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Comparison Different Estimation Methods for the Parameters of Non-Linear Regression

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ABSTRACT
Nonlinear regression models are important tools for solving optimization problems. As traditional techniques would fail to reach satisfactory solutions for the parameter estimation problem. Hence, in this paper, the BAT algorithm to estimate the parameters of Nonlinear Regression models is used . The simulation study is considered to investigate the performance of the proposed algorithm with the maximum likelihood (MLE) and Least square (LS) methods. The results show that the Bat algorithm provides accurate estimation and it is satisfactory for the parameter estimation of the nonlinear regression models than MLE and LS methods depend on Mean Square error.

Keywords: Nonlinear Regression Analysis, Bat algorithm, Simulation, Maximum likelihood method, least square method.

مقارنة طرق التقدير المختلفة لمعلمات الانحدار غير الخطي

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الخلاصه
نماذج الانحدار غيرالخطي هي من الطرق المهمة لحل مشاكل التحسين. حيث أن التقنيات التقليدية قد تفشل في الوصول إلى حلول مناسبة لمشكلة تقدير المعلمات. في هذا البحث تم استعمال خوارزمية الخفافيش لتقدير معاملات نماذج الانحدارغير الخطي. بعد ذلك ، تم استخدام المحاكاة للتحقق من أداء الخوارزمية المقترحة مع طرقتي الاحتمالية القصوى و المربعات الصغرى. أظهرت النتائج أن خوارزمية الخفاش توفر تقديرات دقيقة ومرضية لتقدير المعلمات لنماذج الانحدار غيرالخطي مقارنة بطريقتيMLE و LSبالاعتمادعلى متوسط مربعات الخطا.

1. Introduction

Nonlinear models are used for complex model interrelationships among variables and it plays an important role in various scientific disciplines and engineering. Common examples of nonlinear models include growth, yield density, and dose-response models as well as various

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models that are used to describe physical, biological, industrial, and econometric processes. [1].
There are large number of articles on how to estimate the parameter of nonlinear regression models. In[2], authors examined the nonlinear parameter estimation efficiency under the issue of auto reconditioned errors. In [3], authors used Bayesian parameter estimation, which incorporates a prior distribution function along with the likelihood equation, for estimating the biokinetic parameters for an organic substrate in an unsaturated soil. Some stochastic algorithms to solve the issue of global optimization of nonlinear regression models are used in [4]. These algorithms were applied to estimate the nonlinear regression parameters.
A robust alternative method to the normal Least Squares nonlinear regression method is proved in[5]. In [6], authors developed a new feature-selection method for the nonlinear regression model. The proposed method does not explicitly impose any model assumption on data distribution and it is able to select relevant features supporting complex data structures hidden in a high-dimensional space. In addition, a large-scale simulation study is performed on four synthetic and nine cancer microarray datasets that demonstrate the effectiveness of the proposed method.
However, the nonlinear model makes the estimation of parameters and the statistical analysis of parameter estimates more difficult and more challenging than others. As well, the limitations of these methods for nonlinear parameter estimation. It also is not easily controllable by practitioners and requires much auxiliary information to work properly. These difficulties arise due to a large number of parameters and multimodal nature the objective function.
In order to overcome these difficulties, meta-heuristic algorithms are used. Since it has many advantages including the simplicity of implementation that is reliable, robust, and effective.In [7], authors considered Jaya optimization algorithm for estimating nonlinear metaheuristic algorithm which is named Jaya algorithm then he tested it on a set of benchmark regression problems. In [8] authors used an effective approach based on the Partial Swarm Optimization (PSO)algorithm to enhance the estimation accuracy. The PSO algorithm is tested on the twenty eight models of nonlinear regressionfor various levels of difficulty.
Pan et al.[9]used a Genetic Algorithm (GA) to seek not only spline coefficients but also the degree of the polynomial. The GAs for the parameter estimation of the nonlinear regression models is considered in [10]. They used a large set of test problems with large starting intervals of regression parameters, the analysis of the difficulty levels of the test problems. In [11],authors modified the Firefly Algorithm and Support Vector Regression model for accurate Short Term Load Forecasting, while in [12] authors used differential evolution to find optimal parameters that result in the best accuracy and the minimum errors. The Gravitational Search algorithm for estimating the parameters of nonlinear regression model is used in [13]. Then, a simulation study was conducted to comparative the performance of the proposed algorithm. Hence,in this study used one of Metaheuristic algorithm called bat algorithm (BA) to solve nonlinear regression models. The BAT was widely used in various optimization problems because of its excellent performance [14].
The organization of this paper, Section2 provides the Maximum-likelihood Estimation of Nonlinear regression models; Section 3provides the Least-square Estimation of Nonlinear regression models; Section 4 describes the Bat algorithm; Section 5 consists of a simulation study; a conclusion is provided in section 6.

2.Mathematical Models of Nonlinear Regression

The general form of a regression model is $y = f(x, \beta) + \varepsilon$. where y is the dependent variable, x is a vector of independent variables, β is a vector of parameter(s), and ε with zero mean and variance. This paper used two types of Nonlinear regression models (MGH09, andMeyer4).
The MGH09 model is presented in [15], and the formal of this model as:

$$f\left(x,\beta_p\right)=\frac{\beta_1\left(x^2+x\beta_2\right)}{x^2+x\beta_3+\beta_4}\ ,p=1,2,3,4 \tag{1}$$

On the other hand,the Meyer4 model is presented in[16], and the formal of Meyer4 model as:

$$f(x_i,\beta_s)=\beta_3\left(e^{-\beta_1x_1}+e^{-\beta_2x_2}\right)\ ,s=1,2,3\ ,\ x=1,2 \tag{2}$$

3.Maximum Likelihood Estimation Nonlinear Regression models.

The Maximum likelihood method(MLE) is used to estimate the parameter for two models of Nonlinear regression (MGH 09 and Meyer4) as follows:

3.1. Maximum Likelihood Method to solveMGH 09 Model

From equation (1), the MLE method of estimation MGH 09 model is given as:

$$L=f(x_1,x_2,\dots\dots,x_n,\beta_i,\sigma^2)$$

$$L=(2\pi\sigma^2)^{-\frac{n}{2}}e^{-\frac{\sum_{i=1}^{n}(y_i-f(x,\beta))^2}{2\sigma^2}}$$

$$lnL=-\frac{n}{2}ln2\pi-\frac{n}{2}ln\sigma^2-\frac{\sum_{i=1}^{n}(y_i-f\left(x,\beta_p\right))^2}{2\sigma^2}$$

$$lnL=-\frac{n}{2}ln2\pi-\frac{n}{2}ln\sigma^2-\frac{\sum_{i=1}^{n}(y_i-\frac{\beta_1\left(x^2+x\beta_2\right)}{x^2+x\beta_3+\beta_4})^2}{2\sigma^2}$$

numerical procedures as Newton-Raphson was used to estimate the parameters since the equations are complicated to be solved. Therefore, the equation for this method for the first model is as follows

$$\begin{pmatrix}\hat{\beta}_1\\\hat{\beta}_2\\\hat{\beta}_3\\\hat{\beta}_4\\\hat{\sigma}^2\end{pmatrix}=\begin{pmatrix}\beta_{10}\\\beta_{20}\\\beta_{30}\\\beta_{40}\\\sigma_0^2\end{pmatrix}-\begin{pmatrix}\frac{\partial h_1}{\partial\beta_1}&\frac{\partial h_1}{\partial\beta_2}&\frac{\partial h_1}{\partial\beta_3}&\frac{\partial h_1}{\partial\beta_4}&\frac{\partial h_1}{\partial\sigma^2}\\\frac{\partial h_2}{\partial\beta_1}&\frac{\partial h_2}{\partial\beta_2}&\frac{\partial h_2}{\partial\beta_3}&\frac{\partial h_2}{\partial\beta_4}&\frac{\partial h_2}{\partial\sigma^2}\\\frac{\partial h_3}{\partial\beta_1}&\frac{\partial h_3}{\partial\beta_2}&\frac{\partial h_3}{\partial\beta_3}&\frac{\partial h_3}{\partial\beta_4}&\frac{\partial h_3}{\partial\sigma^2}\\\frac{\partial h_4}{\partial\beta_1}&\frac{\partial h_4}{\partial\beta_2}&\frac{\partial h_4}{\partial\beta_3}&\frac{\partial h_4}{\partial\beta_4}&\frac{\partial h_4}{\partial\sigma^2}\\\frac{\partial h_5}{\partial\beta_1}&\frac{\partial h_5}{\partial\beta_2}&\frac{\partial h_5}{\partial\beta_3}&\frac{\partial h_5}{\partial\beta_4}&\frac{\partial h_5}{\partial\sigma^2}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial lnL}{\partial\beta_1}\\\frac{\partial lnL}{\partial\beta_2}\\\frac{\partial lnL}{\partial\beta_3}\\\frac{\partial lnL}{\partial\beta_4}\\\frac{\partial lnL}{\partial\sigma^2}\end{pmatrix}$$

where $\begin{pmatrix}\beta_{10}\\\beta_{20}\\\beta_{30}\\\beta_{40}\\\sigma_0^2\end{pmatrix}$ represents the vector of the initial parameters.

