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On Annihilator-Extending Modules

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Abstract:
Throughout this work we introduce the notion of Annihilator-closed submodules, and we give some basic properties of this concept. We also introduce a generalization for the Extending modules, namely Annihilator-extending modules. Some fundamental properties are presented as well as we discuss the relation between this concept and some other related concepts.

Keywords: Annihilator Essential submodules, Annihilator-closed submodules, Extending modules, Annihilator-Extending modules.

مقاسات التوسع المرتبطة بتالف الحلقة

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الخلاصه
في هذا البحث قدمنا مفهوم المقاسات الجزئية المغلقة المرتبطة بتالف الحلقة ودرسنا العلاقة بينها وبين المقاسات الجزئية المغلقة .كذلك تمت دراسة بعض الخواص الاساسية لهذا المفهوم .وكذلك قدمنا تعميما لمقاسات التوسع بإسم مقاسات التوسع المرتبطة بتالف الحلقة R وقدمنا بعض الخواص الأساسية حيث تم مناقشة العلاقة بين هذا المفهوم وبعض المفاهيم الاخرى المرتبطة بهذا المفهوم.

Introduction:

Throughout this paper we consider that R is a commutative ring with identity and all modules will be unitary left R-modules . It well known that if N has no proper essential extension in M then a submodule N of an R-module M is called closed submodule that is if there exists a submodule K of M with $N\leq_e K\leq M$ then N=K [1]. A submodule N of an R-module M is called essential submodule of M if for every $K\leq M$ with $N\cap K=0$ then K=0[2]. Many authors have been interested in studying the class of closed submodules and some related concepts see[3-5]. Yousef and Sahira [6] introduced the notion of Annihilator essential submodules as a generalization of essential submodules where a submodule N of an R-module M is called annihilator-essential if $N\cap L=0$ then $ann(L)\leq_e R$ .We also modify the thought of annihilator-closed submodules. If N has no proper annihilator-essential extension in M then a submodule N of an R-module M is called annihilator-closed submodule that is if

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there exist a submodule K of M such that $N\leq_{a.e}K\leq M$ then N=K. In addition we give some basic properties for this concept. In [7-9] authors are interested in study the class of extending modules, and they gave some related concepts, where an R-module M is called Extending module if every submodule of M is essential in a direct summand [10]. In section two we introduce a generalization of extending modules which is called annihilator-extending modules where an R-module M is called annihilator-extending module (shortly by ann-extending) if every submodule of M is annihilator-essential in a direct summand. We also give some basic properties and we study the relation with the other related concepts.

1.Annihilator closed submodules

We recall the definition of Annihilator essential submodules and some basic properties of this definition are given in lemma(1.2). We introduce the notion of annihilator-closed submodules. We study relation between the closed submodules and the annihilator-closed submodules with some basic properties of this concept.

Definition(1.1): [6] A submodule N of an R-module M is called annihilator-essential submodule(shortly by ann-essential) Which is denoted by $N\leq_{a.e}M$ if $N\cap L=0$ then $ann(L)\leq_{e}R$ for every $L\leq M$.

Lemma(1.2):[6]

(1) Every essential submodule is ann-essential submodule.

(2) Let $A\leq B\leq M$. If $A\leq_{a.e}M$ then $B\leq_{a.e} M$.

(3) If $A\cap B$ is ann-essential submodule. Then both of A and B are ann-essential submodule.

(4) If $A\leq_{e}B\leq M$ and $A_1\leq_{a.e}B_1\leq M$ then $A\cap A_1\leq_{a.e}B\cap B_1$.

Definition(1.3): Let N be a submodule of an R-module M , N is called annihilator closed submodule (shortly by ann-closed) in M which is denoted by $N\leq_{a.c} M$ .If N has no proper annihilator essential extension . That is if there exist a submodule K of M such that $N\leq_{a.e} K\leq M$ then N=K.

An ideal I of a ring R is called ann-closed ideal if I has no proper ann-essential extension in R. that is if there exist an ideal J of R with $I\leq_{a.e}J\leq R$ then I=J.

Remarks and Examples(1.4):

(1) Every ann-closed submodule is closed submodule.

Proof: Let N be an ann-closed submodule of M and let K be a submodule of M such that $N\leq_{e} K\leq M$, by lemma(1.2) every essential submodule is ann-essential then $N\leq_{a.e} K\leq M$ .However N is ann-closed submodule in M so N=K .Thus, N is closed submodule in M.

(2) The following example shows that the converse of (1) in general is not true.

Example:

Let $M=Z_6$ as Z- module ,$(\bar{3})$ is closed in Z_6 but not ann-closed submodule since $(\bar{3})$ is ann-essential submodule in Z_6 .

