$. Thus, $ab(w_1 + w_2) \in H_1 \oplus W_2$ and hence $w_1 + abw_2 = x + z$, for some $x \in H_1$ and $z \in W_2$. This implies that $abw_1 - x = z - abw_2 \in W_1 \cap W_2 = 0$, therefore $abw_1 = x \in H_1$ and $(w_1) \leq W_1, W_1$ is a direct summand of W , then $(w_1) \ll W$, [4, Lemma 4.2]. But H_1 is a quasi-small R-submodule of W_1 , then either $aw_1 \in H_1$ or $bw_1 \in H_1$. It follows that $aw = a(w_1 + w_2) \in H_1 \oplus W_2$ or $b(w_1 + w_2) \in H_1 \oplus W_2$. Therefore, $H_1 \oplus W_2$ is a quasi-small prime R-submodule of W .

(\Leftarrow) Let $a, b \in R, w_1 \in W_1$ with $\langle w_1 \rangle \ll W_1$ such that $abw_1 \in H_1$. Since $w_1 + 0 \in W$. Thus, $ab(w_1 + 0) \in H_1 \oplus W_2$ and $\langle w_1 \rangle \ll W_1, W_1$ is a direct summand of W , then $\langle w_1 + 0 \rangle \ll W$, [12, Proposition 1.14]. But $H_1 \oplus W_2$ is a quasi-small prime submodule of W , so either $a(w_1 + 0) \in H_1 \oplus W_2$ or $b(w_1 + 0) \in H_1 \oplus W_2$, it follows that $aw_1 \in H_1$ or $bw_1 \in H_1$. Therefore H_1 is a quasi-small prime submodule of W_1 .

Proof (2): By similar way we can prove (2).

References

- [1] Abdul-Alkalik, A. J., Al-Mothafar, N. S., “I-Nearly Prime Submodules”, *Iraqi Journal of Science*, vol. 60, no. 11, pp. 2468-2472, 2019.
- [2] Mahmood, L. S. “Small Prime Modules and Small Prime Submodules”, *Journal of Al-Nahrain University*, vol. 15, no. 4, pp.191-199, 2012.
- [3] Ramadhan, H. A. and Al-Mothafar, N.S. “On Small Semiprime Submodules”, *Journal of physics: Conference Series*, vol. 1818, 012142, 2021.
- [4] Mohamed, S. H., Muller, B. J., *Continuous and Discreet Modules*, London Math. Soc. LNC 147 Cambridge Univ. Press, Cambridge, 1990.
- [5] Abdul-Razak, M. H. , *Quasi-Prime Modules and Quasi-Prime Submodules*, M.Sc. Thesis, Univ. of Baghdad, 1999.
- [6] Wisam, A. Ali. and Al-Mothafar, N.S. (2021), “On Quasi-Small Prime Modules”, *Journal of physics: Conference Series*, vol. 1818, 012104.
- [7] Dauns, J., *Prime Modules and One –Sided Ideals in “Ring Theory and Algebra III”*, (Proceeding of the Third Oklahoma Conference), B. R. McDonald (editor), Dekker, New York, pp.301-344, 1980.
- [8] Athab, I. A., Prime Submodules and Semiprime Submodules, M.Sc. Thesis, College of Science, University of Baghdad, 1996.
- [9] Mccasland, R. L. and Moore, M.E., “Prime Submodules”, *Communication in Algebra*, vol. 20, no. 6, pp. 1803-1817, 1992.
- [10] Sharpe, D.W. and Vamos, P., *Injective Modules*, Cambridge University Press, 1972.
- [11] Smith, P. F., “Some Remarks on Multiplication Modules”, *Arch. Math.*, vol.50, pp.223-235, 1988.
- [12] Athab, I. A, Some Generalization of Projective Modules, Ph. D. Thesis, College of Science, University of Baghdad, 2004.