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Large-Coessential and Large-Coclosed Submodules

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Abstract

The goal of this research is to introduce the concepts of Large-coessential submodule and Large-coclosed submodule, for which some properties are also considered. Let M be an R-module and K, N are submodules of M such that $K \leq N \leq M$, then K is said to be Large-coessential submodule, if $\frac{N}{K} \ll_L \frac{M}{K}$. A submodule N of M is called Large-coclosed submodule, if K is Large-coessential submodule of N in M, for some submodule K of N, implies that $K = N$.

Keywords: L-small submodule, L-coessential submodule, L-coclosed submodule

المقاسات الجزئية الضد الجوهرية الاساسية والمقاسات الجزئية الضد المغلقة الاساسية

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الخلاصة

الغرض من هذا البحث هو تقديم مفاهيم المقاسات الجزئية الضد الجوهرية الاساسية والمقاسات الجزئية الضد المغلقة الاساسية وسوف نقوم بأستعراض بعض الخواص لهذه المفاهيم. ليكن M مقاس من النمط- R و N , K مقاسات جزئية في M بحيث $K \leq N \leq M$ فأن K يدعى بأنه مقاس جزئي ضد الجوهري الاساسي , اذا كان $\frac{N}{K} \ll_L \frac{M}{K}$. المقاس الجزئي N للمقاس M يدعى بأنه مقاس جزئي ضد المغلق الاساسي اذا كان K مقاس جزئي ضد الجوهري الاساسي من N في المقاس M بحيث K هي مقاس جزئي من N, يؤدي الى انه $K = N$.

1. Introduction

Throughout this paper, R will be a commutative ring with identity. A proper submodule N of an R-module M is called small ($N \ll M$), if for any submodule K of M such that $N + K = M$, implies that $K = M$ [1]. A proper submodule N of an R-module M is called Large (essential) submodule in M, ($N \leq_e M$), if for every non zero submodule K of M, $N \cap K \neq 0$ [1]. A submodule N of M is called closed in M if it has no proper essential extension in M [2]. For $K \leq N \leq M$, K is called coessential submodule of N in M ($K \leq_{ce} N$) if $\frac{N}{K} \ll \frac{M}{K}$, and K is said to be coclosed in M denoted by($K \leq_{cc} M$), if K has no proper coessential submodule in M [2,3]. In an earlier study [4], the concept of Large-small submodule was introduced, such that a proper submodule N of M is called Large-small (L-small) submodule of M, denoted by ($N \ll_L M$). If $N + K = M$, where $K \leq M$, then K is an essential submodule

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of M ($K \leq_e M$). It is clear that every small submodule of M is L-small submodule of M, but the converse is not true. Many authors have been interested in studying different generalizations of coessential and coclosed submodules [5-8]. In this paper, we introduce the concept of Large-coessential submodule as a generalization of coessential submodule, such that a submodule K of an R-module M is said to be Large-coessential submodule, if $\frac{N}{K} \ll_L \frac{M}{K}$, where $K \leq N \leq M$. In section one, we give many properties of this kind of submodule. In section two, we introduce the concept of Large-coclosed submodule, as a generalization of coclosed submodule, such that a submodule N of an R-module M is called Large-coclosed submodule, if K is Large-coessential submodule of N in M, for some submodule K of N, implies that $K = N$. Also, we give some basic properties of this kind of submodules. We give, in Lemma(1.1), some properties of Large-small (L-small) submodule of M, that were introduced earlier [4] and are needed in this paper.

Lemma 1.1[4]: 1- Let $f: M \to M`$ be an epimorphism where $M$ and $M`$ is an R-modules, such that $N \ll_L M`$, then $f^{-1}(N) \ll_L M$.

2- Let M be an R-module and K , N are submodules of M where K is closed in M, such that $K \leq N \leq M$. If $N \ll_L M$, then $K \ll_L M$ and $\frac{N}{K} \ll_L \frac{M}{K}$.

3- Let M be an R-module and K_1, K_2 are submodules of M, then $K_1 \ll_L M$ and $K_2 \ll_L M$ if and only if $K_1 \oplus K_2 \ll_L M$.

4- Let M be an R-module and K, N, and U are submodules of M, such that $K \leq N \leq U \leq M$ and K , N are closed submodules in M. Then, $\frac{U}{K} \ll_L \frac{M}{K}$ if and only if $\frac{U}{N} \ll_L \frac{M}{N}$ and $\frac{N}{K} \ll_L \frac{M}{K}$.

