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# N Sequence Prime Ideals

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**Abstract**
In this paper, the concepts of $n$-sequence prime ideal and $n$-sequence quasi prime ideal are introduced. Some properties of such ideals are investigated. The relations between $n$-sequence prime ideal and each of primary ideal, $n$-prime ideal, quasi prime ideal, strongly irreducible ideal, and $(k, m)$ closed ideal, are studied. Also, the ideals of a principal ideal domain are classified into quasi prime ideals and $n$-sequence quasi prime ideals.

**Keywords:** Prime ascending chain of ideals; Prime with respect to; Length of an ideal; $n$-sequence prime ideal; $n$-sequence quasi prime ideal.

# المثاليات الاولية المتتابعة من نمط $n$

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**الخلاصة**
في هذا البحث نقوم بعرض مفهوم المثاليات الأولية المتتابعة من نمط n و مفهوم المثاليات شبه الأولية المتتابعة من نمط n. نبحث بعض خصائصهما. أضافة الى ذلك نقوم بدراسة علاقة المثاليات الأولية المتتابعة من نمط n مع المثاليات الإبتدائية ، المثاليات الأولية من نمط n ، المثاليات شبه الأولية ، المثاليات الغير قابلة للأختزال بقوة و المثاليات المغلقة من نمط $(k, m)$. قمنا بتصنيف مثاليات الحلقات التي هي ساحة مثاليات رئيسة الى صنفين المثاليات شبه الأولية و المثاليات شبه الأولية المتتابعة من نمط n.

## 1. Introduction

Throughout this paper, R is a commutative ring with identity. Let $I_0 \subset I_1$ be two proper ideals of R. We say that $I_0$ is prime in $I_1$ if for each $a, b$ in $I_1$, $ab \in I_0$ implies $a \in I_0$ or $b \in I_0$ and $I_0$ is prime with respect to $I_1$ if for each $a, b$ in $R$, $ab \in I_0$ implies $a \in I_1$ or $b \in I_1$. An ascending chain of proper ideals $I_0 \subset I_1 \subset I_2 \subset I_3$ ... of R is called a prime ascending chain of ideals if $I_{m-1}$ is a prime ideal in $I_m$ for each $m \in \mathbb{Z}^+$, the set of all positive integers. We also say that $I_0$ is a prime ideal of length $m$ with respect to the prime ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ if $I_0$ is not prime with respect to $I_k$ for each $0 \leq k \leq m-1$, but $I_0$ is prime with respect to $I_m$, while the prime ascending chain $I_0 \subset I_1 \subset I_2 \subset \cdots$ is said to be stabilized at $I_m$ and the ideal $I_m$ is called a stabilizer prime ideal of the chain. Moreover, a non-prime proper ideal $I_0$ of R is called an $n$-sequence prime ideal if $n = min\{t: t \text{ is the length of } I_0$ with respect to a prime ascending chain of ideals of the form $I_0 \subset I_1 \subset I_2 \subset \cdots\}$. Some important results are obtained. It is shown that for each $a \in R$ and $k \in \mathbb{Z}^+$, if $a^k \in I_0$, then

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$a \in I_0$ and consequently, $\sqrt{I_0} = I_0$; see Theorem 2.12 and Corollary 2.16. It is also shown that there are two elements $a, b \in R$ with $ab \in I_0$ but $a^k, b^k \notin I_0$ for each $k \in \mathbb{Z}^+$; see Proposition 2.23. The relations between $n$-sequence prime ideal with some other types of ideals, such as primary ideal, quasi prime ideal, strongly irreducible ideal, and $(k, m)$ closed ideal, are discussed separately; see Proposition 3.1, Proposition 3.3, Theorem 3.10, and Proposition 3.13. Moreover, it is shown that the concept of n-sequence prime ideal is independent with each of weakly prime, weakly irreducible, weakly 2-absorbing, $n$ almost prime, and 2-absorbing ideals. Finally, the concept of $n$-sequence quasi prime ideal is introduced (see 3.16). The family of proper ideals of a principal ideal domain is classified. We show that a proper ideal of a principal ideal domain is either quasi prime or n-sequence quasi prime.

