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Continuous Classical Optimal Control of Triple Nonlinear Parabolic Partial Differential Equations

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Abstract
This paper concerns with the state and proof the existence and uniqueness theorem of triple state vector solution (TSVS) for the triple nonlinear parabolic partial differential equations (TNPPDEs) ,and triple state vector equations (TSVEs), under suitable assumptions. when the continuous classical triple control vector (CCTCV) is given by using the method of Galerkin (MGA). The existence theorem of a continuous classical optimal triple control vector (CCTOCV) for the continuous classical optimal control governing by the TNPPDEs under suitable conditions is proved.

Keywords: Continuous Classical Triple Optimal Control Vector, Nonlinear Triple Parabolic Boundary Value Problem.

سيطرة مستمرة تقليدية مثلى لثلاثي من المعادلات التفاضلية الجزئية غير الخطية المكافئة

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الخلاصة
يهتم هذا البحث بكتابة نص و ببرهان مبرهنة وجود ووحدانية الحل لمتجه الحالة الثلاثي لثلاثي من المعادلات التفاضلية الجزئية المكافئة غير الخطية (معادلات متجه الحالة الثلاثية) بوجود شروط مناسبة وعندما يكون متجه السيطرة التقليدية المستمرة معلوما" باستخدام طريقة كالركن . تم برهان مبرهنة وجود متجه سيطرة امثلية ثلاثية لمسالة السيطرة الامثلية المستمرة التقليدية المسيطرة بثلاثي من المعادلات التفاضلية الجزئية المكافئة غير الخطية بوجود شروط مناسبة.

1. Introduction

The subject of optimal control problem is divided in to two types, the relaxed and the classical optimal control problems, the first type is mostly studied in the last century, while the second one began to study in the beginning of this century. On other hand each of these two types are studied for systems governing by ordinary or partial differential equations. The optimal control problems play an important role in many fields in life problems, different examples for applications of such problems are studied in medicine [1], in aircraft [2], in electric power [3], in economic growth [4], and many other fields.

This role motivates many investigators in the recent years to interest about study the classical optimal control problems OPCTP that are governing by nonlinear ordinary

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differential equations as [5], or by different types of nonlinear parabolic PDEs like " single" nonlinear parabolic PDEs (NLPPDEs) [6], or couple NLPPDEs (CNLPPDEs) [7], or triple linear PPDEs (TLPPDEs) [8]. On the other hand other investigators interested to study the OPCTP for CNLPPDEs and TLPPDEs but involving Neumann boundary conditions (NBCs) as [9] and [10] respectively.

All these investigations encourage us to seek about the OPCTP for triple nonlinear parabolic PDEs (TNLEPDEs). At first, our aim in this work is to state and to prove that the TNLEPDEs with a given CCTCV has a unique TSVS under a suitable conditions, by using the MGA with the compactness theorem (COMTH). The continuity of the Lipschitz operator between the TSVS, and the corresponding CCTCV are proved. Finally, we also prove theorem which ensures the existence CCTOCV for the TNLEPDEs.

2. Problem Description

Let $I=(0,T)$, $T<\infty$, $\Omega\subset\mathbb{R}^3$ be a bounded open region with Lipschitz (LIP) boundary $\Gamma = \partial\Omega$, $Q=\Omega\times I$, $\Sigma=\Gamma\times I$. Consider the following CCTOCP:

The TSVEs is given by the following TNPPDEs:

$$y_{1t}-\Delta y_1+y_1-y_2-y_3=f_1(x,t,y_1,u_1) \quad \text{in } Q \tag{1}$$
$$y_{2t}-\Delta y_2+y_2+y_3+y_1=f_2(x,t,y_2,u_2) \quad \text{in } Q \tag{2}$$
$$y_{3t}-\Delta y_3+y_3+y_1-y_2=f_3(x,t,y_3,u_3) \quad \text{in } Q \tag{3}$$
$$y_1(x,t)=0 \quad \text{on } \Sigma \tag{4}$$
$$y_1(x,0)=y_1^0(x) \quad \text{on } \Omega \tag{5}$$
$$y_2(x,t)=0 \quad \text{on } \Sigma \tag{6}$$
$$y_2(x,0)=y_2^0(x) \quad \text{on } \Omega \tag{7}$$
$$y_3(x,t)=0 \quad \text{on } \Sigma \tag{8}$$
$$y_3(x,0)=y_3^0(x), \quad \text{on } \Omega \tag{9}$$