$$h_1=\frac{\partial lnL}{\partial\beta_1}=\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i(x^2+x\beta_2)}{x^2+x\beta_3+\beta_4}-\frac{\beta_1(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$h_2=\frac{\partial lnL}{\partial\beta_2}=\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i\beta_1x}{x^2+x\beta_3+\beta_4}-\frac{\beta_1^2(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$h_3=\frac{\partial lnL}{\partial\beta_3}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i\beta_1x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{\beta_1^2x(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$h_4 = \frac{\partial lnL}{\partial \beta_4} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i\beta_1(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{\beta_1^2(x^2 + x\beta_2)^2}{(x^2 + x\beta_3 + \beta_4)^3}\right)$$

$$h_5 = \frac{\partial lnL}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}\left(y_i - \frac{\beta_1(x^2 + x\beta_2)}{x^2 + x\beta_3 + \beta_4}\right)^2$$

$$\frac{\partial h_1}{\partial \beta_1} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{(x^2 + x\beta_2)^2}{(x^2 + x\beta_3 + \beta_4)^2}\right)$$

$$\frac{\partial h_1}{\partial \beta_2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i x}{x^2 + x\beta_3 + \beta_4} - \frac{2\beta_1 x(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2}\right)$$

$$\frac{\partial h_1}{\partial \beta_3} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i x(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{2\beta_1 x(x^2 + x\beta_2)^2(x^2 + x\beta_3 + \beta_4)}{(x^2 + x\beta_3 + \beta_4)^4}\right)$$

$$\frac{\partial h_1}{\partial \beta_4} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{2\beta_1(x^2 + x\beta_2)^2(x^2 + x\beta_3 + \beta_4)}{(x^2 + x\beta_3 + \beta_4)^4}\right)$$

$$\frac{\partial h_1}{\partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i(x^2 + x\beta_2)}{x^2 + x\beta_3 + \beta_4} - \frac{\beta_1(x^2 + x\beta_2)^2}{(x^2 + x\beta_3 + \beta_4)^2}\right)$$

$$\frac{\partial h_2}{\partial \beta_1} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i x}{x^2 + x\beta_3 + \beta_4} - \frac{2\beta_1 x(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2}\right)$$

$$\frac{\partial h_2}{\partial \beta_2} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{\beta_1^2 x^2}{(x^2 + x\beta_3 + \beta_4)^2}\right)$$

$$\frac{\partial h_2}{\partial \beta_3} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x^2}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{2\beta_1^2 x^2(x^2 + x\beta_2)^2(x^2 + x\beta_3 + \beta_4)}{(x^2 + x\beta_3 + \beta_4)^4}\right)$$

$$\frac{\partial h_2}{\partial \beta_4} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{2\beta_1^2 x(x^2 + x\beta_2)^2(x^2 + x\beta_3 + \beta_4)}{(x^2 + x\beta_3 + \beta_4)^4}\right)$$

$$\frac{\partial h_2}{\partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x}{x^2 + x\beta_3 + \beta_4} - \frac{\beta_1^2 x(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2}\right)$$

$$\frac{\partial h_3}{\partial \beta_1} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i x(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{2\beta_1(x^2 + x\beta_2)^2}{(x^2 + x\beta_3 + \beta_4)^3}\right)$$

$$\frac{\partial h_3}{\partial \beta_2} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x^2}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{2\beta_1^2 x^2(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^3}\right)$$

$$\frac{\partial h_3}{\partial \beta_3} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{2y_i\beta_1 x^2(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^3} - \frac{3\beta_1^2 x^2(x^2 + x\beta_2)^2}{(x^2 + x\beta_3 + \beta_4)^4}\right)$$

$$\frac{\partial h_3}{\partial \beta_4} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{2y_i\beta_1 x(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^3} - \frac{3\beta_1^2 x(x^2 + x\beta_2)^2}{(x^2 + x\beta_3 + \beta_4)^4}\right)$$

$$\frac{\partial h_3}{\partial \sigma^2} = \frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x(x^2 + x\beta_2)}{(x^2 + x\beta_3 + \beta_4)^2} - \frac{\beta_1^2 x(x^2 + x\beta_2)^2}{(x^2 + x\beta_3 + \beta_4)^3}\right)$$

$$\frac{\partial h_4}{\partial \beta_1}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial h_4}{\partial \beta_2}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1^2 x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial h_4}{\partial \beta_3}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{2y_i\beta_1 x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}-\frac{3\beta_1^2 x(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial h_4}{\partial \beta_4}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{2y_i\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}-\frac{3\beta_1^2(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial h_4}{\partial \sigma^2}=\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{\beta_1^2(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial h_5}{\partial \beta_1}=-\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)}-\frac{\beta_1(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$\frac{\partial h_5}{\partial \beta_2}=-\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i x\beta_1}{(x^2+x\beta_3+\beta_4)}-\frac{\beta_1^2 x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$\frac{\partial h_5}{\partial \beta_3}=\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i x\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{\beta_1^2 x(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial h_5}{\partial \beta_4}=\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(\frac{y_i\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{\beta_1^2(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial h_5}{\partial \sigma^2}=\frac{n}{2\sigma^4}-\frac{1}{\sigma^6}\sum_{i=1}^{n}\left(y_i-\frac{\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)}\right)^2$$

2.2. Maximum likelihood method to solve Meyer 4 Model

From equation (2) the formula for MLE is applied for Meyer4model;

$$\mathrm{lnL}=-\frac{n}{2}ln2\pi-\frac{n}{2}ln\sigma^2-\frac{\sum_{i=1}^{n}\left(y_i-\beta_3\left(e^{-\beta_1 x_1}+e^{-\beta_2 x_2}\right)\right)^2}{2\sigma^2}$$

Thus, the following equation matrixes are applied to estimate the parameters for non-linear regression model by using Newton-Raphson method for the second model.

$$\begin{pmatrix}\hat\beta_1\\ \hat\beta_l\\ \hat\beta_3\\ \hat\sigma^2\end{pmatrix}=\begin{pmatrix}\beta_{10}\\ \beta_{20}\\ \beta_{30}\\ \sigma_0^2\end{pmatrix}-\begin{pmatrix}\frac{\partial h_1}{\partial\beta_1}&\frac{\partial h_1}{\partial\beta_2}&\frac{\partial h_1}{\partial\beta_3}&\frac{\partial h_1}{\partial\sigma^2}\\ \frac{\partial h_2}{\partial\beta_1}&\frac{\partial h_2}{\partial\beta_2}&\frac{\partial h_2}{\partial\beta_3}&\frac{\partial h_2}{\partial\sigma^2}\\ \frac{\partial h_3}{\partial\beta_1}&\frac{\partial h_3}{\partial\beta_2}&\frac{\partial h_3}{\partial\beta_3}&\frac{\partial h_3}{\partial\sigma^2}\\ \frac{\partial h_4}{\partial\beta_1}&\frac{\partial h_4}{\partial\beta_2}&\frac{\partial h_4}{\partial\beta_3}&\frac{\partial h_4}{\partial\sigma^2}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial lnL}{\partial\beta_1}\\ \frac{\partial lnL}{\partial\beta_2}\\ \frac{\partial lnL}{\partial\beta_3}\\ \frac{\partial lnL}{\partial\sigma^2}\end{pmatrix}$$

where $\begin{pmatrix}\beta_{10}\\ \beta_{20}\\ \beta_{30}\\ \sigma_0^2\end{pmatrix}$ represents the vector of the initial parameters.