**2. n-sequence prime ideals**

In this section, we introduce the concept of an $n$-sequence prime ideal of a commutative ring with identity and we illustrate it by some examples. We obtain some results and properties of such ideals.

We start by introducing some new concepts.

**Definition 2.1.** Let $A \subset B$ be two proper ideals of $R$. Then $A$ is said to be a prime ideal with respect to $B$ if for each $a, b$ in R, $ab \in A$ implies $a \in B$ or $b \in B$.

Clearly, if A is a prime ideal of a ring R, then it is prime with respect to any ideal containing it.

**Definition 2.2.** A sequence of proper ideals $I_0 \subset I_1 \subset I_2 \subset I_3$ ... of R is called a prime (resp. p-maximal) ascending chain of ideals if $I_{m-1}$ is a prime (resp. prime and maximal) ideal in $I_m$ for each $m \in \mathbb{Z}^+$. A proper ideal $I_0$ of $R$ is called a prime ideal of length $m$ with respect to the prime ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$, if $I_0$ is not prime with respect to $I_k$ for each $0 \leq k \leq m-1$ but it is prime with respect to $I_m$. Then the prime ascending chain $I_0 \subset I_1 \subset I_2 \subset \ldots$ is said to be stabilize at $I_m$ and the ideal $I_m$ is called the stabilizer ideal of the chain.

**Definition 2.3.** A non-prime proper ideal $I_0$ of R is called an $n$-sequence prime ideal if $n = min\{t: t$ is the length of $I_0$ with respect to a prime ascending chain of ideals of the form $I_0 \subset I_1 \subset I_2 \subset \ldots\}$.

The following remarks are obvious.

**Remark 2.4.** Let $I_0 \subset I_1 \subset I_2 \subset \cdots$ be an ascending chain of ideals of $R$.

1. For each integer $m \geq 0$, there is $\chi \in R$ such that $I_m \subset (I_m \cup \{\chi\}) \subseteq I_{m+1}$, since there exists an element $\chi \in I_{m+1} \backslash I_m$. Moreover, if $I_m$ is maximal in $I_{m+1}$, then $(I_m \cup \{\chi\}) = I_{m+1}$.
2. For each $k \in \mathbb{Z}^+$, there are $k$ elements $\chi_1, \chi_2, \ldots, \chi_{k-1}, \chi_k$ in $R$ such that $(I_0 \cup \{\chi_1, \ldots, \chi_k\}) \subseteq I_k$. Morover, if $I_m$ is maximal in $I_{m+1}$ for each $m$, then $I_k = (I_0 \cup \{\chi_1, \ldots, \chi_k\})$.

**Definition 2.5** [1]**.** A proper ideal $I_0$ of R is quasi prime, if $a$, $b \in R$ with $ab \in I_0$ implies $a \in \sqrt{I_0}$ *or* $b \in \sqrt{I_0}$. Equivalently a proper ideal $I_0$ of R is quasi prime if $\sqrt{I_0}$ is a prime ideal.

**Remark 2.6.**

1. If $B$ is a prime ideal, then every proper ideal of $R$ contained in $B$ is prime with respect to $B$.
2. An ideal $I$ of $R$ is quasi prime if and only if $I$ is prime with respect to $\sqrt{I}$.

**Remark 2.7.**

1. Consider the ideal $I_0 = (m_0)$ of the ring of integers $\mathbb{Z}$ with the prime factorization of $m_0 = p_1 \ldots p_n$ with $p_i$ are distinct primes, $1 \leq i \leq n$. Let $m_i = \frac{m_{i-1}}{p_{l_i}}$ where $p_{l_i} \in \{p_1, \ldots, p_n\}$ for $1 \leq l_i \leq n$ such that $l_i \notin \{l_1, \ldots l_{i-1}\}$. Let $I_i = (m_i)$ be the ideal generated by $m_i$. Then the chain $I_0 \subset I_1 \subset I_2 \subset \cdots$ is a prime ascending chain of ideals and it is stabilized at an ideal

generated by $p_{l_n}$. The number of prime ascending chains of the form $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_{n-1}$ is $n!$ and $I_0$ is an $n$-1-sequence prime ideal.
2. Let $R = \mathbb{Z}[x_1, \dots, x_n]$ be the polynomial ring over $\mathbb{Z}$ with $n$ indeterminates and let $I_0 = (x_1 x_2 x_3 \dots x_n)$, $I_k = (\prod_{i=1}^{n-k} x_i)$ where $n \in \mathbb{Z}^+ \backslash \{1\}$ and $1 \leq k < n$. Then the chain $I_0 \subset I_1 \subset I_2 \subset \cdots$ is a prime ascending chain of ideals of $R$ that is stabilized at the ideal $I_{n-1} = (x_1)$ and $I_0$ is $n - 1$-sequence prime.