Where $x=(x_1,x_2)$, $\vec{y}=(y_1,y_2,y_3)=(y_1(x,t),y_2(x,t),y_3(x,t))\in\left(H_2(Q)\right)^3$ is the triple state vector (TSVS), corresponding to classical triple control vector (CCTCV) $\vec{u}=(u_1,u_2,u_3)=(u_1(x,t),u_2(x,t),u_3(x,t))\in(L^2(Q))^3$ and $(f_1,f_2,f_3)=(f_1(x,t,y_1,u_1),f_2(x,t,y_2,u_2),f_3(x,t,y_3,u_3))\in(L^2(Q))^3$ is a vector of given function defined on $(Q\times\mathbb{R}\times U_1)\times(Q\times\mathbb{R}\times U_2)\times(Q\times\mathbb{R}\times U_3)$ with $U_i\subset\mathbb{R}$, and let $\overrightarrow{W}=W_1\times W_2\times W_3$, $W_i\subset L^2(Q)$, $i=1,2,3$.

The set of admissible CCTCV (ADCCTCV) is

$$\overrightarrow{W}_A=\left\{\vec{w}\in\left(L^2(Q)\right)^3 \mid \vec{w}\in\vec{U} \text{ a.e. in } Q\right\} \text{with } \vec{U}\subset\mathbb{R}^3 \tag{10}$$

The cost function (COF) is

$$G_0(\vec{u})=\sum_{i=1}^{3}\int_Q g_{0i}(x,t,y_i,u_i)\,dxdt \tag{11}$$

The CCTOCV is to find $\vec{u}\in\overrightarrow{W}_A$, s.t.

$$G_0(\vec{u})=min_{\vec{w}\in\overrightarrow{W}_A} G_0(\vec{w}) \tag{12}$$

Let $\vec{V}=V_1\times V_2\times V_3=\{\vec{v}\in(H^1(\Omega))^3 \text{with } v_1=v_2=v_3=0 \text{ on } \partial\Omega\}$.

The notations (v,v), and $\|v\|_0$ refer to the inner product and the norm in $L^2(\Omega)$, respectively. The notations $(v,v)_1$, and $\|v\|_1$ are the inner product and the norm in $H^1(\Omega)$, the $(\vec{v},\vec{v})$ and $\|\vec{v}\|_0$ the inner product and the norm in $(L^2(\Omega))^3$, and $(\vec{v},\vec{v})_1=(v_1,v_1)_1+(v_2,v_2)_1+(v_3,v_3)_1$, $\|\vec{v}\|_1^2=\|v_1\|_1^2+\|v_2\|_1^2+\|v_3\|_1^2$ the inner product and the norm in $\vec{V}$ and $\vec{V}^*$ is the dual of $\vec{V}$, also the notations $\rightharpoonup$, $\longrightarrow$ will indicate to the convergence of a sequence is weakly and strongly respectively.

The weak form (W.F) of the TSVEs (1-9) when $\vec{y}\in H_0^1(\Omega))^3$ is given by

$$\langle y_{1t},v_1\rangle+(\nabla y_1,\nabla v_1)+(y_1,v_1)-(y_2,v_1)-(y_3,v_1)=(f_1,v_1) \tag{13a}$$
$$(y_1^0,v_1)=(y_1(0),v_1), \quad \forall v_1\in V \tag{13b}$$
$$\langle y_{2t},v_2\rangle+(\nabla y_2,\nabla v_2)+(y_2,v_1)+(y_3,v_2)+(y_1,v_2)=(f_2,v_2) \tag{14a}$$

