



N Sequence Prime Ideals

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Abstract

In this paper, the concepts of n -sequence prime ideal and n -sequence quasi prime ideal are introduced. Some properties of such ideals are investigated. The relations between n -sequence prime ideal and each of primary ideal, n -prime ideal, quasi prime ideal, strongly irreducible ideal, and (k, m) closed ideal, are studied. Also, the ideals of a principal ideal domain are classified into quasi prime ideals and n -sequence quasi prime ideals.

Keywords: Prime ascending chain of ideals; Prime with respect to; Length of an ideal; n -sequence prime ideal; n -sequence quasi prime ideal.

المثاليات الاولية المتتابعة من نمط n

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الخلاصة

في هذا البحث نقوم بعرض مفهوم المثاليات الاولية المتتابعة من نمط n و مفهوم المثاليات شبه الاولية المتتابعة من نمط n . نبحث بعض خصائصهما. اضافة الى ذلك نقوم بدراسة علاقة المثاليات الاولية المتتابعة من نمط n مع المثاليات الابتدائية ، المثاليات الاولية من نمط n ، المثاليات شبه الاولية ، المثاليات الغير قابلة للاختزال بقوة و المثاليات المغلقة من نمط (k, m) . قمنا بتصنيف مثاليات الحلقات التي هي ساحة مثاليات رئيسية الى صنفين المثاليات شبه الاولية و المثاليات شبه الاولية المتتابعة من نمط n .

1. Introduction

Throughout this paper, R is a commutative ring with identity. Let $I_0 \subset I_1$ be two proper ideals of R . We say that I_0 is prime in I_1 if for each a, b in I_1 , $ab \in I_0$ implies $a \in I_0$ or $b \in I_0$ and I_0 is prime with respect to I_1 if for each a, b in R , $ab \in I_0$ implies $a \in I_1$ or $b \in I_1$. An ascending chain of proper ideals $I_0 \subset I_1 \subset I_2 \subset I_3 \dots$ of R is called a prime ascending chain of ideals if I_{m-1} is a prime ideal in I_m for each $m \in \mathbb{Z}^+$, the set of all positive integers. We also say that I_0 is a prime ideal of length m with respect to the prime ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \dots$ if I_0 is not prime with respect to I_k for each $0 \leq k \leq m - 1$, but I_0 is prime with respect to I_m , while the prime ascending chain $I_0 \subset I_1 \subset I_2 \subset \dots$ is said to be stabilized at I_m and the ideal I_m is called a stabilizer prime ideal of the chain. Moreover, a non-prime proper ideal I_0 of R is called an n -sequence prime ideal if $n = \min\{t: t \text{ is the length of } I_0 \text{ with respect to a prime ascending chain of ideals of the form } I_0 \subset I_1 \subset I_2 \subset \dots\}$. Some important results are obtained. It is shown that for each $a \in R$ and $k \in \mathbb{Z}^+$, if $a^k \in I_0$, then

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$a \in I_0$ and consequently, $\sqrt{I_0} = I_0$; see Theorem 2.12 and Corollary 2.16. It is also shown that there are two elements $a, b \in R$ with $ab \in I_0$ but $a^k, b^k \notin I_0$ for each $k \in \mathbb{Z}^+$; see Proposition 2.23. The relations between n -sequence prime ideal with some other types of ideals, such as primary ideal, quasi prime ideal, strongly irreducible ideal, and (k, m) closed ideal, are discussed separately; see Proposition 3.1, Proposition 3.3, Theorem 3.10, and Proposition 3.13. Moreover, it is shown that the concept of n -sequence prime ideal is independent with each of weakly prime, weakly irreducible, weakly 2-absorbing, n almost prime, and 2-absorbing ideals. Finally, the concept of n -sequence quasi prime ideal is introduced (see 3.16). The family of proper ideals of a principal ideal domain is classified. We show that a proper ideal of a principal ideal domain is either quasi prime or n -sequence quasi prime.

2. n-sequence prime ideals

In this section, we introduce the concept of an n -sequence prime ideal of a commutative ring with identity and we illustrate it by some examples. We obtain some results and properties of such ideals.

We start by introducing some new concepts.