**Example 2.8.**

**1.** Consider the ideals $I_0 = (2310)$, $I_1 = (210)$, $I_2 = (30)$, $I_3 = (6)$, and $I_4 = (2)$ of $\mathbb{Z}$. Then the chain of ideals $(2310) \subset (210) \subset (30) \subset (6) \subset (2)$ is a prime ascending chain that is stabilized at $I_4$, which shows that $I_0$ is a 4-sequence prime ideal. Moreover, the number of such prime ascending chain is 5!.
Consider the ideals $I_0 = (\chi(\chi+1)(\chi+2)(\chi+3))$, $I_1 = (\chi(\chi+1)(\chi+2))$, $I_2 = (\chi(\chi+1))$, and $I_3 = (\chi)$ of $\mathbb{Z}[\chi]$ as the polynomial ring over $\mathbb{Z}$ with one indeterminate. Then the chain of ideals $(\chi(\chi+1)(\chi+2)(\chi+3)) \subset (\chi(\chi+1)(\chi+2)) \subset (\chi(\chi+1)) \subset (\chi)$ is prime ascending chain that is stabilized at $I_3$, which shows that $I_0$ is a 3-sequence prime ideal. Moreover, the number of prime ascending chains is 4!, which are shown in the following diagram

$$(\chi(\chi+1)(\chi+2)(\chi+3)) \subset \begin{cases} (\chi(\chi+1)(\chi+2)) \subset \begin{cases} (\chi(\chi+1)) \subset \begin{cases} (\chi) \\ (\chi+1) \end{cases} \\ (\chi(\chi+2)) \subset \begin{cases} (\chi) \\ (\chi+2) \end{cases} \\ (\chi+1)(\chi+2) \subset \begin{cases} (\chi+1) \\ (\chi+2) \end{cases} \end{cases} \\ (\chi(\chi+1)(\chi+3)) \subset \begin{cases} (\chi(\chi+1)) \subset \begin{cases} (\chi) \\ (\chi+1) \end{cases} \\ (\chi(\chi+3)) \subset \begin{cases} (\chi) \\ (\chi+3) \end{cases} \\ ((\chi+1)(\chi+3)) \subset \begin{cases} (\chi+1) \\ (\chi+3) \end{cases} \end{cases} \\ (\chi(\chi+2)(\chi+3)) \subset \begin{cases} (\chi(\chi+2)) \subset \begin{cases} (\chi) \\ (\chi+2) \end{cases} \\ (\chi(\chi+3)) \subset \begin{cases} (\chi) \\ (\chi+3) \end{cases} \\ ((\chi+2)(\chi+3)) \subset \begin{cases} (\chi+2) \\ (\chi+3) \end{cases} \end{cases} \\ ((\chi+1)(\chi+2)(\chi+3)) \subset \begin{cases} ((\chi+1)(\chi+2)) \subset \begin{cases} (\chi+1) \\ (\chi+2) \end{cases} \\ ((\chi+1)(\chi+3)) \subset \begin{cases} (\chi+1) \\ (\chi+3) \end{cases} \\ ((\chi+2)(\chi+3)) \subset \begin{cases} (\chi+2) \\ (\chi+3) \end{cases} \end{cases} \end{cases}$$

**Proposition 2.9.** Let $I_0$ be an $n$-sequence prime ideal of $R$. If $a \in R$ with $a^2 \in I_0$, then $a \in I_0$.
Proof. Since $I_0$ is an $n$-sequence prime ideal of $R$, then there is a prime ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ with stabilizer ideal $I_n$. Suppose that $a^2 \in I_0$. Then $a \in I_n$, since $I_0$ is prime with respect to $I_n$. So $a \in I_{n-1}$, since $I_{n-1}$ prime in $I_n$. For the same reason, $a \in I_k$, $0 \leq k \leq n-2$.