(3) It well known that every direct summand is closed submodule[1]. The next example shows that it is not necessary every direct summand is ann-closed submodule

Example:

: Let $M=Z_{45}$ as Z-module , $M=(\bar{5})\oplus(\bar{9})$ both of $(\bar{9})$ and $(\bar{5})$are not ann-closed submodule of M since $(\bar{9}) \leq_{a.e} M$ and $(\bar{5})\leq_{a.e}M$. Also the Z-module $Z_6 =(\bar{2})\oplus(\bar{3})$ both of $(\bar{2})$and $(\bar{3})$ are not ann-closed submodules in M .

(4) Every module is ann-closed submodule of itself.

(5) Since (0) is not ann-essential submodule of any module, then (0) is ann-closed submodule [6].

(6) The intersection of ann-closed submodules of an R-module M is ann-closed submodule in M.

Proof: Let K be a submodule of M with $A\cap B\leq_{a.e}K\leq M$ then by lemma(1.2) $A\leq_{a.e}K\leq M$ and $B\leq_{a.e}K\leq M$,since both of A and B are ann-closed submodules in M then A=K=B hence $A\cap B=K$.

(7) Let M be an R-module and let A be an ann-closed submodule of M .

Next example shows that if B is a submodule of M such that $A\cong B$ then it is not necessary that B is ann-closed submodule of M.

Example : Let M=Z as Z-module , Z is ann-closed in Z and $Z\cong 2Z$ but 2Z is not ann-closed in Z since $2Z\leq_{a.e} Z$.

(8) Let A and B be a submodules of an R-module M such that $A\leq B\leq M$. The following example shows that if B is ann-closed submodule in M then A need not be ann-closed submodule in M.

Example :Let M=Z as Z-module, Z is ann-closed submodule in Z and $3Z\leq Z$ but 3Z is not ann-closed submodule in Z.

(9) Let A and B be a submodules of an R-module M such that $A\leq B\leq M$. The following example shows that if A is ann-closed submodule in M then B need not be ann-closed submodule in M.

Example: Let M=Z as Z-module and the submodules A=(0) and B=2Z . Note that A is ann-closed submodule in Z but B is not ann-closed submodule in Z since $2Z\leq_{a.e}Z$.

(10) If a submodule A of an R-module M is ann-closed and ann-essential submodule then A=M.

Proposition(1.5): Let $N\leq K\leq M$ with N is ann-closed submodule in M . If $K\leq_{a.e} M$ then $\frac{K}{N}\leq_{a.e}\frac{M}{N}$.

Proof: Let $\frac{L}{N}$ be a submodule of $\frac{M}{N}$ with $\frac{K}{N}\cap\frac{L}{N}=0$, then $K\cap L=N$. Since $K\leq_{a.e} M$ and $L\leq_{e}L\leq M$ hence by lemma(1.2) we have $K\cap L\leq_{a.e} M\cap L$. So that $N\leq_{a.e}L\leq M$. But N is ann-closed submodule in M thus, N=L and hence $\frac{L}{N}=0$, we get $ann(\frac{L}{N})= ann(0)=R\leq_{e} R$. Therefore $\frac{K}{N}\leq_{a.e}\frac{M}{N}$.

Proposition(1.6): Let A and B be a submodules of a chained R-module M .If A is ann-closed submodule in B and B is ann-closed submodule in M then A is ann-closed submodule in M.

Proof: Let $L\leq M$ such that $A\leq_{a.e}L\leq M$. Since M is chained then we have two cases:

Case 1 if $L\leq B$, since A is annihilator-closed submodule in B, then A=L and hence $A\leq_{a.c} M$.

Case 2 if $B\leq L$, since $A\leq_{a.e} L$ then by Lemma(1.2) we have $B\leq_{a.e} L$,but B is ann-closed submodule in M then B=L that is $A\leq_{a.e}B$,since A is ann-closed submodule in B then A=B ,hence A is ann-closed submodule in M .

Proposition(1.7): Let A and B be anon zero submodules of an R-module M such that $A\leq B\leq M$. If A is ann-closed submodule in M then A is ann-closed submodule in B.

Proof: Suppose that $A\leq_{a.e} L\leq B$,and we get $L\leq M$. However A is ann-closed submodule in M .Therefore A=L .

Corollary(1.8): Let A and B be a submodules of an R-module M .If $A\cap B$ is ann-closed submodule in M then $A\cap B$ is ann-closed submodule in both of A and B.

Corollary(1.9): Let N and K be ann-closed submodules of an R-module M then both of N and K are ann-closed submodules in N+K .