$$h_1 = \frac{\partial lnL}{\partial \beta_1} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,\beta_3 x_1 e^{-\beta_1 x_1} - \beta_3^2 x_1 e^{-2\beta_1 x_1} - \beta_3^2 x_1 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$h_2 = \frac{\partial lnL}{\partial \beta_2} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,\beta_3 x_2 e^{-\beta_2 x_2} - \beta_3^2 x_2 e^{-(\beta_1 x_1+\beta_2 x_2)} - \beta_3^2 x_2 e^{-2\beta_2 x_2})$$

$$h_3 = \frac{\partial lnL}{\partial \beta_3} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,(e^{-\beta_1 x_1}+e^{-\beta_2 x_2}) - \beta_3(e^{-2\beta_1 x_1} + e^{-2\beta_2 x_2}) - 2\beta_3 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$h_4 = \frac{\partial lnL}{\partial \sigma^2} = -\frac{n}{2\sigma^4} + \sum_{i=1}^{n}\frac{1}{2\sigma^4}(y_i - \beta_3 \mathrm{e}^{-\beta_1 x_1} - \beta_3 e^{-\beta_2 x_2})^2$$

$$\frac{\partial h_1}{\partial \beta_1} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,\beta_3 x_1^2 e^{-\beta_1 x_1} - 2\beta_3^2 x_1^2 e^{-2\beta_1 x_1} - \beta_3^2 x_1^2 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$\frac{\partial h_1}{\partial \beta_2} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(\beta_3^2 x_1 x_2 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$\frac{\partial h_1}{\partial \beta_3} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,x_1 e^{-\beta_1 x_1} - 2\beta_3 x_1 e^{-2\beta_1 x_1} - 2\beta_3 x_1 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$\frac{\partial h_1}{\partial \sigma^2} = \frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i\,\beta_3 x_1 e^{-\beta_1 x_1} - \beta_3^2 x_1 e^{-2\beta_1 x_1} - \beta_3^2 x_1 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$\frac{\partial h_2}{\partial \beta_1} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(\beta_3^2 x_1 x_2 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$\frac{\partial h_2}{\partial \beta_2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,\beta_3 x_2^2 e^{-\beta_2 x_2} - \beta_3^2 x_2^2 e^{-(\beta_1 x_1+\beta_2 x_2)} - 2\beta_3^2 x_2^2 e^{-2\beta_2 x_2})$$

$$\frac{\partial h_2}{\partial \beta_3} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,x_2 e^{-\beta_2 x_2} - 2\beta_3 x_2 e^{-(\beta_1 x_1+\beta_2 x_2)} - 2\beta_3 x_2 e^{-2\beta_2 x_2})$$

$$\frac{\partial h_2}{\partial \sigma^2} = \frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i\,\beta_3 x_2 e^{-\beta_2 x_2} - \beta_3^2 x_2 e^{-(\beta_1 x_1+\beta_2 x_2)} - \beta_3^2 x_2 e^{-2\beta_2 x_2})$$

$$\frac{\partial h_3}{\partial B_1} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,x_1 e^{-\beta_1 x_1} - 2\beta_3 x_1 e^{-2\beta_1 x_1} - 2\beta_3 x_1 e^{-(\beta_1 x_1+\beta_2 x_2)})$$

$$\frac{\partial h_3}{\partial \beta_2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i\,x_2 e^{-\beta_2 x_2} - 2\beta_3 x_2 e^{-(\beta_1 x_1+\beta_2 x_2)} - 2\beta_3 x_2 e^{-2\beta_2 x_2})$$

$$\frac{\partial h_3}{\partial \beta_3} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(e^{-2\beta_1 x_1} - 2e^{-(\beta_1 x_1+\beta_2 x_2)} - e^{-2\beta_2 x_2})$$

$$\frac{\partial h_3}{\partial \sigma^2} = \frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i\,(e^{-\beta_1 x_1} + e^{-\beta_2 x_2}) - \beta_3(e^{-2\beta_1 x_1} + e^{-2\beta_2 x_2}) - 2\beta_3 e^{-(\beta_1 x_1+\beta_2 x_2)}$$

$$\frac{\partial h_4}{\partial \beta_1} = \frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i\,\beta_3 x_1 e^{-\beta_1 x_1} - \beta_3^2 x_1 e^{-2\beta_1 x_1} - \beta_3^2 x_1 e^{-(\beta_1 x_1+\beta_2 x_2)})$$
$$\frac{\partial h_4}{\partial \beta_2} = \frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i\,\beta_3 x_2 e^{-\beta_2 x_2} - \beta_3^2 x_2 e^{-(\beta_1 x_1+\beta_2 x_2)} - \beta_3^2 x_2 e^{-2\beta_2 x_2)})$$
$$\frac{\partial h_4}{\partial \beta_3} = \frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i\,e^{-\beta_1 x_1} + y_i e^{-\beta_2 x_2} - \beta_3 e^{-2\beta_2 x_2} - 2\beta_3 e^{-(\beta_1 x_1+\beta_2 x_2)} - \beta_3 e^{-2\beta_2 x_2})$$
$$\frac{\partial h_4}{\partial \sigma^2} = \frac{n}{2\sigma^4} - \sum_{i=1}^{n}\frac{1}{\sigma^6}(y_i - \beta_3 e^{-\beta_1 x_1} - \beta_3 e^{-\beta_2 x_2})^2$$

3. **Least SquareNonlinear Regression Models.**

To estimate the parameters of MGH09and Meyer 4 model, Least square method**(LS)** is used for these two models of Nonlinear regression in this section as follows:

3.1. Least Square Method to solveMGH09model.

The MGH09model is $f(x;\beta_p) = \frac{\beta_1(x^2+x\beta_2)}{x^2+x\beta_3+\beta_4}$

The formula for LS is for MGH09 Model;

$$Q = \sum_{i=1}^{n}\left[y_i - (f(x,\beta_p))\right]^2$$
$$Q = \sum_{i=1}^{n}\left[y_i - \frac{\beta_1(x^2+x\beta_2)}{x^2+x\beta_3+\beta_4}\right]^2$$

numerical procedures as Newton-Raphson was used to estimate the parameters ,since the equations are complicated to be solved. Therefore, the equation for this method for the first model is as follows

$$\begin{pmatrix}\hat\beta_1\\ \hat\beta_l\\ \hat\beta_3\\ \hat\beta_4\end{pmatrix} = \begin{pmatrix}\beta_{10}\\ \beta_{20}\\ \beta_{30}\\ \beta_{40}\end{pmatrix} - \begin{pmatrix}\frac{\partial h_1}{\partial\beta_1} & \frac{\partial h_1}{\partial\beta_2} & \frac{\partial h_1}{\partial\beta_3} & \frac{\partial h_1}{\partial\beta_4}\\ \frac{\partial h_2}{\partial\beta_1} & \frac{\partial h_2}{\partial\beta_2} & \frac{\partial h_2}{\partial\beta_3} & \frac{\partial h_2}{\partial\beta_4}\\ \frac{\partial h_3}{\partial\beta_1} & \frac{\partial h_3}{\partial\beta_2} & \frac{\partial h_3}{\partial\beta_3} & \frac{\partial h_3}{\partial\beta_4}\\ \frac{\partial h_4}{\partial\beta_1} & \frac{\partial h_4}{\partial\beta_2} & \frac{\partial h_4}{\partial\beta_3} & \frac{\partial h_4}{\partial\beta_4}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial Q}{\partial\beta_1}\\ \frac{\partial Q}{\partial\beta_2}\\ \frac{\partial Q}{\partial\beta_3}\\ \frac{\partial Q}{\partial\beta_4}\end{pmatrix}$$

where $\begin{pmatrix}\beta_{10}\\ \beta_{20}\\ \beta_{30}\\ \beta_{40}\end{pmatrix}$represents the vector of the initial parameters.