Definition 2.1. Let $A \subset B$ be two proper ideals of R . Then A is said to be a prime ideal with respect to B if for each a, b in R , $ab \in A$ implies $a \in B$ or $b \in B$.

Clearly, if A is a prime ideal of a ring R , then it is prime with respect to any ideal containing it.

Definition 2.2. A sequence of proper ideals $I_0 \subset I_1 \subset I_2 \subset I_3 \dots$ of R is called a prime (resp. p -maximal) ascending chain of ideals if I_{m-1} is a prime (resp. prime and maximal) ideal in I_m for each $m \in \mathbb{Z}^+$. A proper ideal I_0 of R is called a prime ideal of length m with respect to the prime ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \dots$, if I_0 is not prime with respect to I_k for each $0 \leq k \leq m - 1$ but it is prime with respect to I_m . Then the prime ascending chain $I_0 \subset I_1 \subset I_2 \subset \dots$ is said to be stabilize at I_m and the ideal I_m is called the stabilizer ideal of the chain.

Definition 2.3. A non-prime proper ideal I_0 of R is called an n -sequence prime ideal if $n = \min\{t: t \text{ is the length of } I_0 \text{ with respect to a prime ascending chain of ideals of the form } I_0 \subset I_1 \subset I_2 \subset \dots\}$.

The following remarks are obvious.

Remark 2.4. Let $I_0 \subset I_1 \subset I_2 \subset \dots$ be an ascending chain of ideals of R .

1. For each integer $m \geq 0$, there is $\chi \in R$ such that $I_m \subset (I_m \cup \{\chi\}) \subseteq I_{m+1}$, since there exists an element $\chi \in I_{m+1} \setminus I_m$. Moreover, if I_m is maximal in I_{m+1} , then $(I_m \cup \{\chi\}) = I_{m+1}$.
2. For each $k \in \mathbb{Z}^+$, there are k elements $\chi_1, \chi_2, \dots, \chi_{k-1}, \chi_k$ in R such that $(I_0 \cup \{\chi_1, \dots, \chi_k\}) \subseteq I_k$. Moreover, if I_m is maximal in I_{m+1} for each m , then $I_k = (I_0 \cup \{\chi_1, \dots, \chi_k\})$.

Definition 2.5 [1]. A proper ideal I_0 of R is quasi prime, if $a, b \in R$ with $ab \in I_0$ implies $a \in \sqrt{I_0}$ or $b \in \sqrt{I_0}$. Equivalently a proper ideal I_0 of R is quasi prime if $\sqrt{I_0}$ is a prime ideal.

Remark 2.6.

1. If B is a prime ideal, then every proper ideal of R contained in B is prime with respect to B .
2. An ideal I of R is quasi prime if and only if I is prime with respect to \sqrt{I} .

Remark 2.7.

1. Consider the ideal $I_0 = (m_0)$ of the ring of integers \mathbb{Z} with the prime factorization of $m_0 = p_1 \dots p_n$ with p_i are distinct primes, $1 \leq i \leq n$. Let $m_i = \frac{m_{i-1}}{p_{l_i}}$ where $p_{l_i} \in \{p_1, \dots, p_n\}$ for $1 \leq l_i \leq n$ such that $l_i \notin \{l_1, \dots, l_{i-1}\}$. Let $I_i = (m_i)$ be the ideal generated by m_i . Then the chain $I_0 \subset I_1 \subset I_2 \subset \dots$ is a prime ascending chain of ideals and it is stabilized at an ideal