**Corollary 2.10.** Let $I_0$ be a proper ideal of $R$. If $a \in R$ with $a^2 \in I_0$ but $a \notin I_0$, then $I_0$ is not an $n$-sequence prime ideal of $R$.

**Proposition 2.11.** Let $I_0$ be an $n$-sequence prime ideal of $R$. For any two elements $x, a$ in $R$, if $xa^2 \in I_0$, then $xa \in I_0$.

**Proof**. Let $xa^2 \in I_0$. Then $(xa)^2 \in I_0$. By proposition 2.9, $xa \in I_0$.

**Theorem 2.12.** Let $I_0$ be an $n$-sequence prime ideal of $R$. If $a \in R$ with $a^k \in I_0$, where $k > 1$, then $a \in I_0$.

**Proof.** Suppose that $a^k \in I_0$ for some $k > 1$ and let $k_1 = \begin{cases} k & if\ k\ is\ even \\ k+1 & if\ k\ is\ odd \end{cases}$. Then $a^{k_1} \in I_0$. By Proposition 2.9, $a^{(\frac{k_1}{2})} \in I_0$. If $\frac{k_1}{2} = 1$, then the proof is complete. If $\frac{k_1}{2} > 1$, let $k_2 = \begin{cases} \frac{k_1}{2} & if\ \frac{k_1}{2}\ is\ even \\ \frac{k_1}{2} + 1 & if\ \frac{k_1}{2}\ is\ odd \end{cases}$. Then $a^{k_2} \in I_0$. Also by Proposition 2.9, $a^{(\frac{k_2}{2})} \in I_0$. By iterating these steps, we obtain $a \in I_0$.

**Definition 2.13** [2]**.** If $I_0$ is an ideal of $R$, then the radical of $I_0$ denoted by $\sqrt{I_0}$ is $\sqrt{I_0} = \{x \in R;\ x^n \in I_0 \text{ for some } n \in \mathbb{Z}^+\}$, which is an ideal of $R$.

**Definition 2.14** [3]**.** The nilradical of R (radical($R$)) is the set of all nilpotent elements in R which forms an ideal of R. Equivalently, radical($R$) $= \sqrt{(0)}$ is the radical of the zero ideal.

**Corollary 2.15**. Let $I_0$ be an ideal of $R$ and $a \in R$. If $a^k \in I_0$, $k > 1$, and $a \notin I_0$, then $I_0$ is not an $n$-sequence prime ideal of $R$. Equivalently, if $a \in \sqrt{I_0}$ and $a \notin I_0$, then $I_0$ is not an $n$-sequence prime ideal of $R$.

**Corollary 2.16.** If $I_0$ is an ideal of R such that $\sqrt{I_0} \neq I_0$, then $I_0$ is not an $n$ -sequence prime. Equivalently, if $I_0$ is an n-sequence prime ideal, then $\sqrt{I_0} = I_0$.

The following remark shows that the converse of Corollary 2.16. is not true in general.

**Remark 2.17.** If $I_0$ is an ideal of R and $\sqrt{I_0} = I_0$, then $I_0$ may not be an $n$-sequence prime ideal, for example the ideal $I_0 = (2)$ of $\mathbb{Z}$, then $I_0$ is a prime ideal, so $\sqrt{I_0} = \sqrt{(2)} = (2)$, but $I_0$ is not an $n$-sequence prime ideal.

**Example 2.18.** By Corollary 2.15, we obtain that

1. The ideal $I_0 = (p^k)$, $p$ is a prime number and $k > 1$ is not an $n$-sequence prime ideal of $\mathbb{Z}$.
2. For each prime number $p$ and $k \in \mathbb{Z}^+$, the ring $\mathbb{Z}_{p^k}$ has no $n$-sequence prime ideal.
3. The ideal $(x^k)$, $k > 1$ of the ring R=$\mathbb{Z}[x]$ is not $n$-sequence prime.