$$k_1 = \frac{\partial Q}{\partial \beta_1} = -2\sum_{i=1}^{n}\left(\frac{y_i(x^2+x\beta_2)}{x^2+x\beta_3+\beta_4} - \frac{\beta_1(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^2}\right)$$
$$k_2 = \frac{\partial Q}{\partial \beta_2} = -2\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x}{x^2+x\beta_3+\beta_4} - \frac{\beta_1^2(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}\right)$$
$$k_3 = \frac{\partial Q}{\partial \beta_3} = 2\sum_{i=1}^{n}\left(\frac{y_i\beta_1 x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2} - \frac{\beta_1^2 x(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$k_4=\frac{\partial Q}{\partial \beta_4}=2\sum_{i=1}^{n}\left(\frac{y_i\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{\beta_1^2(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial k_1}{\partial \beta_1}=2\sum_{i=1}^{n}\left(\frac{(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$\frac{\partial k_1}{\partial \beta_2}=-2\sum_{i=1}^{n}\left(\frac{y_ix}{x^2+x\beta_3+\beta_4}-\frac{2\beta_1x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$\frac{\partial k_1}{\partial \beta_3}=2\sum_{i=1}^{n}\left(\frac{y_ix(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1x(x^2+x\beta_2)^2(x^2+x\beta_3+\beta_4)}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial k_1}{\partial \beta_4}=2\sum_{i=1}^{n}\left(\frac{y_i(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1(x^2+x\beta_2)^2(x^2+x\beta_3+\beta_4)}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial k_2}{\partial \beta_1}=-2\sum_{i=1}^{n}\left(\frac{y_ix}{x^2+x\beta_3+\beta_4}-\frac{2\beta_1x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$\frac{\partial k_2}{\partial \beta_2}=2\sum_{i=1}^{n}\left(\frac{\beta_1^2x^2}{(x^2+x\beta_3+\beta_4)^2}\right)$$

$$\frac{\partial k_2}{\partial \beta_3}=2\sum_{i=1}^{n}\left(\frac{y_i\beta_1x^2}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1^2x^2(x^2+x\beta_2)^2(x^2+x\beta_3+\beta_4)}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial k_2}{\partial \beta_4}=2\sum_{i=1}^{n}\left(\frac{y_i\beta_1x}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1^2x(x^2+x\beta_2)^2(x^2+x\beta_3+\beta_4)}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial k_3}{\partial \beta_1}=2\sum_{i=1}^{n}\left(\frac{y_ix(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial k_3}{\partial \beta_2}=2\sum_{i=1}^{n}\left(\frac{y_i\beta_1x^2}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1^2x^2(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial k_3}{\partial \beta_3}=-2\sum_{i=1}^{n}\left(\frac{2y_i\beta_1x^2(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}-\frac{3\beta_1^2x^2(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial k_3}{\partial \beta_4}=-2\sum_{i=1}^{n}\left(\frac{2y_i\beta_1x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}-\frac{3\beta_1^2x(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial k_4}{\partial \beta_1}=2\sum_{i=1}^{n}\left(\frac{y_i(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial k_4}{\partial \beta_2}=2\sum_{i=1}^{n}\left(\frac{y_i\beta_1x}{(x^2+x\beta_3+\beta_4)^2}-\frac{2\beta_1^2x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}\right)$$

$$\frac{\partial k_4}{\partial \beta_3}=-2\sum_{i=1}^{n}\left(\frac{2y_i\beta_1x(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}-\frac{3\beta_1^2x(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^4}\right)$$

$$\frac{\partial k_4}{\partial \beta_4}=-2\sum_{i=1}^{n}\left(\frac{2y_i\beta_1(x^2+x\beta_2)}{(x^2+x\beta_3+\beta_4)^3}-\frac{3\beta_1^2(x^2+x\beta_2)^2}{(x^2+x\beta_3+\beta_4)^4}\right)$$

3.2. Least Square Method to solve Meyer 4 model.

The formula for LS is for Meyer 4model;

$$Q = \sum_{i=1}^{n} \left[y_i - \beta_3 (e^{-\beta_1 x_1} + e^{-\beta_2 x_2}) \right]^2$$

Thus, the following equation matrixes are applied to estimate the parameters for non-linear regression model by using Newton-Raphson method for the second model.

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{pmatrix} = \begin{pmatrix} \beta_{10} \\ \beta_{20} \\ \beta_{30} \end{pmatrix} - \begin{pmatrix} \frac{\partial k_1}{\partial \beta_1} & \frac{\partial k_1}{\partial \beta_2} & \frac{\partial k_1}{\partial \sigma^2} \\ \frac{\partial k_2}{\partial \beta_1} & \frac{\partial k_2}{\partial \beta_2} & \frac{\partial k_2}{\partial \sigma^2} \\ \frac{\partial k_3}{\partial \beta_1} & \frac{\partial k_3}{\partial \beta_2} & \frac{\partial k_3}{\partial \beta_3} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial Q}{\partial \beta_1} \\ \frac{\partial Q}{\partial \beta_2} \\ \frac{\partial Q}{\partial \beta_3} \end{pmatrix}$$

where $\begin{pmatrix} \beta_{10} \\ \beta_{20} \\ \beta_{30} \end{pmatrix}$ represents the vector of the initial parameters.

$$k_1 = \frac{\partial Q}{\partial \beta_1} = 2 \sum_{i=1}^{n} \left[y_i \beta_3 x_1 e^{-\beta_1 x_1} - \beta_3^2 x_1 e^{-2\beta_1 x_1} - \beta_3^2 x_1 e^{-(\beta_1 x_1 + \beta_2 x_2)} \right]$$

$$k_2 = \frac{\partial Q}{\partial \beta_2} = 2 \sum_{i=1}^{n} \left[y_i \beta_3 x_2 e^{-\beta_2 x_2} - \beta_3^2 x_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} - \beta_3^2 x_2 e^{-2\beta_2 x_2} \right]$$

$$k_3 = \frac{\partial Q}{\partial \beta_3} = -2 \sum_{i=1}^{n} \left[y_i (e^{-\beta_1 x_1} + e^{-\beta_2 x_2}) - \beta_3 (e^{-2\beta_1 x_1} + e^{-2\beta_2 x_2}) - 2\beta_3 e^{-(\beta_1 x_1 + \beta_2 x_2)} \right]$$

$$\frac{\partial k_1}{\partial \beta_1} = -2 \sum_{i=1}^{n} \left[y_i \beta_3 x_1^2 e^{-\beta_1 x_1} - 2\beta_3^2 x_1^2 e^{-2\beta_1 x_1} - \beta_3^2 x_1^2 e^{-(\beta_1 x_1 + \beta_2 x_2)} \right]$$

$$\frac{\partial k_1}{\partial \beta_2} = 2 \sum_{i=1}^{n} \left[\beta_3^2 x_1 x_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} \right]$$

$$\frac{\partial k_1}{\partial \beta_3} = 2 \sum_{i=1}^{n} \left[y_i x_1 e^{-\beta_1 x_1} - 2\beta_3 x_1 e^{-2\beta_1 x_1} - 2\beta_3 x_1 e^{-(\beta_1 x_1 + \beta_2 x_2)} \right]$$

$$\frac{\partial k_2}{\partial \beta_1} = 2 \sum_{i=1}^{n} \left[\beta_3^2 x_1 x_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} \right]$$

$$\frac{\partial k_2}{\partial \beta_2} = -2 \sum_{i=1}^{n} \left[y_i \beta_3 x_2^2 e^{-\beta_2 x_2} - \beta_3^2 x_2^2 e^{-(\beta_1 x_1 + \beta_2 x_2)} - 2\beta_3^2 x_2^2 e^{-2\beta_2 x_2} \right]$$

$$\frac{\partial k_2}{\partial \beta_3} = 2 \sum_{i=1}^{n} \left[y_i x_2 e^{-\beta_2 x_2} - 2\beta_3 x_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} - 2\beta_3 x_2 e^{-2\beta_2 x_2} \right]$$

$$\frac{\partial k_3}{\partial B_1} = 2 \sum_{i=1}^{n} \left[y_i x_1 e^{-\beta_1 x_1} - 2\beta_3 x_1 e^{-2\beta_1 x_1} - 2\beta_3 x_1 e^{-(\beta_1 x_1 + \beta_2 x_2)} \right]$$

$$\frac{\partial k_3}{\partial \beta_2} = 2 \sum_{i=1}^{n} \left[y_i x_2 e^{-\beta_2 x_2} - 2\beta_3 x_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} - 2\beta_3 x_2 e^{-2\beta_2 x_2} \right]$$

$$\frac{\partial k_3}{\partial \beta_3} = 2\sum_{i=1}^{n}\left[e^{-2\beta_1 x_1} - 2e^{-(\beta_1 x_1 + \beta_2 x_2)} - e^{-2\beta_2 x_2}\right]$$

4.BATAlgorithm

BAT algorithm is a nature-inspired algorithm that belongs to the SI family. The standard BAT Algorithm was created by **[17]**. BAT Algorithm works on the echolocation of microbats and it uses echo of bats for seeking food. They discover their way in the night by radiating the sound signal called sonar/echolocation which used that signal to detect the object or obstacles surrounding them. Yang focused on three rules for the implementation of the BAT: Firstly, bats fly randomly with fixed frequency towards the specified location with specific velocity, however, the loudness and wavelength can vary. Thus, bats automatically adjust their wavelengths according to their target. Secondly, to measure the distance to the specific point, all bats use echolocation. Thirdly, the author considered that loudness is varied from maximum to minimum rather than any other way. BATAlgorithm uses automatic zooming to try to balance exploration and exploitation during the search process by mimicking the variation of pulse emission rates and loudness of bats when searching for prey.The steps of BAT Algorithm are introduced as follows:

Step1. At the first, $f(x,\beta)$ is used as a fitness function of the Bat algorithm, initialize population of Bat xi, the objective function ,velocity v_i.Determine pulse frequency f_i at x_i. Loudness A and pulse rate r_i are initialized.