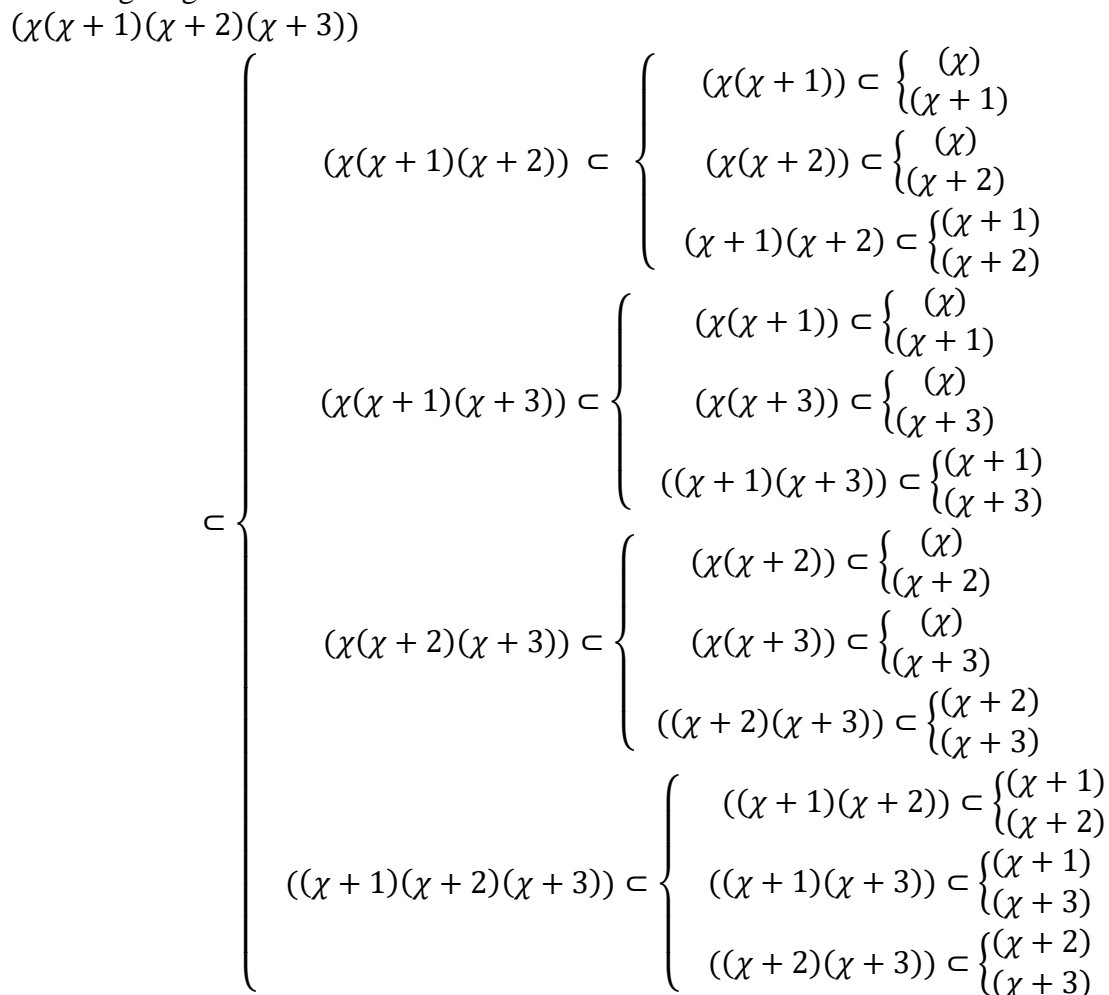
generated by p_{l_n} . The number of prime ascending chains of the form $I_0 \subset I_1 \subset I_2 \subset \dots \subset I_{n-1}$ is $n!$ and I_0 is an $n-1$ -sequence prime ideal.

2. Let $R = \mathbb{Z}[x_1, \dots, x_n]$ be the polynomial ring over \mathbb{Z} with n indeterminates and let $I_0 = (x_1 x_2 x_3 \dots x_n)$, $I_k = (\prod_{i=1}^{n-k} x_i)$ where $n \in \mathbb{Z}^+ \setminus \{1\}$ and $1 \leq k < n$. Then the chain $I_0 \subset I_1 \subset I_2 \subset \dots$ is a prime ascending chain of ideals of R that is stabilized at the ideal $I_{n-1} = (x_1)$ and I_0 is $n - 1$ -sequence prime.

Example 2.8.

1. Consider the ideals $I_0 = (2310)$, $I_1 = (210)$, $I_2 = (30)$, $I_3 = (6)$, and $I_4 = (2)$ of \mathbb{Z} . Then the chain of ideals $(2310) \subset (210) \subset (30) \subset (6) \subset (2)$ is a prime ascending chain that is stabilized at I_4 , which shows that I_0 is a 4-sequence prime ideal. Moreover, the number of such prime ascending chain is $5!$.