Now, we prove the following Lemma that we used in this paper.

Lemma 1.2: Let M be an R-module and K, N are submodules of M such that $K \leq N \leq M$. If $\frac{N}{K} \ll_L \frac{M}{K}$, then $N \ll_L M$.

Proof: Let $\pi: M \to \frac{M}{K}$ be a natural epimorphism and since $\frac{N}{K} \ll_L \frac{M}{K}$, then by Lemma(1.1), we get $N = \pi^{-1}(\frac{N}{K}) \ll_L M$, hence $N \ll_L M$.

2. Large-Coessential submodule

In this section we introduce the concept of Large- coessential submodule and many of its properties.

Definition 2.1: Let M be an R-module and K , N are submodules of M such that $K \leq N \leq M$, then K is called Large-coessential (L-coessential) submodule of N in M ($K \leq_{L.ce} N$) if $\frac{N}{K} \ll_L \frac{M}{K}$.

Remarks and Examples 2.2

1- Every coessential submodule is L-coessential submodule.

Proof: Let K be a coessential submodule of M and $K \leq N$ such that $\frac{N}{K} \ll \frac{M}{K}$, then by [4], $\frac{N}{K} \ll_L \frac{M}{K}$ and hence K is L-coessential submodule.

2- The converse of (1) is not true, as in the following example: In Z as Z-module, $\{\bar{0}\}$ is L-coessential submodule of 2Z in Z, since $\frac{2Z}{\{\bar{0}\}} \backsimeq 2Z \ll_L \frac{Z}{\{\bar{0}\}} \backsimeq Z$. But 2Z is not small in Z by [4] , so $\{\bar{0}\}$ is not coessential submodule of 2Z .

3- In Z_4 as Z-module, $\{\bar{0}\}$ is L-coessential submodule of $\{\bar{0}, \bar{2}\}$ in Z_4, since $\frac{\{\bar{0}, \bar{2}\}}{\{\bar{0}\}} \backsimeq \{\bar{0}, \bar{2}\} \ll_L \frac{Z_4}{\{\bar{0}\}} \backsimeq Z_4$ and since $\{\bar{0}, \bar{2}\} + Z_4 = Z_4$ and Z_4 is essential in Z_4.

4- In Z as Z-module, 4Z is L-coessential submodule of 2Z in Z, since $\frac{2Z}{4Z} \backsimeq \{\bar{0}, \bar{2}\} \ll_L \frac{Z}{4Z} \backsimeq Z_4$ by (3).

5- In Z_6 as Z-module, $\{\bar{0}\}$ is not L-coessential submodule of $\{\bar{0},\bar{3}\}$ in Z_6 , since $\frac{\{\bar{0},\bar{3}\}}{\{\bar{0}\}} \simeq \{\bar{0},\bar{3}\}$ and $\frac{Z_6}{\{\bar{0}\}} \simeq Z_6$ and hence $\{\bar{0},\bar{3}\}$ is not L-small in Z_6 by [4].
6- In Z_8 as Z-module, $\{\bar{0},\bar{4}\}$ is L-coessential submodule of $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ in Z_8 , since $\frac{\{\bar{0},\bar{2},\bar{4},\bar{6}\}}{\{\bar{0},\bar{4}\}} \simeq \{\bar{0},\bar{4}\} \ll_L \frac{Z_8}{\{\bar{0},\bar{4}\}} \simeq \{\bar{0},\bar{2},\bar{4},\bar{6}\}$, since $\{\bar{0},\bar{4}\} + \{\bar{0},\bar{2},\bar{4},\bar{6}\} = \{\bar{0},\bar{2},\bar{4},\bar{6}\}$ and $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ is essential in $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$.
7- Let M be an R-module and K, N are submodules of M such that $K \leq N \leq M$. If $\frac{M}{K}$ is semisimple module, then K is coessential submodule of N in M if and only if K is L-coessential submodule of N in M .

Proposition 2.3: Let M be an R-module and N be a submodule of M, then $N \ll_L M$ if and only if $\{\bar{0}\} \leq_{L.ce} N$ in M .