$(y_2^0, v_2) = (y_2(0), v_2), \quad \forall v_2 \in V$ (14b)
$\langle y_{3t}, v_3 \rangle + (\nabla y_3, \nabla v_3) + (y_3, v_3) + (y_1, v_3) - (y_2, v_3) = (f_3, v_3)$ (15a)
$(y_2^0, v_3) = (y_3(0), v_3), \quad \forall v_3 \in V$ (15b)

Assumptions (A):

(i) f_i is Carathéodory type (CAT) on $Q \times (\mathbb{R} \times \mathbb{R})$, satisfies

$|f_i(x,t,y_i,u_i)| \leq \eta_i(x,t) + c_i|y_i| + \acute{c}_i|u_i|$

where $(x,t) \in Q, y_i, u_i \in \mathbb{R}, c_i, \acute{c}_i > 0$ *and* $\eta_i \in L^2(Q)$ $\forall$ i =1,2,3

(ii) f_i is Lip w.r.t. y_i, i.e. $|f_i(x,t,y_i,u_i) - f_i(x,t,\bar{y}_i,u_i)| \leq L_i|y_i - \bar{y}_i|$

where $(x,t) \in Q, y_i, \bar{y}_i, u_i \in \mathbb{R}$ and $L_i > 0$, $\forall$ $i = 1,2,3$.

Theorem 2.1 (Projection Theorem) [7]: Let $\mathcal{F}$ be a closed linear subspace of a Hilbert space $\mathcal{H}$, then for any $h \in \mathcal{H}$ there is a unique $u_0 \in \mathcal{F}$, s.t. $\|h - u_0\| \leq \|h - u\|, \quad \forall u \in \mathcal{F}$. Furthermore, $h - u_0$ is orthogonal to the subspace $\mathcal{F}$, i.e. $\langle h - u_0, u \rangle = 0, \forall u \in \mathcal{F}$.

Theorem 2.2 (Alaoglu's theorem) [7]: Let $\{k_n\}_{n\in\mathbb{N}}$ be a bounded sequence in a Hilbert space $\mathcal{H}$, then there is a subsequence of $\{k_n\}_{n\in\mathbb{N}}$, which converges weakly to some $u \in \mathcal{H}$.

Main Results

3. The TSVS:

Theorem (3.1): Existence and Uniqueness Of The W.F: With Assumptions (A) for each $\vec{u} \in (L^2(\Omega))^3$, the W.F of TSVEs (13-15) has a unique solution $\vec{y} = (y_1, y_2, y_3), \vec{y} \in (L^2(I,V))^3, s.t\ \vec{y}_t = (y_{1t}, y_{2t}, y_{3t}) \in (L^2(I,V^*))^3$

Proof: Let $V_n$be the set of piecewise affine function in Ω, $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n})$ with $v_{in} \in V_n$,$\forall$ $i = 1,2,3$ and $\vec{y}_n$=$(y_{1n}, y_{2n}, y_{3n}), \forall\, n$, then the solution of $\vec{y} = (y_1, y_2, y_3)$ can be approximated by

$y_{1n} = \sum_{j=1}^{n} c_{1j}(t)\, v_{1j}(x)$ (16)
$y_{2n} = \sum_{j=1}^{n} c_{2j}(t)\, v_{2j}(x)$ (17)
$y_{3n} = \sum_{j=1}^{n} c_{3j}(t)\, v_{3j}(x)$ (18)

Where $c_{ij}(t)$ is known as function of t $\forall i = 1,2,3$ &$\forall$ $j = 1,2,3, \dots, n$.