Consider the ideals $I_0 = (\chi(\chi + 1)(\chi + 2)(\chi + 3))$, $I_1 = (\chi(\chi + 1)(\chi + 2))$, $I_2 = (\chi(\chi + 1))$, and $I_3 = (\chi)$ of $\mathbb{Z}[\chi]$ as the polynomial ring over \mathbb{Z} with one indeterminate. Then the chain of ideals $(\chi(\chi + 1)(\chi + 2)(\chi + 3)) \subset (\chi(\chi + 1)(\chi + 2)) \subset (\chi(\chi + 1)) \subset (\chi)$ is prime ascending chain that is stabilized at I_3 , which shows that I_0 is a 3-sequence prime ideal. Moreover, the number of prime ascending chains is $4!$, which are shown in the following diagram



Proposition 2.9. Let I_0 be an n -sequence prime ideal of R . If $a \in R$ with $a^2 \in I_0$, then $a \in I_0$. Proof. Since I_0 is an n -sequence prime ideal of R , then there is a prime ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \dots$ with stabilizer ideal I_n . Suppose that $a^2 \in I_0$. Then $a \in I_n$, since I_0 is prime with respect to I_n . So $a \in I_{n-1}$, since I_{n-1} prime in I_n . For the same reason, $a \in I_k$, $0 \leq k \leq n - 2$.

Corollary 2.10. Let I_0 be a proper ideal of R . If $a \in R$ with $a^2 \in I_0$ but $a \notin I_0$, then I_0 is not

an n -sequence prime ideal of R .

Proposition 2.11. Let I_0 be an n -sequence prime ideal of R . For any two elements x, a in R , if $xa^2 \in I_0$, then $xa \in I_0$.

Proof. Let $xa^2 \in I_0$. Then $(xa)^2 \in I_0$. By proposition 2.9, $xa \in I_0$.

Theorem 2.12. Let I_0 be an n -sequence prime ideal of R . If $a \in R$ with $a^k \in I_0$, where $k > 1$, then $a \in I_0$.

Proof. Suppose that $a^k \in I_0$ for some $k > 1$ and let $k_1 = \begin{cases} k & \text{if } k \text{ is even} \\ k + 1 & \text{if } k \text{ is odd} \end{cases}$. Then

$a^{k_1} \in I_0$. By Proposition 2.9, $a^{\binom{k_1}{2}} \in I_0$. If $\frac{k_1}{2} = 1$, then the proof is complete. If $\frac{k_1}{2} > 1$, let

$k_2 = \begin{cases} \frac{k_1}{2} & \text{if } \frac{k_1}{2} \text{ is even} \\ \frac{k_1}{2} + 1 & \text{if } \frac{k_1}{2} \text{ is odd} \end{cases}$. Then $a^{k_2} \in I_0$. Also by Proposition 2.9, $a^{\binom{k_2}{2}} \in I_0$. By iterating

these steps, we obtain $a \in I_0$.

Definition 2.13 [2]. If I_0 is an ideal of R , then the radical of I_0 denoted by $\sqrt{I_0}$ is

$\sqrt{I_0} = \{x \in R; x^n \in I_0 \text{ for some } n \in \mathbb{Z}^+\}$, which is an ideal of R .

Definition 2.14 [3]. The nilradical of R ($\text{radical}(R)$) is the set of all nilpotent elements in R which forms an ideal of R . Equivalently, $\text{radical}(R) = \sqrt{(0)}$ is the radical of the zero ideal.

Corollary 2.15. Let I_0 be an ideal of R and $a \in R$. If $a^k \in I_0, k > 1$, and $a \notin I_0$, then I_0 is not an n -sequence prime ideal of R . Equivalently, if $a \in \sqrt{I_0}$ and $a \notin I_0$, then I_0 is not an n -sequence prime ideal of R .

Corollary 2.16. If I_0 is an ideal of R such that $\sqrt{I_0} \neq I_0$, then I_0 is not an n -sequence prime. Equivalently, if I_0 is an n -sequence prime ideal, then $\sqrt{I_0} = I_0$.