Step2. By adjusting the frequency, new solutions are generated and updating velocities and positions/solutions.

Step3. If (random > r)

From the best solutions, select the solution and around the selected best solution generate a neighborhood solution.

Step4. Else Fly random to create a new solution.

Step5. If (random <A &&(x_i)<f(x_0)) , whereas $f(\cdot)$= objective function. Acknowledge the new solution increases and diminishes A.

Step6. Find the current best (x_0) by ranking the bats.

Step7. While (iteration< maximum number of emphases)

Post procedure outcomes and representation. The algorithm terminates with the best aggregate solution.

5.Simulation Study

In order to verify the performance of the BAT algorithm, Maximum Likelihood estimation, and Least Square method, simulation study is used based on Mean Squares Error (MSE) to estimate the parameters of two nonlinear regression models (Meyer4 and MGH09).

The proposed estimation methods in nonlinear regression models have been implemented using a variety of samples (20, 40,80,160, and 200). To obtain the numerical results,Matlab version 2015 will be used.The following steps of the Monte Carlo simulation explanation the statistical outcomes for each model based on Mean Squared Errors criteria parameters estimated.

5.1Simulation of MGH09 model

1- Initialize all the parameters of BES, BAT, and GSA algorithms.

2- Used $\frac{\sum_{i=1}^{n}\left[y_i - \left(\frac{ß_1(x^2 + xß_2)}{x^2 + xß_3 + ß_4}\right)\right]^2}{n}$ as fitness function for each algorithm.

3- Utilize different set parameters for MGH09model as: ($\beta_1, \beta_2, \beta_3, \beta_4$) = (0.15, 0.15, 0.8, 0.9) and (0.2, 0.3, 0.8, 0.2).
4-Calculate the values of response variable y_i depend on x_i which was generated according to the exponential distribution($exp(2)$), while the random variable e_i is generated according to $N(0, \sigma^2)$ for all methods (BAT, LS, MLE) .
5- Calculate the $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4 and$ MSE based on L=1000 replicate and run 10.

5.2 Simulation of Meyer4 model

1- Initialize all the parameters of BES, BAT, and GSA algorithms.
2- Used $\frac{\sum_{i=1}^{n}[y_i-(\beta_1 e^{\beta_3 x_1}+\beta_2 e^{\beta_2 x_2})]^2}{n}$ as fitness function for each algorithm.
3- Utilize different set parameters for Meyer 4 model as: ($\beta_1, \beta_2, \beta_3,$) =(10, 50, 50), ,and (1, 50, 50).
4-Calculate the values of response variable y_i depend on x_i which was generated according to the exponential distribution($exp(2)$), while the random variable e_i is generated according to $N(0, \sigma^2)$ for all methods (BAT, LS, MLE) .
5- Calculate the $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ and MSE based on L=1000 replicate and run 10.

6. The Results of Simulation Study

Tables 1and 2 illustrate the results of the estimated parameters and MSE for each parameter of MGH09 model of nonlinear regression. While tables 3 and 4 show the results of the second model (Meyer 4) that has three parameters. For more illustration, Figures 1and 20 provide comparative analyses of all the various performance parameter values for each sample of BAT, MLE, and LS methods.According to all tables and figures, it can be seen that the BAT algorithm provide less MSE and the estimations are closer to the real parameter values than the other methods (MLE and LS). Therefore, the BAT method may be considered as an effective parameter estimation method for nonlinear regression models.

Table 1-Comparative results of BAT, MLE , and LS for the MGH09 model when $\beta_1 = 0.15,\ \beta_2 = 0.15, \beta_3 = 0.8\ , and\ \beta_4 = 0.9$

n	$Methods$	$stistics$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	MSE
20	LS	$Estimated$	0.151715159	0.070508541	0.789650025	0.739838828	$3.58E-07$
		MSE	$5.23E-06$	0.007820011	0.004898751	0.027919114	
	MLE	$Estimated$	0.503455552	0.514273951	0.516265037	0.503927242	0.097771942
		MSE	0.124931094	0.13270331	0.080513207	0.156873637	
	Bat	$Estimated$	0.148549094	0.213336906	0.786232831	1.043366965	$7.79E-08$
		MSE	$3.27E-06$	0.007148174	0.000223314	0.035744628	
40	LS	$Estimated$	0.149919176	0.211676319	0.836633217	0.971752361	$1.81E-06$
		MSE	$8.92E-07$	0.004042677	0.003055091	0.00594633	
	MLE	$Estimated$	0.502000523	0.507939148	0.508959763	0.501989638	0.092438427
		MSE	0.12390437	0.128120442	0.084704498	0.158412249	
	Bat	$Estimated$	0.150672764	0.128824239	0.809409688	0.849316892	$5.49E-09$
		MSE	$2.09E-06$	0.002795185	0.000250631	0.014792295	
80	LS	$Estimated$	0.151995087	0.062919763	0.806959866	0.695075355	$1.66E-06$
		MSE	$3.06E-05$	0.011541864	0.007668218	0.049959188	
	MLE	$Estimated$	0.501050922	0.504288928	0.504898254	0.501074494	0.09686519
		MSE	0.12323675	0.125520673	0.087085095	0.159141562	
	Bat	$Estimated$	0.15191094	0.075263906	0.82012359	0.723705484	$5.92E-08$
		MSE	$3.66E-06$	0.005624217	0.000406272	0.031242596	
160	LS	$Estimated$	0.156705049	0.10631912	0.942651351	0.788487643	$1.20E-06$
		MSE	$4.50E-05$	0.013211774	0.023638276	0.057172483	
	MLE	$Estimated$	0.500466861	0.50171244	0.501903667	0.500497614	0.085709727
		MSE	0.122827021	0.123701646	0.088861429	0.159602157	

200	Bat	Estimated	0.150039697	0.147183153	0.799819444	0.89401494	$5.20E-10$
		MSE	$6.28E-08$	0.000101087	$5.04E-06$	0.000489982	
	LS	Estimated	0.151611996	0.119253134	0.824012703	0.834385866	$1.80E-07$
		MSE	$5.51E-06$	0.001966269	0.0013042	0.009205831	
	MLE	Estimated	0.500397391	0.501546604	0.501723086	0.500433399	0.094120613
		MSE	0.122778332	0.123585016	0.088969118	0.159653469	
	Bat	Estimated	0.149847968	0.156540931	0.798712836	0.914240911	$7.07E-09$
		MSE	$5.61E-07$	0.001058505	$3.58E-05$	0.00554906	