The W.F of the TSVEs (13-15) is approximated w.r.t. *x*, using the MGA $\forall v_i \in V$, $i = 1,2,3$:

$\langle y_{1nt}, v_1 \rangle + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) - (y_{3n}, v_1) = (f_1(y_{1n}, u_1), v_1)$ (19a)
$(y_{1n}^0, v_1) = (y_1^0, v_1)$ (19b)
$\langle y_{2nt}, v_2 \rangle + (\nabla y_{2n}, \nabla v_2) + (y_{2n}, v_2) + (y_{3n}, v_2) + (y_1, v_2) = (f_2(y_{2n}, u_2), v_2)$ (20a)
$(y_{2n}^0, v_2) = (y_2^0, v_2)$ (20b)
$\langle y_{3nt}, v_3 \rangle + (\nabla y_{3n}, \nabla v_3) + (y_{3n}, v_3) + (y_{1n}, v_3) - (y_{2n}, v_3) = (f_3(y_{3n}, u_3), v_3)$ (21a)
$(y_{3n}^0, v_3) = (y_3^0, v_3)$ (21b)

Where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n \subset V \subset L^2(\Omega)$ is the projection of y_i^0 w.r.t. the norm $\|\,.\,\|_0$, i.e. $(y_{in}^0, v_i) = (y_i^0, v_i), \Leftrightarrow \|y_{in}^0 - v_i\|_0 \leq \|y_i^0 - v_i\|_0, \quad \forall i = 1,2,3$ and $\forall v_i \in V_n$.

By substituting ((16)-(18)) in ((19) - (21)) and setting $v_1 = v_{1i}$, $v_2 = v_{2i}$, and $v_3 = v_{3i}$ we get the following system, which has a unique solution $\vec{y}_n$ because of all the coefficient matrices are continuous.

$A\acute{C}_1(t) + DC_1(t) - EC_2(t) - KC_3(t) = b_1(\bar{V}_1^T(x)\, C_1(t))$ (19a')
$AC_1(0) = b_1^0$ (19b')
$B\acute{C}_2(t) + FC_2(t) + MC_3(t) + HC_1(t) = b_2(\bar{V}_2^T(x)\, C_2(t))$ (18a')
$BC_2(0) = b_2^0$ (18b')
$P\acute{C}_3(t) + OC_3(t) + SC_1(t) - ZC_2(t) = b_3(\bar{V}_3^T(x)\, C_3(t))$ (19a')
$PC_3(0) = b_3^0$ (19$\acute{b}$)

Where $A = (a_{ij})_{n\times n}$, , $a_{ij} = (v_{1j}, v_{1i}), D = (d_{ij})_{n\times n}, d_{ij} = [(\nabla v_{1j}, \nabla v_{1i}) + (v_{1j}, v_{1i})]$, $E = (e_{ij})_{n\times n}$, $e_{ij} = (v_{2j}, v_{1i})$, $K = (k_{ij})_{n\times n}$, $k_{ij} = (v_{3j}, v_{1i})$, $C_l(t) = (C_{lj}(t))_{n\times 1}, \acute{C}_l(t) = \left(\acute{C}_{lj}(t)\right)_{n\times 1}, C_l(0) = (C_{lj}(0))_{n\times 1}, b_l = (b_{li})_{n\times 1}, b_{li} =$

$(f_l(\bar{V}_l^T\, C_l(t), u_l), v_{li}), \bar{V}_l = (v_l)_{n\times 1}$, $b_l^0 = (b_{li}^0)$, $(b_{li}^0) = (y_l^0, v_{li})$, , $B = \left(b_{ij}\right)_{n\times n}$ $b_{ij} = (v_{2j}, v_{2i})$, $F = \left(f_{ij}\right)_{n\times n}$, $f_{ij} = \left[\left(\nabla v_{2j}, \nabla v_{2i}\right) + \left(v_{2j}, v_{2i}\right)\right]$, $, M = \left(m_{ij}\right)_{n\times n}, m_{ij} = \left(v_{3j}, v_{2i}\right), H = \left(h_{ij}\right)_{n\times n}, h_{ij} = (v_{1j}, v_{2i})$, $P = (p_{ij})_{n\times n}$, $p_{ij} = \left(v_{3j}, v_{3i}\right), O = (o_{ij})_{n\times n}, o_{ij} = \left[\left(\nabla v_{3j}, \nabla v_{3i}\right) + \left(v_{3j}, v_{3i}\right)\right], S = \left(s_{ij}\right)_{n\times n}, s_{ij} = \left(v_{1j}, v_{3i}\right), Z = \left(z_{ij}\right)_{n\times n}, z_{ij} = \left(v_{2j}, v_{3i}\right), \forall l = 1,2,3$.