The following remark shows that the converse of Corollary 2.16. is not true in general.

Remark 2.17. If I_0 is an ideal of R and $\sqrt{I_0} = I_0$, then I_0 may not be an n -sequence prime ideal, for example the ideal $I_0 = (2)$ of \mathbb{Z} , then I_0 is a prime ideal, so $\sqrt{I_0} = \sqrt{(2)} = (2)$, but I_0 is not an n -sequence prime ideal.

Example 2.18. By Corollary 2.15, we obtain that

1. The ideal $I_0 = (p^k), p$ is a prime number and $k > 1$ is not an n -sequence prime ideal of \mathbb{Z} .
2. For each prime number p and $k \in \mathbb{Z}^+$, the ring \mathbb{Z}_{p^k} has no n -sequence prime ideal.
3. The ideal $(x^k), k > 1$ of the ring $R = \mathbb{Z}[x]$ is not n -sequence prime.

Proposition 2.19. Let I_0 be an n -sequence prime ideal of R . Then the radical of R is contained in I_0 .

Proof. If x is a nilpotent element of R , then $x^n = 0 \in I_0$ for some $n \in \mathbb{Z}^+$. By Theorem 2.12, $x \in I_0$, which means the radical of R is contained in I_0 .

Proposition 2.20. Let I_0 be an n -sequence prime ideal of R . If $\chi = \chi_1^{\alpha_1} \chi_2^{\alpha_2} \dots \chi_k^{\alpha_k} \in I_0$, then $\chi_1 \chi_2 \dots \chi_k \in I_0$, where $\chi_i \in R$ and $\alpha_i \in \mathbb{Z}^+$ for each $1 \leq i \leq k$. Equivalently, if $\chi = \chi_1^{\alpha_1} \chi_2^{\alpha_2} \dots \chi_k^{\alpha_k} \in I_0$ and $\chi_1 \chi_2 \dots \chi_k \notin I_0$, then I_0 is not an n -sequence prime ideal of R .

Proof. Let $\chi_1^{\alpha_1} \chi_2^{\alpha_2} \dots \chi_k^{\alpha_k} \in I_0$ and $\alpha = \text{Max}\{\alpha_i; 1 \leq i \leq k\}$. Then $\chi_1^\alpha \chi_2^\alpha \dots \chi_k^\alpha = (\chi_1 \chi_2 \dots \chi_k)^\alpha \in I_0$. By Theorem 2.12, $\chi_1 \chi_2 \dots \chi_k \in I_0$.

Proposition 2.21. Let I_0 be an n -sequence prime ideal of R and $I_0 \subset I_1 \subset I_2 \subset \dots$ be a prime ascending chain of ideals with stabilizer ideal I_n . Then, for each $m \in \mathbb{Z}^+$, there exists an element $a \in R$ such that $a \in I_m$, but $a^k \notin I_{m-1}$. Moreover, if b divides a , then $b^k \notin I_{m-1}$ for each $k \in \mathbb{Z}^+$.

Proof. Since $I_{m-1} \subset I_m$ for each $m \in \mathbb{Z}^+$, then there exists an element $a \in I_m$ but $a \notin I_{m-1}$. If $a^2 \in I_{m-1}$, then $a \in I_{m-1}$, since I_{m-1} is prime in I_m , which is a contradiction with the assumption $a \notin I_{m-1}$. Hence, $a^2 \notin I_{m-1}$. If $a^k = aa^{k-1} \in I_{m-1}$ for some $k > 2$, then

$a \in I_{m-1}$ or $a^{k-1} \in I_{m-1}$, since I_{m-1} is prime in I_m . Since $a \notin I_{m-1}$, then $a^{k-1} \in I_{m-1}$. By iterating this step, we obtain $a \in I_{m-1}$, which is a contradiction. Therefore, $a^k \notin I_{m-1}$ for each $k \in \mathbb{Z}^+$. Now, suppose that b divides a , then there exists an element x in R such that $a = xb$. If $b^k \in I_{m-1}$ for some $k \in \mathbb{Z}^+$, then $x^k b^k \in I_{m-1}$, implies that $a^k \in I_{m-1}$. This is a contradiction with $a^k \notin I_{m-1}$.