Proof:(⟹) Suppose that $N \ll_L M$, hence by Lemma(1.1), we have $\frac{N}{\{\bar{0}\}} \ll_L \frac{M}{\{\bar{0}\}}$, so $\{\bar{0}\} \leq_{L.ce} N$ in M.
(⟸) Let $\{\bar{0}\} \leq_{L.ce} N$ in M and let $N + K = M$, where K is submodule of M , so $\frac{N+K}{\{\bar{0}\}} = \frac{M}{\{\bar{0}\}}$, hence $\frac{N}{\{\bar{0}\}} + \frac{K}{\{\bar{0}\}} = \frac{M}{\{\bar{0}\}}$. Also, since $\{\bar{0}\} \leq_{L.ce} N$ in M, then $\frac{N}{\{\bar{0}\}} \ll_L \frac{M}{\{\bar{0}\}}$, so $\frac{K}{\{\bar{0}\}} \leq_e \frac{M}{\{\bar{0}\}}$, hence $K \leq_e M$ and then $N \ll_L M$.

Theorem 2.4: Let M be an R-module and K , N , U are submodules of M such that $K \leq N \leq U \leq M$ and K is closed in M, then $K \leq_{L.ce} U$ in M if and only if $U + N = M$, implies that $N \leq_e M$.

Proof: (⟹) Let $K \leq_{L.ce} U$ in M and $\frac{U}{K} + \frac{N}{K} = \frac{M}{K}$. Since $\frac{U}{K} \ll_L \frac{M}{K}$, hence by Lemma(1.2), we have $U \ll_L M$ and $U + N = M$, so $N \leq_e M$.
(⟸) Let $U + N = M$, so $\frac{U}{K} + \frac{N}{K} = \frac{M}{K}$. Since $N \leq_e M$ and K is closed in M, then we have $\frac{N}{K} \leq_e \frac{M}{K}$ [2], hence $\frac{U}{K} \ll_L \frac{M}{K}$ and we get $K \leq_{L.ce} U$ in M.

Proposition 2.5: Let M be an R-module and K , N, and U are submodules of M such that $K \leq N \leq U \leq M$, then $N \leq_{L.ce} U$ in M if and only if $\frac{N}{K} \leq_{L.ce} \frac{U}{K}$ in $\frac{M}{K}$.

Proof: (⟹) Suppose that $N \leq_{L.ce} U$ in M, hence $\frac{U}{N} \ll_L \frac{M}{N}$. Since $\frac{U}{N} \simeq \frac{U/K}{N/K}$ and $\frac{M}{N} \simeq \frac{M/K}{N/K}$ by the Third isomorphism Theorem, then $\frac{U/K}{N/K} \ll_L \frac{M/K}{N/K}$ and hence $\frac{N}{K} \leq_{L.ce} \frac{U}{K}$ in $\frac{M}{K}$.
(⟸) Supppose that $\frac{N}{K} \leq_{L.ce} \frac{U}{K}$ in $\frac{M}{K}$, hence $\frac{U/K}{N/K} \ll_L \frac{M/K}{N/K}$ and by using the Third isomorphism Theorem, we get $\frac{U}{N} \simeq \frac{U/K}{N/K} \ll_L \frac{M/K}{N/K} \simeq \frac{M}{N}$, hence $\frac{U}{N} \ll_L \frac{M}{N}$, then $N \leq_{L.ce} U$ in M.

Proposition 2.6: Let M be an R-module and K , N, and U are submodules of M, such that $K \leq N \leq U \leq M$ and K , N are closed in M, then $K \leq_{L.ce} U$ in M if and only if $K \leq_{L.ce} N$ in M and $N \leq_{L.ce} U$ in M.

Proof: (⟹) Suppose that $K \leq_{L.ce} U$ in M, then $\frac{U}{K} \ll_L \frac{M}{K}$ and by Lemma(1.1), we have $\frac{N}{K} \ll_L \frac{M}{K}$ and $\frac{U}{N} \ll_L \frac{M}{N}$, hence $K \leq_{L.ce} N$ in M and $N \leq_{L.ce} U$ in M.
(⟸) Supppose that $K \leq_{L.ce} N$ in M and $N \leq_{L.ce} U$ in M, hence $\frac{N}{K} \ll_L \frac{M}{K}$ and $\frac{U}{N} \ll_L \frac{M}{N}$ and by Lemma(1.1), we get the result.

Proposition 2.7: Let M be an R-module and K, N, U, and H are submodules of M, such that $K \leq N \leq U \leq H \leq M$ and $K + U$ is closed in M. If $K \leq_{L.ce} N$ in M and $U \leq_{L.ce} H$ in M, then $K + U \leq_{L.ce} N + H$ in M.