The norm $\|\vec{y}_n^0\|_0$ ***is bounded:*** Since $\vec{y}^0 \in (L^2(\Omega))^3$ then there is $\{\vec{v}_n^0\}, \vec{v}_n^0 \in \vec{V}_n$ *s.t* $\vec{v}_n^0 \to \vec{y}^0$ in $(L^2(\Omega))^3$, from theorem 2.1 and ((19b)-(21b)) one has $\vec{y}_n^0 \to \vec{y}^0$ in $(L^2(\Omega))^3$,and $\|\vec{y}_n^0\|_0 \le b_1$.

The norm $\|\vec{y}_n(t)\|_{L^\infty\left(I,L^2(\Omega)\right)}$ ***and*** $\|\vec{y}_n(t)\|_Q$ ***are bounded:*** Setting $v_i = y_{in}$, for $i = 1,2,3$ in (19a), (20a),and (20a), integrating both sides (IBS) w.r.t. t from 0 to T, adding them, this gives

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n \rangle\, dt + \int_0^T \|\vec{y}_n\|_1^2\, dt = \int_0^T [f_1(y_{1n}, u_1), y_{1n}) + f_2(y_{2n}, u_2), y_{2n}) + f_3(y_{3n}, u_3), y_{3n})]\, \mathrm{dt} \tag{22}$$

Since the 2$^{\text{nd}}$ term of the L.H.S. of (22) is positive, then using Lemma 1.2 in [11] for the 1$^{\text{st}}$ term of it, taking T= t ∈ [0. T], finally applying Assum (A-i) for the R.H.S. of (22), one has

$$\begin{aligned}\int_0^t \frac{d}{dt} ||\vec{y}_n(t)||_0^2\, dt \le{}& \int_0^t \int_\Omega (\eta_1^2 + |y_{1n}|^2) dxdt + 2\int_0^t \int_\Omega c_1 |y_{1n}|^2\, dxdt + \acute{c}_1 \int_0^t \int_\Omega (|u_1|^2 + |y_{1n}|^2) dxdt \\ &+ \int_0^t \int_\Omega (\eta_2^2 + |y_{2n}|^2)\, dxdt + 2\int_0^t \int_\Omega c_2 |y_{2n}|^2\, dxdt + 2\int_0^t \int_\Omega c_2 |y_{2n}|^2\, dxdt \\ &+ \acute{c}_2 \int_0^t \int_\Omega (|u_2|^2 + |y_{2n}|^2) dxdt + \int_0^t \int_\Omega (\eta_3^2 + |y_{3n}|^2) dxdt + \int_0^t \int_\Omega (\eta_3^2 + |y_{3n}|^2) dxdt \\ &+ 2\int_0^t \int_\Omega c_3 |y_{3n}|^2\, dxdt + \acute{c}_3 \int_0^t \int_\Omega (|u_3|^2 + |y_{3n}|^2) dxdt \end{aligned} \tag{23}$$

Since $\|\eta_i\|_Q \le \acute{b}_i$, $\|u_i\|_Q \le c_{i1}$,$\forall i = 1,2,3$ and $||\vec{y}_n(t)||_0^2 \le b$,then (23) becomes

$\|\vec{y}_n(t)\|_0^2 \le c_1^* + c_7 \int_0^t ||\vec{y}_n||_0^2\, dt$ with $c_1^* = c_{11} + c_{21} + c_{31} + b + \acute{b}_1 + \acute{b}_2 + \acute{b}_3$.

Where $c_7 = \max(c_4, c_5, c_6)$ with $c_4 = 1 + \acute{c}_1 + 2c_1$, $c_5 = 1 + \acute{c}_2 + 2c_2$ and $c_6 = (1 + \acute{c}_3 + 2c_3)$.