Proposition 2.22. Let I_0 be an n -sequence prime ideal of R and $I_0 \subset I_1 \subset I_2 \subset \dots$ be a prime ascending chain of ideals with stabilizer ideal I_n . If $a, b \in R$ with $ab \in I_0$ and $a, b \notin I_{n-1}$, then I_n contains exactly one of a or b .

Proof. Let $a, b \in R$ with $ab \in I_0$ and $a, b \notin I_{n-1}$. Since I_n is stabilizer, I_0 is prime with respect to I_n . Then $a \in I_n$ or $b \in I_n$. Suppose that both a and b are in I_n . Since I_{n-1} is prime in I_n , then $a \in I_{n-1}$ or $b \in I_{n-1}$, which is a contradiction with the assumption $a, b \notin I_{n-1}$.

Proposition 2.23. Let I_0 be an n -sequence prime ideal of R . Then there are two elements $a, b \in R$ with $ab \in I_0$ but $a^k, b^k \notin I_0$ for each $k \in \mathbb{Z}^+$. Moreover, if $c \in R$ divides a or b , then $c^k \notin I_0$ for each $k \in \mathbb{Z}^+$.

Proof. Since I_0 is an n -sequence prime ideal of R , then I_0 is not a prime ideal. This means that there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin I_0$. By Corollary 2.15, if $a^k \in I_0$ for some $k \in \mathbb{Z}^+$, then I_0 is not an n -sequence prime ideal, which is a contradiction. Similarly, we get a contradiction if $b^k \in I_0$. Suppose that $c \in R$ divides a and $c^k \in I_0$ for some $k \in \mathbb{Z}^+$. Then there is an elements x in R such that $a = xc$. If $c^k \in I_0$, then $x^k c^k \in I_0$, implies that $a^k \in I_0$. This is a contradiction with $a^k \notin I_0$.

Corollary 2.24. If I_0 is an n -sequence prime ideal of R , then there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin \sqrt{I_0}$.

Corollary 2.25. Let I_0 be an n -sequence prime ideal of R and $I_0 \subset I_1 \subset I_2 \subset \dots$ be a prime ascending chain of ideals with stabilizer ideal I_n . Then $I_n \neq \sqrt{I_0}$.

Proof. Since I_0 is an n -sequence prime ideal of R , then by Corollary 2.24, there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin \sqrt{I_0}$. Since I_n is the stabilizer ideal of the given prime ascending chain, then $a \in I_n$ or $b \in I_n$. This means that there is an element in I_n but not in $\sqrt{I_0}$. Therefore, $I_n \neq \sqrt{I_0}$.

3. Relations between n -sequence prime ideals and some types of ideals

In this section, we study the relation between an n -sequence prime ideal and each of primary ideal, n -prime ideal, quasi prime ideal, strongly irreducible ideal, and (k, m) closed ideal. It is shown that the concept of n -sequence prime ideal is independent of each of weakly prime, weakly irreducible, weakly 2-absorbing, n almost prime, and 2-absorbing ideals. Moreover, we introduce the concept of n -sequence quasi prime ideal and classify the family of proper ideals for a principal ideal domain. We show that a proper ideal of a principal ideal domain is either quasi prime or n -sequence quasi prime.

Proposition 3.1. If I_0 is an n -sequence prime ideal of R , then it is not a primary ideal. Equivalently, if I_0 is a primary ideal, then it is not an n -sequence prime ideal.

Proof. By Proposition 2.23, there are two elements $a, b \in R$ with $ab \in I_0$ but $a^k, b^k \notin I_0$ for each $k \in \mathbb{Z}^+$. Therefore, I_0 is not a primary ideal.

The following is an example for an ideal which is neither primary nor n sequence prime.

Example 3.2. Consider the ideal $I = (x, y^2 z^3)$ of the polynomial ring $\mathbb{Z}[x, y, z]$. Then I is neither n -sequence prime nor primary.

Proposition 3.3. If I_0 is an n -sequence prime ideal of R , then it is not a quasi prime ideal. Equivalently, if I_0 is a quasi prime ideal of R , then it is not an n -sequence prime ideal.