Proof: Suppose that $K \leq_{L.ce} N$ in M and $U \leq_{L.ce} H$ in M, hence $\frac{N}{K} \ll_L \frac{M}{K}$ and $\frac{H}{U} \ll_L \frac{M}{U}$. Thus we have $N \ll_L M$ and $H \ll_L M$ by Lemma(1.2), hence $N + H \ll_L M$ and $K + U$ is closed in M. Thus we have, $\frac{N+H}{K+U} \ll_L \frac{M}{K+U}$ by Lemma(1.1), hence $K + U \leq_{L.ce} N + H$ in M.

Corollary 2.8: Let M be an R-module and K, N, and U are submodules of M such that $K \leq N \leq U \leq M$. If $K \leq_{L.ce} N$ in M, then $K + U \leq_{L.ce} N + U$ in M.

Proof: Let $U \leq M$, since $U \leq_{L.ce} U$ in M and $K \leq_{L.ce} N$ in M, then by proposition(2.7), we get $K + U \leq_{L.ce} N + U$ in M .

Proposition 2.9: Let M be an R-module and K , N, and U are submodules of M, such that $K \leq N \leq U \leq M$ and K is closed in M. If $K \leq_{L.ce} N$ in M and $U \ll_L M$, then $K \leq_{L.ce} N \bigoplus U$.

Proof: Let $\frac{H}{K}$ be a submodule of $\frac{M}{K}$ such that $\frac{N+U}{K} + \frac{H}{K} = \frac{M}{K}$. Hence, $\frac{N}{K} + \frac{U}{K} + \frac{H}{K} = \frac{M}{K}$, so we get $N + U + H = M$. Since $U \ll_L M$, then $N + H \leq_e M$ and since K is closed in M, hence $\frac{N+H}{K} \leq_e \frac{M}{K}$ by [2]. Also, since $K \leq_{L.ce} N$ in M , then $\frac{N}{K} \ll_L \frac{M}{K}$, hence we get $\frac{U}{K} + \frac{H}{K} = \frac{U+H}{K} \leq_e \frac{M}{K}$. Therefore, $(\frac{N+H}{K}) \cap (\frac{U+H}{K}) \leq_e \frac{M}{K}$ by [1], hence $\frac{(N+H)\cap(U+H)}{K} \leq_e \frac{M}{K}$ and then $\frac{(N\cap U)+H}{K} \leq_e \frac{M}{K}$. Hence, we get $\frac{H}{K} \leq_e \frac{M}{K}$, so $\frac{N\bigoplus U}{K} \ll_L \frac{M}{K}$, hence $K \leq_{L.ce} N \bigoplus U$.

Proposition 2.10: Let M be an R-module and K , N, and U are submodules of M such that $K \leq N \leq U \leq M$ and K is closed in M. If $N = K + U$ and $U \ll_L M$, then $K \leq_{L.ce} N$ in M.

Proof: Let $\frac{H}{K}$ be a submodule of $\frac{M}{K}$ such that $\frac{N}{K} + \frac{H}{K} = \frac{M}{K}$, hence $N + H = M$. Also, since $N = K + U$, so $M = N + H = (K + U) + H = U + H$, hence $M = U + H$. Since $U \ll_L M$, we get $H \leq_e M$ and K is closed in M, then $\frac{H}{K} \leq_e \frac{M}{K}$ by [2], hence $\frac{N}{K} \ll_L \frac{M}{K}$, so $K \leq_{L.ce} N$ in M .

Proposition 2.11: Let $f: M \to N$ be an epimorphism where M and N are R-modules. If $K \leq_{L.ce} U$ in N such that $f^{-1}(K)$ is closed in M ,then $f^{-1}(K) \leq_{L.ce} f^{-1}(U)$ in M.

Proof: Let $\frac{H}{f^{-1}(K)}$ be a submodule of $\frac{M}{f^{-1}(K)}$ such that $\frac{f^{-1}(U)}{f^{-1}(K)} + \frac{H}{f^{-1}(K)} = \frac{M}{f^{-1}(K)}$, so $f^{-1}(U) + H = M$ and hence $U + f(H) = N$, so $\frac{U}{K} + \frac{f(H)}{K} = \frac{N}{K}$. Also, since $K \leq_{L.ce} U$ in N, then $\frac{U}{K} \ll_L \frac{N}{K}$ and hence $\frac{f(H)}{K} \leq_e \frac{N}{K}$, then $f(H) \leq_e N$. Thus, $H = f^{-1}(f(H)) \leq_e M$. Since $f^{-1}(K)$ is closed in M, then $\frac{H}{f^{-1}(K)} \leq_e \frac{M}{f^{-1}(K)}$ by [2], so $\frac{f^{-1}(U)}{f^{-1}(K)} \ll_L \frac{M}{f^{-1}(K)}$ and hence $f^{-1}(K) \leq_{L.ce} f^{-1}(U)$ in M.