We use the Bellman- Gronwall (BGIN) inequality to get

$||\vec{y}_n(t)||_0^2 \le c_1^* e^{c_7 T} = b^2(c)$, $\forall\, t \in [0, T]$ we can easily obtain the following

$||\vec{y}_n(t)||_{L^\infty\left(I.L^2(\Omega)\right)} \le b\,(c)$ and $||\vec{y}_n(t)||_Q \le b_1(c)$.

The norm $||\vec{y}_n(t)||_{L^2(I,V)}$ ***is bounded:*** By using the same previous steps in (21), but with $t = T$, and $\|\vec{y}_n(T)\|_0^2$ is positive, one can easily obtain that

$\|\vec{y}_n\|_{L^2(I,V)} = \int_0^T \|\vec{y}_n\|_1^2\, dt \le b_2^2(c) = 0.5(\acute{\mathrm{b}}_1 + \acute{\mathrm{b}}_2 + \acute{\mathrm{b}}_3 + \acute{c}_1 \mathrm{d}_1 + \acute{c}_2 \mathrm{d}_2 + \acute{c}_3 \mathrm{d}_3 + \mathrm{c}_7 \mathrm{b}_1(\mathrm{c}))$.

The convergence of the solution: Let $\{\vec{V}_n\}$ be a sequence of subspace of $\vec{V}$ s.t $\forall\, \vec{v} = (v_1, v_2, v_3) \in \vec{V}$, there is a sequence $\{\vec{v}_n\}$, $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}) \in \vec{V}_n, \forall\, n$ and $\vec{v}_n \to \vec{v}$ in $\vec{V}$ $\Rightarrow \vec{v}_n \longrightarrow \vec{v}$ in $(L^2(\Omega))^3$.

Since for any n, ($\vec{V}_n \subset \vec{V}$), problem ((19)-21)) has a unique solution $\vec{y}_n$, hence corresponding to the sequence of subspaces $\{\vec{V}_n\}$, there is a sequence of approximation problems (19-21), so by substituting $\vec{v} = \vec{v}_n \in \vec{V}_n$ for $n = 1,2,3$, one gets

$$\langle y_{1nt}, v_{1n} \rangle + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) - (y_{3n}, v_{1n}) = (f_1(y_{1n}, u_1), v_{1n}) \tag{24a}$$
$$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n}), \tag{24b}$$
$$\langle y_{2nt}, v_{2n} \rangle + (\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n}) + (y_{3n}, v_{2n}) + (y_{1n}, v_{2n}) = (f_2(y_{2n}, u_2), v_{2n}) \tag{25a}$$
$$(y_{2n}^0, v_{2n}) = (y_2^0, v_{2n}), \tag{25b}$$
$$\langle y_{3nt}, v_{3n} \rangle + (\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n}) + (y_{1n}, v_{3n}) - (y_{2n}, v_{3n}) = (f_3(y_{3n}, u_3), v_{3n}) \tag{26a}$$
$$(y_{3n}^0, v_{3n}) = (y_3^0, v_{3n}), \tag{26b}$$

Which has a sequence of solutions $\{\vec{y}_n\}_{n=1}^{\infty}$, from the previous steps we get that $\|\vec{y}_n\|_{L^2(Q)}$, and $\|\vec{y}_n\|_{L^2(I,V)}$ are bounded. By theorem 2.2, there is a subsequence of $\{\vec{y}_n\}_{n\in N}^{\infty}$, , such that $\vec{y}_n \rightharpoonup \vec{y}$ in $(L^2(\Omega))^3$ and $\vec{y}_n \rightharpoonup \vec{y}$ in $L^2(I,V))^3$.
From the Assumptions (A-i), and the bounded of the norms, one gets through the COMTH in [11], $\vec{y}_n \to \vec{y}$ in$(L^2(Q))^3$.
Now, consider the W.F ((24)-(26)), from the MGA for any arbitrary $\vec{v} \in \vec{V}$ there exists a sequence $\{\vec{v}_n\}, \vec{v}_n \in \vec{V}_n, \forall n$ s.t $\vec{v}_n \to \vec{v}$ in V (then in $L^2(\Omega)$), so MBS of ((24)a-(26)a) by $\varphi_i(t) \in C^1[0,T]$ with $\varphi_i(T) = 0, \forall i = 1,2,3$, IBS w.r.t. t from 0 to T, and then we integrate (IBPs) the1^{st} term in the L.H.S. of each obtained equation, one obtains