Proof. Let I_0 be an n -sequence prime ideal. Then by Corollary 2.24, there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin \sqrt{I_0}$. Therefore, I_0 is not a quasi prime ideal.

The following is an example of an ideal which is neither n -sequence prime nor quasi prime.

Example 3.4. The ideal (xy^2) of $\mathbb{Z}[x, y]$ is neither n -sequence prime nor quasi prime.

Definition 3.5 [4]. A proper ideal I_0 of R is 2-prime (resp. m -prime, $m \in \mathbb{Z}^+$) if $ab \in I_0$, implies $a^2 \in I_0$ or $b^2 \in I_0$ (resp. $a^m \in I_0$ or $b^m \in I_0$).

Proposition 3.6. If I_0 is an n -sequence prime ideal of R , then it is not an m -prime ideal, for each $m \in \mathbb{Z}^+$.

Proof. The proof is similar to the proof of Proposition 3.1.

The following definitions are needed.

Definition 3.7 [5]. A proper ideal I_0 of R is weakly prime (resp. almost prime and n almost prime), if $a, b \in R$, with $ab \in I_0 \setminus \{0\}$ (resp. $ab \in I_0 \setminus I_0^2$ and $ab \in I_0 \setminus I_0^n$; $n > 2$), implies $a \in I_0$ or $b \in I_0$.

Definition 3.8 [6]. A proper ideal I of R is said to be a 2-absorbing (resp. weakly 2-absorbing) ideal of R if $a, b, c \in R$ and $abc \in I$ (resp. $abc \in I \setminus \{0\}$), then $ab \in I$ or $bc \in I$ or $ac \in I$.

Definition 3.9 [7], [8]. Let I be a proper ideal of R . Then I is strongly irreducible (resp. weakly irreducible), if for each pair of ideals A and B of R , $A \cap B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ (resp. $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$) and I is strongly 2-irreducible, if for each ideals A, B and C of R , $A \cap B \cap C \subseteq I$ implies $A \cap B \subseteq I$ or $A \cap C \subseteq I$ or $B \cap C \subseteq I$.

Theorem 3.10. If I_0 is an n -sequence prime ideal of R , then it is not a strongly irreducible ideal. Equivalently, if I_0 is a strongly irreducible ideal of R , then it is not an n -sequence prime ideal.

Proof. Let I_0 be an n -sequence prime ideal of R . Then there are two elements $a, b \in R$ with $ab \in I_0$ but $a, b \notin I_0$. So $(ab) \subseteq I_0$ and, consequently, $\sqrt{(ab)} \subseteq \sqrt{I_0}$. By Corollary 2.16, $I_0 = \sqrt{I_0}$, then $\sqrt{(ab)} \subseteq I_0$. Clearly, $(ab) = (a)(b)$ and $\sqrt{(a)(b)} = \sqrt{(a)} \cap \sqrt{(b)} = \sqrt{(a)} \cap \sqrt{(b)}$ [2], then $\sqrt{(a)} \cap \sqrt{(b)} \subseteq I_0$. On the other hand, $a, b \notin I_0$, then $\sqrt{(a)}, \sqrt{(b)} \notin I_0$. Therefore, I_0 is not strongly irreducible.

Remark 3.11. The concept of n -sequence prime ideal is independent with each of weakly prime, weakly 2-absorbing, weakly irreducible, 2-absorbing, almost prime, strongly 2-irreducible, and n almost prime ideals. For example the ideal (30) of \mathbb{Z} is a 2-sequence prime ideal but it is not any one of weakly prime, weakly 2-absorbing, weakly irreducible, 2-absorbing, almost prime, strongly 2-irreducible, and n almost prime ideals. On the other hand, the ideal (5) of \mathbb{Z} is a weakly prime, weakly 2-absorbing, weakly irreducible, 2-absorbing, almost prime, strongly 2-irreducible, and n almost prime ideal, but it is not an n -sequence prime ideal for each $n \geq 1$.

Definition 3.12 [9]. Let $m, k \in \mathbb{Z}^+$ with $1 \leq m < k$. A proper ideal I of R is a (k, m) closed ideal if whenever $a^k \in I$ for some $a \in R$ implies $a^m \in I$.