**Proposition 2.19.** Let $I_0$ be an $n$-sequence prime ideal of $R$. Then the radical of $R$ is contained in $I_0$.

**Proof**. If $x$ is a nilpotent element of $R$, then $x^n = 0 \in I_0$ for some $n \in \mathbb{Z}^+$. By Theorem 2.12, $x \in I_0$, which means the radical of R is contained in $I_0$.

**Proposition 2.20**. Let $I_0$ be an $n$-sequence prime ideal of $R$. If $\chi = {\chi_1}^{\alpha_1}{\chi_2}^{\alpha_2} \ldots {\chi_k}^{\alpha_k} \in I_0$, then $\chi_1\chi_2 \ldots \chi_k \in I_0$, where $\chi_i \in R$ and $\alpha_i \in \mathbb{Z}^+$ for each $1 \leq i \leq k$. Equivalently, if $\chi = {\chi_1}^{\alpha_1}{\chi_2}^{\alpha_2} \ldots {\chi_k}^{\alpha_k} \in I_0$ and $\chi_1\chi_2 \ldots \chi_k \notin I_0$, then $I_0$ is not an $n$-sequence prime ideal of $R$.

**Proof.** Let ${\chi_1}^{\alpha_1}{\chi_2}^{\alpha_2} \ldots {\chi_k}^{\alpha_k} \in I_0$ and $\alpha = Max\{\alpha_i;\ 1 \leq i \leq k\}$. Then ${\chi_1}^{\alpha}{\chi_2}^{\alpha} \ldots {\chi_k}^{\alpha} = (\chi_1\chi_2 \ldots \chi_k)^{\alpha} \in I_0$. By Theorem 2.12, $\chi_1\chi_2 \ldots \chi_k \in I_0$.

**Proposition 2.21.** Let $I_0$ be an $n$-sequence prime ideal of $R$ and $I_0 \subset I_1 \subset I_2 \subset \cdots$ be a prime ascending chain of ideals with stabilizer ideal $I_n$. Then, for each $m \in \mathbb{Z}^+$, there exists an element $a \in R$ such that $a \in I_m$, but $a^k \notin I_{m-1}$. Moreover, if $b$ divides $a$, then $b^{\,k} \notin I_{m-1}$ for each $k \in \mathbb{Z}^+$.

**Proof.** Since $I_{m-1} \subset I_m$ for each $m \in \mathbb{Z}^+$, then there exists an element $a \in I_m$ but $a \notin I_{m-1}$. If $a^2 \in I_{m-1}$, then $a \in I_{m-1}$, since $I_{m-1}$ is prime in $I_m$, which is a contradiction with the assumption $a \notin I_{m-1}$. Hence, $a^2 \notin I_{m-1}$. If $a^k = aa^{k-1} \in I_{m-1}$ for some $k > 2$, then

$a \in I_{m-1}$ or $a^{k-1} \in I_{m-1}$, since $I_{m-1}$ is prime in $I_m$. Since $a \notin I_{m-1}$, then $a^{k-1} \in I_{m-1}$. By iterating this step, we obtain $a \in I_{m-1}$, which is a contradiction. Therefore, $a^k \notin I_{m-1}$ for each $k \in \mathbb{Z}^+$. Now, suppose that $b$ divides $a$, then there exists an element $x$ in $R$ such that $a = xb$. If $b^k \in I_{m-1}$ for some $k \in \mathbb{Z}^+$, then $x^k b^k \in I_{m-1}$, implies that $a^k \in I_{m-1}$. This is a contradiction with $a^k \notin I_{m-1}$.

**Proposition 2.22.** Let $I_0$ be an $n$ -sequence prime ideal of R and $I_0 \subset I_1 \subset I_2 \subset \cdots$ be a prime ascending chain of ideals with stabilizer ideal $I_n$. If $a, b \in R$ with $ab \in I_0$ and $a, b \notin I_{n-1}$, then $I_n$ contains exactly one of $a$ or $b$.

**Proof**. Let $a, b \in R$ with $ab \in I_0$ and $a, b \notin I_{n-1}$. Since $I_n$ is stabilizer, $I_0$ is prime with respect to $I_n$. Then $a \in I_n$ or $b \in I_n$. Suppose that both $a$ and $b$ are in $I_n$. Since $I_{n-1}$ is prime in $I_n$, then $a \in I_{n-1}$ or $b \in I_{n-1}$, which is a contradiction with the assumption $a, b \notin I_{n-1}$.