Lemma(1.10):[4] Let M be a non zero multiplication R-module with only one maximal submodule N .If $N\neq(0)$ then N is essential (and hence ann-essential) submodule of M.

Proposition(1.11): Let M be a non zero multiplication R-module with only one non zero maximal submodule N then N cannot be ann-closed submodule in M.

Proof: Assume that N is ann-closed submodule in M , since M is multiplication module then by previous lemma $N\leq_{a.e} M$ .But N is ann-closed submodule in M then N=M which is a contradiction with maximally of N .

Proposition(1.12): Let M be a faithful multiplication R-module .If IM is an ann-closed submodule in M then I is ann-closed ideal in R.

Proof: Suppose that $I\leq_{a.e} J\leq R$, then by [6,proposition(1.8)] $IM\leq_{a.e} JM\leq M$ .But $IM \leq_{a.c} M$ then IM=JM and hence I=J .Thus, I is ann-closed ideal in R.

Corollary(1.13): Let M and N be a faithful multiplication R-module and an ann-closed submodule in M, respectively then $(N:_R M)$ is ann-closed ideal in R .

Proof: Since M is faithful multiplication R-module then $N=(N:_R M)M$ [11], put $(N:_R M)=I$ then we get IM is ann-closed submodule in M then by previous proposition we get the result.

Proposition(1.14): Let M and N be an R-module and a non zero submodule of M , respectively then there exist an ann-closed submodule H in M such that $N\leq_{a.e} H$.

Proof: Consider the set F={K: K is a submodule of M such that $N\leq_{a.e} K$}. It is clear that $F\neq \emptyset$ since N∈F . Let $\{N_\alpha\}_{\alpha\in\Lambda}$ be a chain in F . Since $N_\alpha \leq M$ for every $\alpha \in \Lambda$ then $\bigcup_{\alpha\in\Lambda} N_\alpha \leq M$, and since $N_\alpha \leq \bigcup_{\alpha\in\Lambda} N_\alpha \leq K$ with $N_\alpha \leq_{a.e} K$ for every $\alpha \in \Lambda$ then by Lemma(1.2) we get $\bigcup_{\alpha\in\Lambda} N_\alpha \leq_{a.e} K$ .Therefore $\bigcup_{\alpha\in\Lambda} N_\alpha \in F$,by zorns lemma F has maximal element say H . Now we claim that H is ann-closed submodule in M so that assume that there exist a submodule B in M such that $H\leq_{a.e} B\leq M$ .Since $N\leq_{a.e} H\leq_{a.e} B$ then by Lemma(1.2) we get $N\leq_{a.e} B$ then B∈F this leads to contradiction with maximality of H. Thus, H=B that is H is ann-closed submodule in M such that $N\leq_{a.e} H$.

2.Annihilator-Extending Modules.

In this section we introduce the concept of annihilator-extending modules as a generalization of extending modules and we also give some basic properties for it.

Definition(2.1): An R-module M is called annihilator extending module (shortly ann-extending module) if every submodule of M is ann-essential in a direct summand .

A ring R is called ann-extending ring if every ideal of R is ann-essential in direct summand .

Remarks and Examples(2.2):

(1) The Z-module Z_6 is ann-extending module since every submodule is ann-essential in a direct summand .

(2) Every extending module is ann-extending module .

Proof: It follows directly from Remarks and Examples (1.4,1).

Next example shows that the converse in general is not true.

Example: Let $M=Z_8\oplus Z_2$ as Z-module , M has eleven submodules which are $N_1=(\bar{0},\bar{0})$, $N_2=(\bar{1},\bar{0})$, $N_3=(\bar{0},\bar{1})$, $N_4=(\bar{1},\bar{1})$, $N_5=(\bar{2},\bar{0})$ $N_6=(\bar{2},\bar{1})$, $N_7=(\bar{4},\bar{0})$, $N_8=(\bar{4},\bar{1})$, $N_9=((\bar{0},\bar{1}),(\bar{4},\bar{0}))$, $N_{10}=((\bar{2},\bar{0}),(\bar{4},\bar{1}))$, N_{11}=M. N_2 and N_4 are direct summands, N_2 and N_8 are direct summands. W is ann-extending module but not extending module since N_6 is not essential in any direct summand of M.

(3) Every semi simple module is ann-extending module , for example Z_2 , Z_3 ,Z_6 ,Z_{10} ,Z_{30} .

Proof: Since every semi simple module is extending module see[8] So that the result follows from (2) .

(4) Every uniform module is an ann-extending module and indecomposable module , for example Z ,Z_4 ,Z_8 , Z_{16} .