Proposition 2.12: Let M be an R-module and K , N, and U are submodules of M, then the followings are equivalent:

1- If $K \leq_{L.ce} K + N$, then $K \cap N \leq_{L.ce} N$.

2- If $K \leq_{L.ce} N$ and $V \leq M$, then $K \cap V \leq_{L.ce} N \cap V$.

3- If $K \leq_{L.ce} N$ and $W \leq_{L.ce} U$, then $K \cap W \leq_{L.ce} N \cap U$.

Proof: $(1) \Longrightarrow (2)$ Let $K \leq_{L.ce} N$ and $V \leq M$. Since $K + (N \cap V) \leq N$, then by proposition(2.6), we get $K \leq_{L.ce} K + (N \cap V)$ in M. Hence from (1), $K \cap (N \cap V) \leq_{L.ce} N \cap V$ in M, so $K \cap V \leq_{L.ce} N \cap V$.

$(2) \Longrightarrow (3)$ Let $K \leq_{L.ce} N$ in M and $W \leq M$, hence from (2), $K \cap W \leq_{L.ce} N \cap W$. Also, $W \leq_{L.ce} U$ and $N \leq M$, hence from(2), $N \cap W \leq_{L.ce} N \cap U$, then by proposition(2.6), we get $K \cap W \leq_{L.ce} N \cap U$.

$(3) \Longrightarrow (1)$ Let $K \leq_{L.ce} K + N$. Since $N \leq_{L.ce} N$, then from (3) we get $K \cap N \leq_{L.ce} (K + N) \cap N$ and hence, $K \cap N \leq_{L.ce} N$.

3. Large-Coclosed submodule

In this section we introduce the concept of Large-coclosed submodule and some of its properties.

Definition 3.1: Let M be an R-module and N be a submodule of M, then N is called Large-coclosed (L-coclosed) submodule of M ($N \leq_{L.cc} M$) if $K \leq_{L.ce} N$ in M for some submodule K of N, implies that $K = N$. Equivalently, N is called Large-coclosed (L-coclosed) submodule of M, if N has no proper L-coessential submodule of M.

Let N, K be submodules of M such that $N \leq K \leq M$, then N is called Large-coclosure (L-coclosure) submodule of K in M, if $N \leq_{L.ce} K$ in M and $N \leq_{L.cc} M$.

Remarks and Examples 3.2

1- Every L-coclosed submodule is coclosed submodule.

Proof: Let N be L-coclosed submodule of M and $K \leq N$, such that $\frac{N}{K} \ll \frac{M}{K}$. Hence by [4], $\frac{N}{K} \ll_L \frac{M}{K}$, so $K \leq_{L.ce} N$ in M. Since N is L-coclosed of M, then $K = N$ and hence N is coclosed submodule.

2- The converse of (1) is not true, as in the following example: In Z_6 as Z-module: $\{\bar{0}, \bar{2}, \bar{4}\}$ is coclosed submodule of Z_6 , since $\{\bar{0}\}$ is the only submodule of $\{\bar{0}, \bar{2}, \bar{4}\}$, such that $\frac{\{\bar{0},\bar{2},\bar{4}\}}{\{\bar{0}\}} \backsimeq \{\bar{0}, \bar{2}, \bar{4}\}$, $\frac{Z_6}{\{\bar{0}\}} \backsimeq Z_6$, and $\{\bar{0}, \bar{2}, \bar{4}\}$ is not small in Z_6 . Also, $\{\bar{0}\} \neq \{\bar{0}, \bar{2}, \bar{4}\}$, but $\{\bar{0}, \bar{2}, \bar{4}\}$ is not L-coclosed, since $\{\bar{0}, \bar{2}, \bar{4}\}$ is L-small in Z_6, but $\{\bar{0}\} \neq \{\bar{0}, \bar{2}, \bar{4}\}$.

3- In Z_6 as Z-module: $\{\bar{0}, \bar{3}\}$ is L-coclosed of Z_6, since $\{\bar{0}\}$ is not L-coessential submodule of $\{\bar{0}, \bar{3}\}$, by (2.2), and $\{\bar{0}\} \neq \{\bar{0}, \bar{3}\}$.