Table 2-Comparative results of BAT, MLE , and LS for the MGH09 model when $\beta_1 = 0.2, \ \beta_2 = 0.3, \beta_3 = 0.8, and \ \beta_4 = 0.2$

n	Methods	stistics	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	MSE
20	LS	Estimated	0.198563461	0.468772465	0.905673197	0.328778925	$2.03E-06$
		MSE	$2.44E-05$	0.028514556	0.017903652	0.016745801	
	MLE	Estimated	0.50186797	0.506898013	0.50680456	0.502575481	0.062084243
		MSE	0.091124293	0.042807823	0.085964722	0.091551958	
	Bat	Estimated	0.199519526	0.413233696	0.899975185	0.269016981	$7.84E-09$
		MSE	$2.32E-07$	0.012984713	0.010151105	0.0048238	
40	LS	Estimated	0.190402415	0.575780451	0.837990239	0.419488378	$1.00E-05$
		MSE	$9.42E-05$	0.105566648	0.027690421	0.048837246	
	MLE	Estimated	0.500892564	0.503649422	0.503529505	0.50142033	0.047927947
		MSE	0.090536349	0.041473533	0.087895262	0.090854218	
	Bat	Estimated	0.188543252	4.781834675	4.384471386	2.700098913	$2.64E-08$
		MSE	0.000240548	37.60341081	23.87559879	11.63024106	
80	LS	Estimated	0.192445928	0.417313839	0.74364182	0.319119165	$2.67E-07$
		MSE	$9.11E-05$	0.03647124	0.003287909	0.030839115	
	MLE	Estimated	0.500500862	0.50214085	0.502167598	0.500763698	0.058516506
		MSE	0.090300768	0.040860944	0.08870416	0.090458803	
	Bat	Estimated	0.200221975	0.263182704	0.768716141	0.176780915	$2.64E-08$
		MSE	$3.15E-07$	0.014812522	0.011340491	0.0056579	
160	LS	Estimated	0.194048612	0.39539069	0.750958061	0.308020023	$9.70E-06$
		MSE	$9.89E-05$	0.057665018	0.003529179	0.048873838	
	MLE	Estimated	0.500257908	0.501018979	0.501014312	0.500370736	0.064787254
		MSE	0.090154811	0.040408631	0.089392442	0.090222579	
	Bat	Estimated	0.199308653	0.559823904	1.036200604	0.356046495	$6.62E-09$
		MSE	$5.11E-07$	0.077826224	0.064579881	0.027993725	
200	LS	Estimated	0.194372218	0.595399609	0.921271119	0.44205817	$8.88E-06$
		MSE	$4.16E-05$	0.088207748	0.020904921	0.061813882	
	MLE	Estimated	0.500206645	0.500846374	0.500872262	0.500277783	0.062613775
		MSE	0.09012403	0.040339266	0.089477404	0.090166747	
	Bat	Estimated	0.200832814	0.104011897	0.62775209	0.0785641	$6.42E-08$
		MSE	$6.94E-07$	0.038679778	0.029902624	0.014843757	

Table 3-Comparative results of BAT, MLE , and LS for the Meyer4 model when $\beta_1 = 10 \ , \beta_2 = 50, and \ \beta_3 = 50$

n	Methods	stistics	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	MSE
20	LS	Estimated	9.601801235	61.69430033	47.42875965	0.084151766
		MSE	0.300712974	281.7793273	11.91632063	

		Estimated	644.9123274	53.2057941	91.16952661	32.96049751
	MLE	MSE	757904.2858	20.54555895	3389.87724	
		Estimated	40.77863295	30.51205804	37.65358385	0.005841764
	Bat	MSE	1896.364672	968.9647544	301.8748017	
		Estimated	17.20641518	835.1202472	112.2786867	0.393892627
	LS	MSE	65.33656475	1202206.495	6336.183068	
40		Estimated	15.1373951	29.07870338	24.77681078	22.13971932
	MLE	MSE	255.5305314	1283.265827	1250.094672	
		Estimated	14.46648289	62.01490792	66.9357689	0.280061576
	Bat	MSE	45.78501846	163.667919	702.5526821	
		Estimated	10.36987182	70.32012848	51.96116488	1.500427752
	LS	MSE	0.136848401	697.192864	4.549972719	
80		Estimated	2331.548161	48.80022162	48.81901695	12.6768616
	MLE	MSE	10626904.82	2.228792745	2.28095017	
		Estimated	10.40732954	52.81723565	51.19028625	$1.15E-10$
	Bat	MSE	0.331817044	15.87327676	2.833374472	
		Estimated	10.59445534	70.13748203	53.0878844	1.853603832
	LS	MSE	0.706759437	811.0397154	19.07054029	
160		Estimated	28.39299492	46.84835493	43.55531141	18.3707913
	MLE	MSE	381.0647442	11.78593885	41.69177217	
		Estimated	10.04168984	53.23687808	50.03402584	0.129632506
	Bat	MSE	0.318023517	95.2415262	7.848983234	
		Estimated	23612.54439	520.3206061	441.1436789	53.70572547
	LS	MSE	915364914.2	442970.5675	306550.4344	
200		Estimated	37.65425215	47.70607539	46.45960516	25.34661378
	MLE	MSE	774.2029511	6.517380482	13.74240482	
		Estimated	10.00001372	50.00004079	50.00002088	$9.80E-12$
	Bat	MSE	$6.23E-10$	$4.49E-09$	$2.17E-09$	

Table 4-Comparative results of BAT, MLE , and LS for the Meyer4 model when $\beta_1 = 1\ ,\ \ \beta_2 = 50, and\ \ \beta_3 = 50$

n	$Methods$	$stistics$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	MSE
20	LS	Estimated	2.640488273	64.86573739	77.93616157	98.20988846
		MSE	3.132766393	321.9869279	812.9705708	
	MLE	Estimated	29118.54404	1104.31883	1626.64333	403.555018
		MSE	1612145864	1156921.388	4958970.048	
	Bat	Estimated	1.000144036	48.8111722	49.99394881	$9.05E-06$
		MSE	$6.88E-08$	4.926722114	$6.23E-05$	
40	LS	Estimated	1.565023002	84.0517136	67.46828932	70.36958123
		MSE	0.416929665	1221.869978	357.1447187	
	MLE	Estimated	2412.907265	5173.794398	86.57625167	773.5051462
		MSE	6860994.85	32348231.51	1832.142222	
	Bat	Estimated	1.080969708	43.01511685	52.76995633	0.012260554
		MSE	0.013221261	117.874176	15.05022078	

80	*LS*	*Estimated*	1.292583284	53.61577582	56.75021682	58.3270173
		MSE	0.38362602	843.5831244	325.6323612	
	MLE	*Estimated*	17700.76907	1159.045942	413.7649029	637.9854332
		MSE	377497635.9	1403907.164	174581.5355	
	Bat	*Estimated*	1.045633113	46.39675834	51.38826128	0.919571062
		MSE	0.002568365	272.2285177	3.940678818	
160	*LS*	*Estimated*	1.635436548	74.68754607	67.28221937	102.3223959
		MSE	0.828082639	609.9223894	597.6783009	
	MLE	*Estimated*	14766.11883	8805.778536	388.6323896	609.1474993
		MSE	342054705.7	77015642.63	195727.3446	
	Bat	*Estimated*	1.588641927	72.81417328	71.80293305	0.717093788
		MSE	0.694574769	552.8664555	934.9449684	
200	*LS*	*Estimated*	1.854464171	50.29221977	76.24920478	85.7141631
		MSE	0.744818707	273.2774207	701.2680198	
	MLE	*Estimated*	0.0001	0.0001	0.0001	573.7242251
		MSE	0.99980001	2499.99	2499.99	
	Bat	*Estimated*	1.319945551	16.42867519	57.89187143	68.82122505
		MSE	0.28576261	1127.350388	255.8722624	

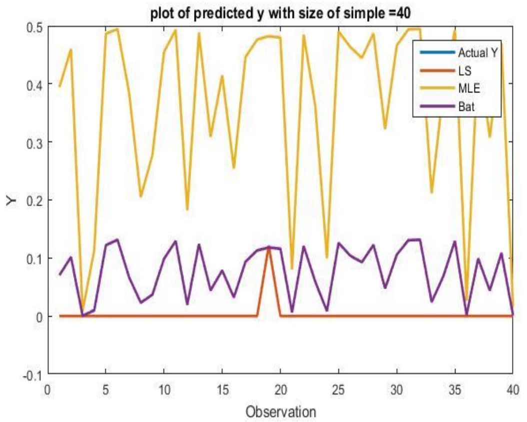

Figure 1-Comparative BAT, MLE and LS methods for MGH09 model when of sample size=20