$$-\int_0^T (y_{1n}, v_{1n})\, \acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n})\varphi_1(t) + (y_{1n}, v_{1n})\varphi_1(t) - (y_{2n}, v_{1n})\varphi_1(t) - (y_{3n}, v_{1n})\varphi_1(t)]\, dt = \int_0^T f_1(y_{1n}, u_1), v_{1n})\, \varphi_1(t)dt + (y_{1n}^0, v_{1n})\varphi_1(0) \tag{27}$$

$$-\int_0^T (y_{2n}, v_{2n})\, \acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n})\varphi_2(t) + (y_{2n}, v_{2n})\varphi_2(t) + (y_{3n}, v_{2n})\varphi_2(t) + (y_{1n}, v_{2n})\varphi_2(t)]\, dt = \int_0^T f_2(y_{2n}, u_2), v_{2n})\, \varphi_2(t)dt + (y_{2n}^0, v_{2n})\varphi_2(0) \tag{28}$$

$$-\int_0^T (y_{3n}, v_{3n})\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n})\varphi_3(t) + (y_{3n}, v_{3n})\varphi_3(t) + (y_{1n}, v_{3n})\varphi_3(t) - (y_{2n}, v_{3n})\varphi_3(t)]\, dt = \int_0^T f_3(y_{3n}, u_3), v_{3n})\, \varphi_3(t)dt + (y_{3n}^0, v_{3n})\varphi_3(0) \tag{29}$$

Since $y_n \rightharpoonup y$ in L^2 (Q))^3, $y_n^0 \to y^0$ in $L^2(\Omega))^3$ and

$$\left.\begin{array}{l} v_n \to \vec{v} \text{ in } L^2(\Omega))^3 \\ v_n \to \vec{v} \text{ in } V \end{array}\right\} \to \left\{\begin{array}{l} v_{in}\, \acute{\varphi}_i \to v_i \acute{\varphi}_i \text{ in } L^2(Q) \\ v_{in}\, \varphi_i \to v_i \varphi_i \text{ in } L^2(I,V) \end{array}\right.$$

Then

$$\int_0^T (y_{1n}, v_{1n})\, \acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n})\varphi_1(t) + (y_{1n}, v_{1n})\varphi_1(t) - (y_{2n}, v_{1n})\varphi_1(t) - (y_{3n}, v_{1n})\varphi_1(t)]\, dt \to \int_0^T (y_1, v_1)\acute{\varphi}_1(t)\, dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) - (y_3, v_1)\varphi_1(t)]\, dt \tag{30}$$

$$\int_0^T (y_{2n}, v_{2n})\, \acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n})\varphi_2(t) + (y_{2n}, v_{2n})\varphi_2(t) + (y_{3n}, v_{2n})\varphi_2(t) + (y_{1n}, v_{2n})\varphi_2(t)]\, dt \to \int_0^T (y_2, v_2)\acute{\varphi}_2(t)\, dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]\, dt \tag{31}$$

$$\int_0^T (y_{3n}, v_{3n})\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n})\varphi_3(t) + (y_{3n}, v_{3n})\varphi_3(t) + (y_{1n}, v_{3n})\varphi_3(t) - (y_{2n}, v_{3n})\varphi_3(t)]\, dt \to \int_0^T (y_3, v_3)\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3)\varphi_3(t) + (y_3, v_3)\varphi_3(t) + (y_1, v_3)\varphi_3(t) - (y_2, v_3)\varphi_3(t)]\, dt$$
(32)