**Proposition 2.23.** Let $I_0$ be an $n$-sequence prime ideal of $R$. Then there are two elements $a, b \in R$ with $ab \in I_0$ but $a^k, b^k \notin I_0$ for each $k \in \mathbb{Z}^+$. Moreover, if $c \in R$ divides $a$ or $b$, then $c^k \notin I_0$ for each $k \in \mathbb{Z}^+$.

**Proof.** Since $I_0$ is an $n$-sequence prime ideal of $R$, then $I_0$ is not a prime ideal. This means that there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin I_0$. By Corollary 2.15, if $a^k \in I_0$ for some $k \in \mathbb{Z}^+$, then $I_0$ is not an $n$-sequence prime ideal, which is a contradiction. Similarly, we get a contradiction if $b^k \in I_0$. Suppose that $c \in R$ divides $a$ and $c^k \in I_0$ for some $k \in \mathbb{Z}^+$. Then there is an elements $x$ in $R$ such that $a = xc$. If $c^k \in I_0$ , then $x^k c^k \in I_0$, implies that $a^k \in I_0$. This is a contradiction with $a^k \notin I_0$.

**Corollary 2.24.** If $I_0$ is an $n$-sequence prime ideal of $R$, then there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin \sqrt{I_0}$.

**Corollary 2.25.** Let $I_0$ be an $n$-sequence prime ideal of $R$ and $I_0 \subset I_1 \subset I_2 \subset \cdots$ be a prime ascending chain of ideals with stabilizer ideal $I_n$. Then $I_n \neq \sqrt{I_0}$.

**Proof.** Since $I_0$ is an $n$-sequence prime ideal of $R$, then by Corollary 2.24, there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin \sqrt{I_0}$. Since $I_n$ is the stabilizer ideal of the given prime ascending chain, then $a \in I_n$ or $b \in I_n$. This means that there is an element in $I_n$ but not in $\sqrt{I_0}$. Therefore, $I_n \neq \sqrt{I_0}$.

## 3. Relations between n-sequence prime ideals and some types of ideals

In this section, we study the relation between an n-sequence prime ideal and each of primary ideal, $n$-prime ideal, quasi prime ideal, strongly irreducible ideal, and $(k, m)$ closed ideal. It is shown that the concept of n-sequence prime ideal is independent of each of weakly prime, weakly irreducible, weakly 2-absorbing, $n$ almost prime, and 2-absorbing ideals. Moreover, we introduce the concept of n-sequence quasi prime ideal and classify the family of proper ideals for a principal ideal domain. We show that a proper ideal of a principal ideal domain is either quasi prime or n-sequence quasi prime.

**Proposition 3.1.** If $I_0$ is an $n$-sequence prime ideal of $R$, then it is not a primary ideal. Equivalently, if $I_0$ is a primary ideal, then it is not an $n$-sequence prime ideal.

**Proof.** By Proposition 2.23, there are two elements $a, b \in R$ with $ab \in I_0$ but $a^k, b^k \notin I_0$ for each $k \in \mathbb{Z}^+$. Therefore, $I_0$ is not a primary ideal.

The following is an example for an ideal which is neither primary nor n sequence prime.

**Example 3.2**. Consider the ideal $I = (x, y^2 z^3)$ of the polynomial ring $\mathbb{Z}[x, y, z]$. Then $I$ is neither $n$-sequence prime nor primary.

**Proposition 3.3.** If $I_0$ is an $n$-sequence prime ideal of $R$, then it is not a quasi prime ideal. Equivalently, if $I_0$ is a quasi prime ideal of $R$, then it is not an $n$-sequence prime ideal.

**Proof.** Let $I_0$ be an $n$ -sequence prime ideal. Then by Corollary 2.24, there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin \sqrt{I_0}$. Therefore, $I_0$ is not a quasi prime ideal.

The following is an example of an ideal which is neither $n$-sequence prime nor quasi prime.