4- In Z_4 as Z-module: $\{\bar{0}, \bar{2}\}$ is not L-coclosed of Z_4, since $\{\bar{0}\} \leq_{L.ce} \{\bar{0}, \bar{2}\}$ by (2.2), but $\{\bar{0}\} \neq \{\bar{0}, \bar{2}\}$.

5- In Z_8 as Z-module: $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ in not L-coclosed of Z_8, since $\{\bar{0}, \bar{4}\} \leq_{L.ce} \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ by (2.2), but $\{\bar{0}, \bar{4}\} \neq \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.

6- In Z as Z-module: 2Z is not L-coclosed of Z, since $4Z \leq_{L.ce} 2Z$ by (2.2), but $4Z \neq 2Z$.

7- Let M be an R-module and K, N are submodules of M such that $K \leq N \leq M$. If $\frac{M}{K}$ is semisimple module, then N is coclosed submodule of M if and only if N is L-coclosed submodule .

Proposition 3.3: Let M be an R-module and K , U, and N are submodules of M such that $U \leq K \leq N \leq M$, then $N \leq_{L.cc} M$ if and only if $\frac{N}{U} \leq_{L.cc} \frac{M}{U}$.

Proof:(⇒) let $\frac{K}{U} \leq \frac{N}{U}$ and $\frac{K}{U} \leq_{L.ce} \frac{N}{U}$ in $\frac{M}{U}$, so by proposition(2.5) we get $K \leq_{L.ce} N$ in M. Since $N \leq_{L.cc} M$, then $K = N$ and hence $\frac{K}{U} = \frac{N}{U}$.

(⇐) Let $K \leq_{L.ce} N$ in M , so by proposition(2.5) we get $\frac{K}{U} \leq_{L.ce} \frac{N}{U}$ in $\frac{M}{U}$. Since $\frac{N}{U} \leq_{L.cc} \frac{M}{U}$, then $\frac{K}{U} = \frac{N}{U}$ and hence $K = N$.

Proposition 3.4: Let M be an R-module and N be a nonzero submodule of M, then either $N \ll_L M$ or $N \leq_{L.cc} M$, but not both.

Proof: Suppose that N be a nonzero submodule of M and let N be not L-coclosed submodule. Then, there exists a proper submodule K of N such that $K \leq_{L.ce} N$, hence $\frac{N}{K} \ll_L \frac{M}{K}$ and by Lemma(1.2), we get $N \ll_L M$. Now, if $N \leq_{L.cc} M$ and by supposing that $N \ll_L M$, let $\{\bar{0}\} \leq N$ such that $\{\bar{0}\} \leq_{L.ce} N$, so $\frac{N}{\{\bar{0}\}} \ll_L \frac{M}{\{\bar{0}\}}$. Since $N \leq_{L.cc} M$, then $\{\bar{0}\} = N$, but this is a contradiction, hence N is not L-small in M.

Lemma 3.5: Let M be an R-module and U, K, and N are submodules of M such that $U \leq K \leq N \leq M$. If $\frac{K}{U} \ll_L \frac{N}{U}$ and $\frac{N}{U} \ll_L \frac{M}{U}$, then $\frac{K}{U} \ll_L \frac{M}{U}$.

Proof: Let $\frac{H}{U}$ be a submodule of $\frac{M}{U}$ such that $\frac{K}{U}+\frac{H}{U}=\frac{M}{U}$. Hence $K+H=M,$ so $N+H=M,$ hence $\frac{N}{U}+\frac{H}{U}=\frac{M}{U}$. Since $\frac{N}{U} \ll_L \frac{M}{U}$, then $\frac{H}{U} \leq_e \frac{M}{U}$ and hence $\frac{K}{U} \ll_L \frac{M}{U}$.

Proposition 3.6: Let M be an R-module and U, K, and N are submodules of M such that $U \leq K \leq N \leq M$. If $K \leq_{L.cc} M$ and $\frac{N}{U} \ll_L \frac{M}{U}$, then $K \leq_{L.cc} N$.

Proof: Let $U \leq_{L.ce} K$ in N, hence $\frac{K}{U} \ll_L \frac{N}{U}$. Since $\frac{N}{U} \ll_L \frac{M}{U}$ then by Lemma(3.5), we get $\frac{K}{U} \ll_L \frac{M}{U}$, hence $U \leq_{L.ce} K$ in M. Also, since $K \leq_{L.cc} M,$ then $U=K,$ so $K \leq_{L.cc} N$.

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