Proposition 3.13. If I_0 is an n -sequence prime ideal of R , then it is a (k, m) closed ideal for some $m, k \in \mathbb{Z}^+$ and $1 \leq m < k$.

Proof. Let I_0 be an n -sequence prime ideal. To show that I_0 is (k, m) closed, we have to show that if $a^k \in I_0$ for some $a \in R$ and $k \in \mathbb{Z}^+$, then $a^m \in I_0$ for each $1 \leq m < k$. Suppose that $a^k \in I_0$ for some $a \in R$ and $k \in \mathbb{Z}^+$. Then by Corollary 2.12, $a \in I_0$, which implies that $a^m \in I_0$ for each $m \in \mathbb{Z}^+$, in particular $a^m \in I_0$, for each $1 \leq m < k$. Then I_0 is a (k, m) closed ideal.

The converse of the above proposition is not true in general as it is shown in the following example.

Example 3.14. The ideal (x) of $\mathbb{Z}[x, y]$ is a (k, m) closed ideal for each $m, k \in \mathbb{Z}^+$ with $1 \leq m < k$, but it is not an n -sequence prime ideal.

One can study n -sequence prime ideals in some type of rings and study its relation with some other types of ideals given in [10, 11, 12].

Recall that a non-zero non-unit element p of a commutative ring R is said to be prime if for

a, b in R with $p|ab$ implies $p|a$ or $p|b$, we prove the following result.

Remark 3.15. Let R be a principal ideal domain and p_1, \dots, p_n be n distinct prime elements of R . Then the ideal $(\prod_{i=1}^n p_i)$ is prime in $(\prod_{i=1}^{n-1} p_i)$.

Proof. Let $a, b \in (\prod_{i=1}^{n-1} p_i)$ and $ab \in (\prod_{i=1}^n p_i)$. Then there is $x \in R$ such that $ab = x \prod_{i=1}^n p_i$. Then p_n divides ab . So p_n divides a or b . On the other hand, $\prod_{i=1}^{n-1} p_i$ divides each of a, b , since $a, b \in (\prod_{i=1}^{n-1} p_i)$. So, $\prod_{i=1}^n p_i$ divides a or b . Therefore, $a \in (\prod_{i=1}^n p_i)$ or $b \in (\prod_{i=1}^n p_i)$.

Now, we introduce the concept of n -sequence quasi prime ideal.

Definition 3.16. A proper ideal I_0 of R is n -sequence quasi prime if $\sqrt{I_0}$ is an n -sequence prime ideal.

Theorem 3.17. Let I_0 be a proper ideal of a principal ideal domain R . Then either I_0 is a quasi prime ideal or it is an n -sequence quasi prime ideal.

Proof. Let $I_0 = (a)$ be a non-zero ideal of R . If $a = p^\alpha$, where p is a prime element and $\alpha \in \mathbb{Z}^+$, then clearly the ideal I_0 is a quasi prime ideal. If $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = \prod_{i=1}^k p_i^{\alpha_i}$ where p_i 's are distinct primes and $k > 1$, and $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$, then $\sqrt{I_0} = \sqrt{(\prod_{i=1}^k p_i^{\alpha_i})} = (\prod_{i=1}^k p_i)$. Let $J_0 = \sqrt{I_0}$ and $J_h = (\prod_{i=1}^{k-h} p_i)$ where $1 \leq h < k$. By Remark 3.15, $J_0 \subset J_1 \subset \dots$ is a prime ascending chain of ideals and stabilized at $J_{k-1} = (p_1)$. So that $J_0 = \sqrt{I_0}$ is $(k-1)$ -sequence prime ideal. This means that I_0 is a $(k-1)$ -sequence quasi prime ideal. Now, if $I_0 = (0)$, then it is a prime ideal, so it is a quasi prime but it is not an n -sequence prime ideal. Therefore, I_0 is either quasi prime or n -sequence quasi prime.

Corollary 3.18. Let I_0 be a proper ideal of a principal ideal domain R . Then either $\sqrt{I_0}$ is a prime ideal or it is an n -sequence prime ideal.

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