**Example 3.4**. The ideal $(xy^2)$ of $\mathbb{Z}[x,y]$ is neither $n$-sequence prime nor quasi prime.
**Definition 3.5** [4]**.** A proper ideal $I_0$ of $R$ is 2-prime (resp. $m$-prime, $m \in \mathbb{Z}^+$) if $ab \in I_0$, implies $a^2 \in I_0$ or $b^2 \in I_0$(resp. $a^m \in I_0$ or $b^m \in I_0$).
**Proposition 3.6.** If $I_0$ is an $n$-sequence prime ideal of $R$, then it is not an $m$-prime ideal, for each $m \in \mathbb{Z}^+$.
**Proof.** The proof is similar to the proof of Proposition 3.1.
The following definitions are needed.
**Definition 3.7** [5]**.** A proper ideal $I_0$ of R is weakly prime (resp. almost prime and $n$ almost prime), if $a,\ b \in R$, with $ab \in I_0 \backslash \{0\}$ (resp. $ab \in I_0 \backslash {I_0}^2$ and $ab \in I_0 \backslash {I_0}^n;\ n > 2$), implies $a \in I_0$ or $b \in I_0$.
**Definition 3.8** [6]. A proper ideal $I$ of $R$ is said to be a 2-absorbing(resp. weakly 2-absorbing) ideal of $R$ if $a,b,c \in R$ and $abc \in I$ (resp. $abc \in I\backslash\{0\}$), then $ab \in I$ or $bc \in I$ or $ac \in I$.
**Definition 3.9** [7], [8]**.** Let $I$ be a proper ideal of $R$. Then $I$ is strongly irreducible (resp. weakly irreducible), if for each pair of ideals $A$ and $B$ of R, $A \cap B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$(resp. $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$) and $I$ is strongly 2-irreducible, if for each ideals $A, B$ and $C$ of $R, A \cap B \cap C \subseteq I$ implies $A \cap B \subseteq I$ or $A \cap C \subseteq I$ or $B \cap C \subseteq I$.
**Theorem 3.10.** If $I_0$ is an $n$-sequence prime ideal of $R$, then it is not a strongly irreducible ideal. Equivalently, if $I_0$ is a strongly irreducible ideal of $R$, then it is not an $n$-sequence prime ideal.
**Proof.** Let $I_0$ be an $n$-sequence prime ideal of $R$. Then there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin I_0$. So $(ab) \subseteq I_0$ and, consequently, $\sqrt{(ab)} \subseteq \sqrt{I_0}$. By Corollary 2.16, $I_0 = \sqrt{I_0}$, then $\sqrt{(ab)} \subseteq I_0$. Clearly, $(ab) = (a)(b)$ and $\sqrt{(a)(b)} = \sqrt{(a) \cap (b)} = \sqrt{(a)} \cap \sqrt{(b)}$ [2], then $\sqrt{(a)} \cap \sqrt{(b)} \subseteq I_0$. On the other hand, $a, b \notin I_0$, then $\sqrt{(a)}, \sqrt{(b)} \nsubseteq I_0$. Therefore, $I_0$ is not strongly irreducible.
**Remark 3.11.** The concept of n-sequence prime ideal is independent with each of weakly prime, weakly 2-absorbing, weakly irreducible, 2-absorbing, almost prime, strongly 2-irreducible, and $n$ almost prime ideals. For example the ideal (30) of $\mathbb{Z}$ is a 2-sequence prime ideal but it is not any one of weakly prime, weakly 2-absorbing, weakly irreducible, 2-absorbing, almost prime, strongly 2-irreducible, and $n$ almost prime ideals. On the other hand, the ideal (5) of $\mathbb{Z}$ is a weakly prime, weakly 2-absorbing, weakly irreducible, 2-absorbing, almost prime, strongly 2-irreducible, and $n$ almost prime ideal, but it is not an n-sequence prime ideal for each $n \geq 1$.
**Definition 3.12** [9]**.** Let $m, k \in \mathbb{Z}^+$ with $1 \leq m < k$. A proper ideal $I$ of $R$ is a $(k, m)$ closed ideal if whenever $a^k \in I$ for some $a \in R$ implies $a^m \in I$.
**Proposition 3.13.** If $I_0$ is an $n$-sequence prime ideal of $R$, then it is a $(k, m)$ closed ideal for some $m, k \in \mathbb{Z}^+$ and $1 \leq m < k$.
**Proof.** Let $I_0$ be an n-sequence prime ideal. To show that $I_0$ is $(k, m)$ closed, we have to show that if $a^k \in I_0$ for some $a \in R$ and $k \in \mathbb{Z}^+$, then $a^m \in I_0$ for each $1 \leq m < k$. Suppose that $a^k \in I_0$ for some $a \in R$ and $k \in \mathbb{Z}^+$. Then by Corollary 2.12, $a \in I_0$, which implies that $a^m \in I_0$ for each $m \in \mathbb{Z}^+$, in particular $a^m \in I_0$, for each $1 \leq m < k$. Then $I_0$ is a $(k, m)$ closed ideal.
The converse of the above proposition is not true in general as it is shown in the following example.
**Example 3.14.** The ideal $(x)$ of $\mathbb{Z}[x,y]$ is a $(k, m)$ closed ideal for each $m, k \in \mathbb{Z}^+$ with $1 \leq m < k$, but it is not an n-sequence prime ideal.
One can study n-sequence prime ideals in some type of rings and study its relation with some other types of ideals given in [10, 11, 12].
Recall that a non-zero non-unit element $p$ of a commutative ring R is said to be prime if for