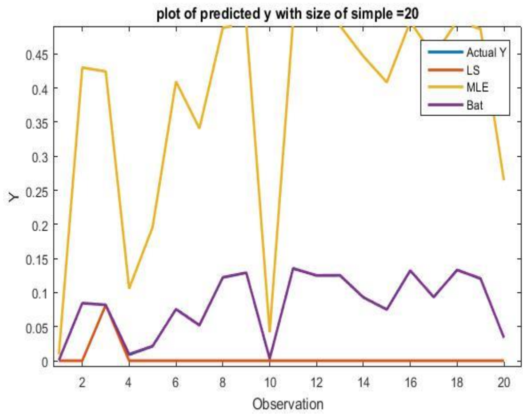

Figure 2- Comparative BAT, MLE and LS methods for MGH09 model when of sample size=40

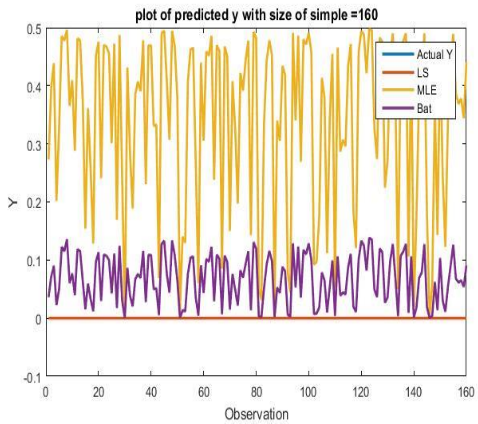

Figure 3- Comparative BAT, MLE and LS methods for MGH09model when of sample size=80

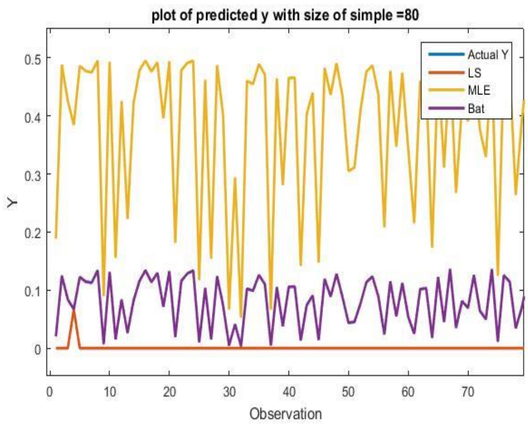

Figure 4- Comparative BAT, MLE and LS methods for MGH09 model when of sample size=160

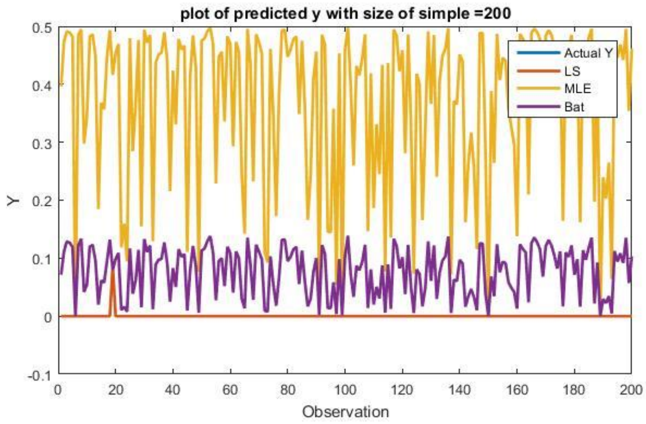

Figure 5- Comparative BAT, MLE and LS methods for MGH09 model when of sample size=200

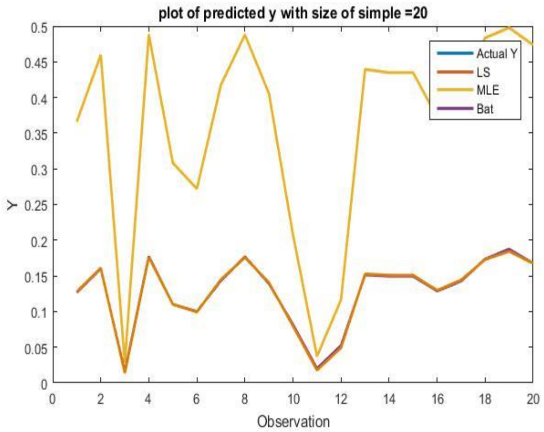

Figure 6- Comparative BAT, MLE and LS methods for MGH09model when of sample size=20

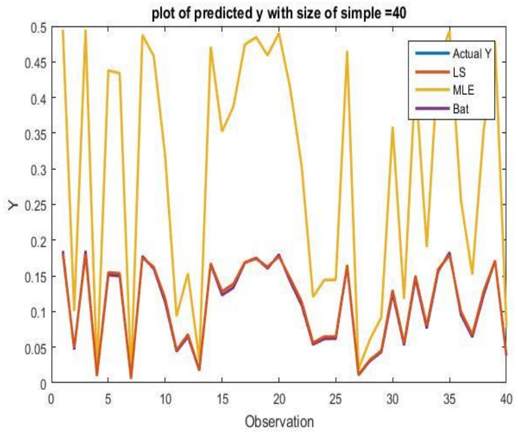

Figure 7-Comparative BAT, MLE and LS methods methods for MGH09 model when of sample size=40

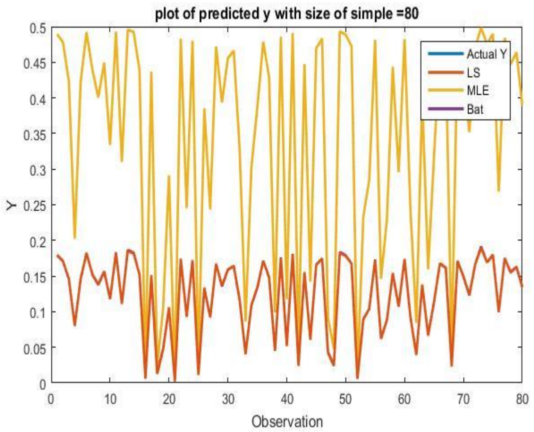

Figure 8-Comparative BAT, MLE and LS methods for MGH09model when of sample

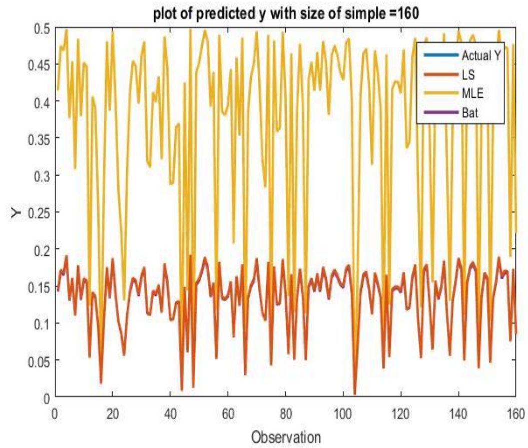

Figure 9- Comparative BAT, MLE and LS methods size=80for MGH09 model when of sample size=160

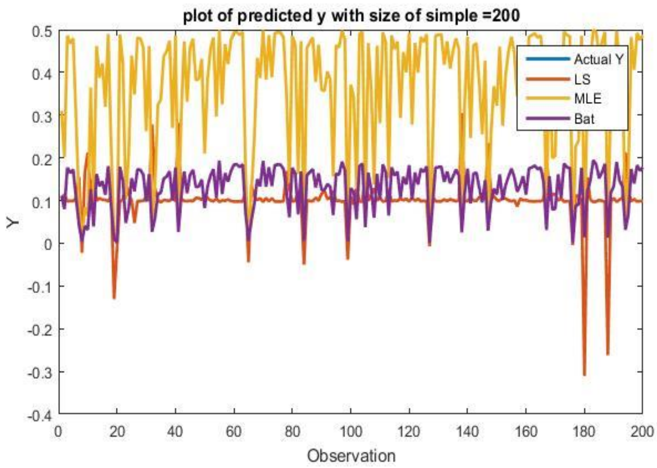

Figure 10- Comparative BAT, MLE and LS methods for MGH09 model when of sample size=200

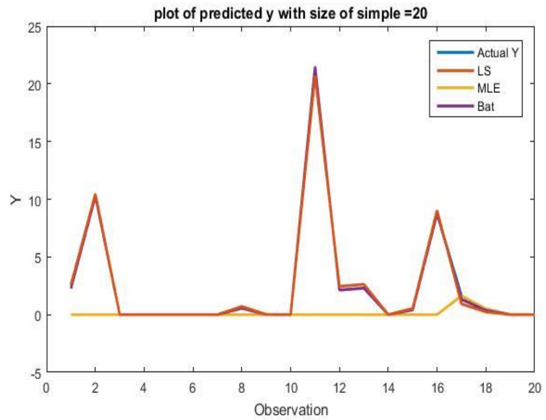

Figure 11- Comparative BAT, MLE and LS methods for Meyer4 model when of sample size=20