Proof: Since every uniform module is an extending module[8] then the result follows directly from (2) .Now, assume that M is decomposable module then there exist two non zero submodules N and K such that M=N⊕K so both of N and K are not essential submodules which is a contradiction since M is uniform module . The converse in general is not true for example : consider the Z-module Z_6 is ann-extending module since every submodule is ann-essential in direct summand but Z_6 is not uniform module.

Recall that: an R-module M is called annihilator-uniform(shortly by ann-uniform) module if every submodule of M is ann-essential submodule[6].

(5) Every ann-uniform module is an ann-extending module.

Proof: Let N be a submodule of M .If N=(0) then clearly that N is ann-essential in (0) which is a direct summand of M . If N≠(0) ,since M is ann-uniform module then $N \leq_{a.e} M$, but M is direct summand of it self ,so M is ann-extending module . Next example shows that the converse in general is not true.

Example: Let M= Z_{24} as Z_{24} module , M is ann-extending but not ann-uniform module.

(6) Every π- injective module is an ann-extending module .

(7) Every module over a semi simple ring is an ann-extending module .

Proof: let R be a semi simple ring and let M be an R-module ,then by [1,Theorem:1.18 p 29] M is injective module and hence M is an extending module [10] and by (2) M is an ann-extending module .

(8) An ann-essential submodule of an ann-extending module need not be ann-extending module for example: Let M be an R-module such that M is not ann-extending module , The injective hull of M is an injective module and hence it is an extending module , but $M \leq_e E(M)$ [1, proposition 1.11 p 22] so $M \leq_{a.e} E(M)$ so we find an ann-essential submodule of an ann-extending module but not ann-extending module .

(9) If both of M_1 and M_2 are isomorphic modules and M_1 is an ann-extending module then M_2 is an ann-extending module .

Proposition(2.3): Let M be an indecomposable R-module Then M is ann-extending module if and only if M is ann-uniform module .

Proof: Let (0)≠N≤M, since M is ann-extending module then $N \leq_{a.e} H$ where H is a direct summand of M ,but M is indecomposable module then either H=(0) or H=M , but N≠(0) therefore H=M and hence M is ann-uniform module. The converse is clear by (5) .

Proposition(2.4): Let M be an R-module .then M is ann-extending module if and only if every ann-closed submodule in M is a direct summand of M .

Proof: Let N be an ann-closed submodule in M , since M is ann-extending module then $N \leq_{a.e} H$ where H is a direct summand of M ,but N is ann-closed submodule in M then N=H ,that is N is a direct summand of M . Now let N be a submodule of M . If N= (0) then N is an ann-essential submodule in a direct summand of M . If N≠(0) then by proposition (1.14) there exist an ann-closed submodule H in M such that $N \leq_{a.e} H$ and by assumption H is a direct summand of M .Therefore M is an ann-extending module .

Proposition(2.5): Let A and B be a submodules of an ann-extending R-module M . If A∩B is ann-closed submodule in M then A∩B is a direct summand of A and B .

Proof: Since A∩B is ann-closed submodule in M then by proposition(2.5) A∩B is a direct summand of M , but A∩B ≤A≤M then by[12,lemma (2.3.4)] we have A∩B is a direct summand of A .

In similar way we can prove that A∩B is a direct summand of B .

Proposition(2.6); Let M be a finitely generated faithful multiplication R-module .If R is an ann-extending module then M is an ann-extending module.

Proof: Let N be an ann-closed submodule in M ,since M is multiplication R-module then $N=(N:_R M)M$ [11] .since $N\leq_{a.c} M$ then by proposition(1.12) $(N:_R M)\leq_{a.c} R$,but R is ann-extending module hence $(N:_R M)$ is a direct summand of R .thus, $R=(N:_R M)\oplus J$ where J is an ideal of S,and hence $M=RM=((N:_R M)\oplus J)M=(N:_R M)M\oplus JM$ .since M is faithful multiplication module then by [11] $(N:_R M)M\cap JM= ((N:_R M)\cap J)M$,we have $(N:_R M)\cap J=(0)$ hence $(N:_R M)M\cap JM=(0)M=(0)$ therefore $M=(N:_R M)M\oplus JM$ that is $(N:_R M)M=N$ is a direct summand of M.

Remark(2.7): Let M be an R-module such that for every submodule N of M. There exist an ann-closed submodule H of M with $N\leq_e H$ then M is an extending module if and only if M is an ann-extending module .

Proof: ⇒) It is clear so that the details are omitted .

⇐) We assume that M is an ann-extending module , and let $N\leq M$,by hypothesis there exist an ann-closed submodule H of M such that $N\leq_e H$,since M is an ann-extending module then H is a direct summand of M and hence M is an extending module.

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