$a, b$ in R with $p|ab$ implies $p|a$ or $p|b$, we prove the following result.

**Remark 3.15.** Let R be a principal ideal domain and $p_1, \dots, p_n$ be $n$ distinct prime elements of R. Then the ideal $(\prod_{i=1}^{n} p_i)$ is prime in $(\prod_{i=1}^{n-1} p_i)$.

**Proof.** Let $a, b \in (\prod_{i=1}^{n-1} p_i)$ and $ab \in (\prod_{i=1}^{n} p_i)$. Then there is $x \in R$ such that $ab = x \prod_{i=1}^{n} p_i$. Then $p_n$ divides $ab$. So $p_n$ divides $a$ or $b$. On the other hand, $\prod_{i=1}^{n-1} p_i$ divides each of $a, b$, since $a, b \in (\prod_{i=1}^{n-1} p_i)$. So, $\prod_{i=1}^{n} p_i$ divides $a$ or $b$. Therefore, $a \in (\prod_{i=1}^{n} p_i)$ or $b \in (\prod_{i=1}^{n} p_i)$.

Now, we introduce the concept of n-sequence quasi prime ideal.

**Definition 3.16.** A proper ideal $I_0$ of R is n-sequence quasi prime if $\sqrt{I_0}$ is an n-sequence prime ideal.

**Theorem 3.17.** Let $I_0$ be a proper ideal of a principal ideal domain $R$. Then either $I_0$ is a quasi prime ideal or it is an $n$-sequence quasi prime ideal.

**Proof.** Let $I_0 = (a)$ be a non-zero ideal of $R$. If $a = p^\alpha$, where $p$ is a prime element and $\alpha \in \mathbb{Z}^+$, then clearly the ideal $I_0$ is a quasi prime ideal. If $a = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_k}^{\alpha_k} = \prod_{i=1}^{k} {p_i}^{\alpha_i}$ where $p_i$'s are distinct primes and $k > 1$, and $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$, then $\sqrt{I_0} = \sqrt{\left(\prod_{i=1}^{k} {p_i}^{\alpha_i}\right)} = \left(\prod_{i=1}^{k} p_i\right)$. Let $J_0 = \sqrt{I_0}$ and $J_h = \left(\prod_{i=1}^{k-h} p_i\right)$ where $1 \leq h < k$. By Remark 3.15, $J_0 \subset J_1 \subset \dots$ is a prime ascending chain of ideals and stabilized at $J_{k-1} = (p_1)$. So that $J_0 = \sqrt{I_0}$ is $(k\text{-}1)$-sequence prime ideal. This means that $I_0$ is a $(k\text{-}1)$-sequence quasi prime ideal. Now, if $I_0 = (0)$, then it is a prime ideal, so it is a quasi prime but it is not an $n$-sequence prime ideal. Therefore, $I_0$ is either quasi prime or n-sequence quasi prime.

**Corollary 3.18.** Let $I_0$ be a proper ideal of a principal ideal domain $R$. Then either $\sqrt{I_0}$ is a prime ideal or it is an $n$-sequence prime ideal.

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