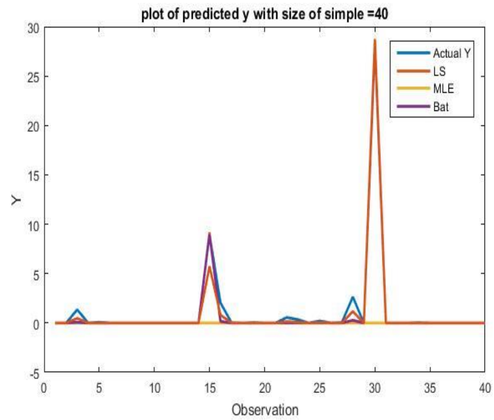

Figure 12-Comparative BAT, MLE and LS methods for Meyer4 model when of sample size=40

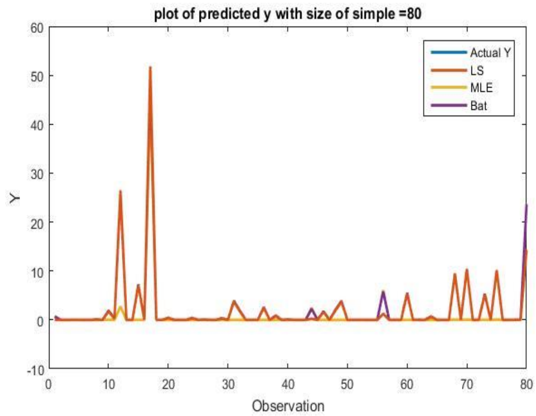

Figure 13- Comparative BAT, MLE and LS methods for Meyer4 model when of sample size=80

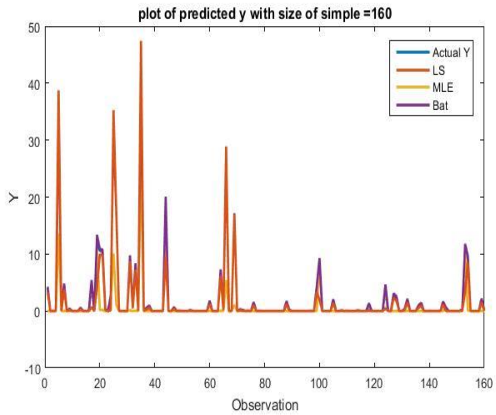

Figure 14-Comparative BAT, MLE and LS methods for Meyer4 model when of sample size=160

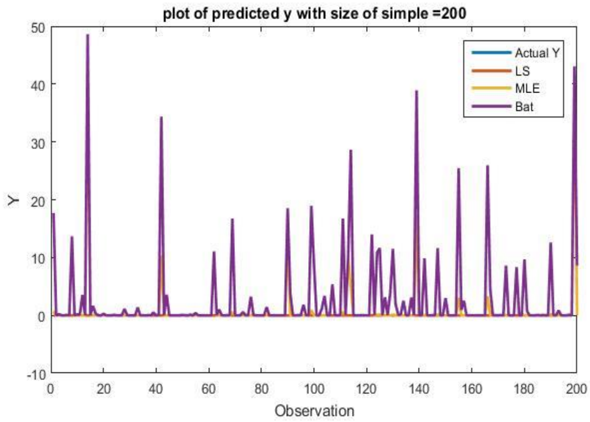

Figure 15- Comparative BAT, MLE and LS methodsfor Meyer4 model when of sample size=200

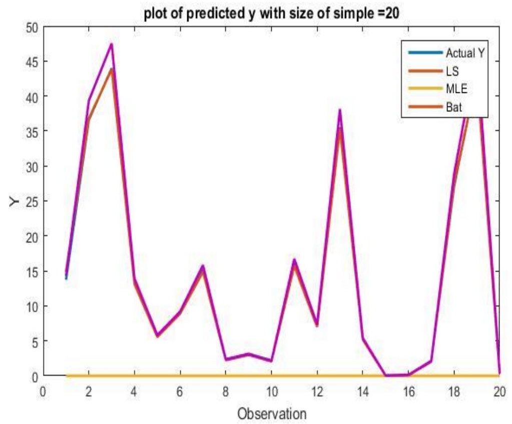

Figure 16- Comparative BAT, MLE and LS methods for Meyer4 model when of sample size=20

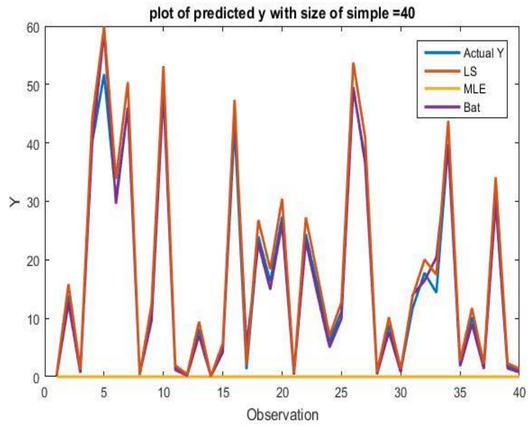

Figure 17- Comparative BAT, MLE and LS methods for Meyer4 model when of sample size=40

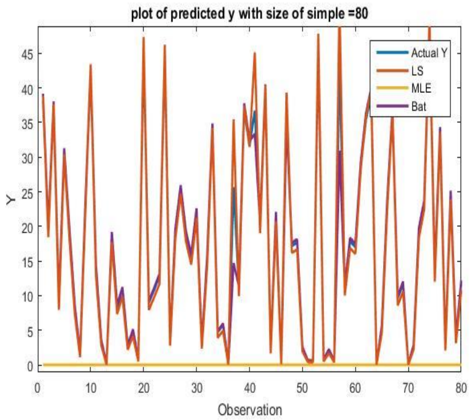

Figure 18-Comparative BAT, MLE and LS methods for Meyer4 model when of samplesize=80

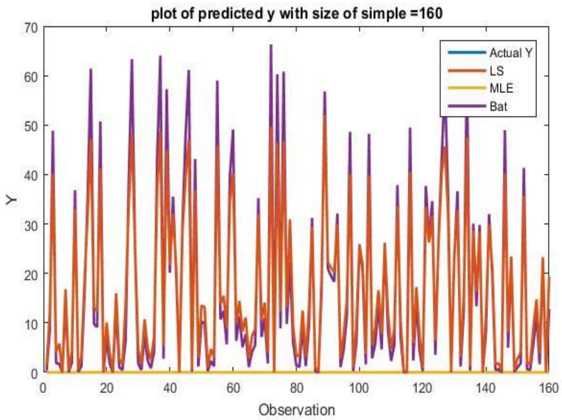

Figure 19- Comparative BAT, MLE and LS for Meyer4 model when of sample size=160

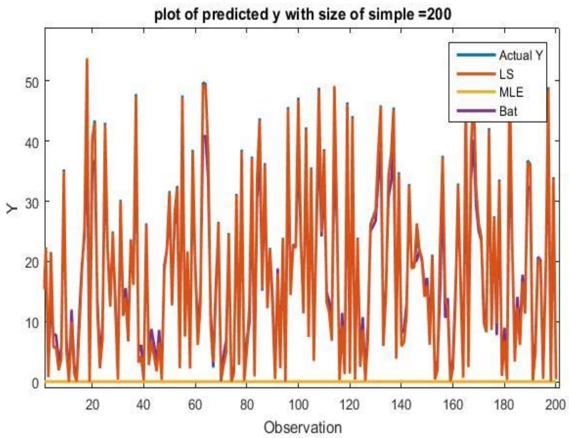

Figure 20- Comparative BAT, MLE and LS methods for Meyer4 model when of sample size=200

7.CONCLUSIONS

In this paper, BAT Algorithm has been utilized as an alternative approach for estimating the parameters of nonlinear regression models. Two types of nonlinear regression models Meyer4, and MGH09 are used, which have a different number of parameters . A simulation analysis is employed to investigate and compare the performance of the proposed methods. The results have been shown that the BAT Algorithm delivers good results compared to the classical estimator of LS and MLE methods. Furthermore other studies may be used other algorithms for comparison with the algorithms that used in the research, such as the Artificial Bee or ant Colony algorithms.

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