Now, let $w_{in} = v_i\, \varphi_i$, $\forall i = 1,2,3$, then $w_{in} \to w_i$ in $L^2(Q)$ with $w_i = v_i\, \varphi_i$, from applying the Assumptions (A-i), then using proposition 3.1 in [12], we get $\int_Q f_i(x,t,y_{in},u_i) w_{in} dxdt$ is continuous w.r.t (y_{in}, u_i, w_{in}), but $y_{in} \to y_i$ in $(L^2(Q))^3$ and $w_{in} \to w_i$ in $L^2(Q)$, therefore

$$\int_0^T (f_i(y_{in} u_i), v_{in})\, \varphi_i(t)dt \longrightarrow \int_0^T (f_i(y_i u_i), v_i)\, \varphi_i(t)dt\, , \forall i = 1,2,3 \tag{33}$$

From ((30-32)) and (33), (27-29) becomes

$$-\int_0^T (y_1, v_1)\acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) - (y_3, v_1)\varphi_1(t)]dt = \int_0^T (f_1(y_1, u_1) v_1)\, \varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0) \tag{34}$$

$$-\int_0^T (y_2, v_2)\acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)dt = \int_0^T (f_2(y_2, u_2), v_2)\, \varphi_2(t)dt + (y_2^0, v_2)\varphi_2(t) \tag{35}$$

$$-\int_0^T (y_3, v_3)\acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3)\varphi_3(t) + (y_3, v_3)\varphi_3(t) + (y_1, v_3)\varphi_3(t) - (y_2, v_3)\varphi_3(t)dt = \int_0^T (f_3(y_3, u_3), v_3)\, \varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0) \tag{36}$$

Case1: Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall i = 1,2,3$ in (34-36), IBPs for the1^{st}

terms in the L.H.S. of each one of the obtained equations, this yields

$$\int_0^T \langle y_{1t}, v_1\rangle\, \varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) - (y_3, v_1)\varphi_1(t)]dt = \int_0^T (f_1(y_1, u_1)v_1)\, \varphi_1(t)dt \quad (37)$$

$$\int_0^T \langle y_{2t}, v_2\rangle\, \varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]\, dt = \int_0^T (f_2(y_2, u_2), v_2)\, \varphi_2(t)dt \quad (38)$$

$$\int_0^T \langle y_{3t}, v_3\rangle\, \varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3)\varphi_3(t) + (y_3, v_3)\varphi_1(t) + (y_1, v_3)\varphi_1(t) - (y_2, v_3)\varphi_3(t)]\, dt = \int_0^T (f_3(y_3, u_3), v_3)\, \varphi_3(t)d \quad (39)$$

Which gives

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) - (y_3, v_1) = (f_1(y_1, u_1), v_1)$, a.e. in I
$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_3, v_2) + (y_1, v_2) = (f_2(y_2, u_2), v_2)$, a.e. in I
$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) + (y_3, v_3) + (y_1, v_3) - (y_2, v_3) = (f_3(y_3, u_3), v_3)$, a.e. in I

i.e. $\vec{y}$ is a solution of the TSVEs ((13)a-(15)a).

Case 2: Choose $\varphi_i \in C^1[0,T]$, $\forall i = 1,2,3$, s.t $\varphi_i(T) = 0\ \&\varphi_i(0) \neq 0$,
IBPs for 1st term in the L.H.S. of (37-39)**,** to get

$$-\int_0^T (y_1, v_1)\acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) - (y_3, v_1)\varphi_1(t)]dt = \int_0^T (f_1(y_1, u_1)v_1)\, \varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0) \quad (40)$$

$$-\int_0^T (y_2, v_2)\acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)dt = \int_0^T (f_2(y_2, u_2), v_2)\, \varphi_2(t)dt + (y_2(0), v_2)\varphi_2(t) \quad (41)$$

$$-\int_0^T (y_3, v_3)\acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3)\varphi_3(t) + (y_3, v_3)\varphi_3(t) + (y_1, v_3)\varphi_3(t) -(y_2, v_3)\varphi_3(t)dt = \int_0^T (f_3(y_3, u_3), v_3)\, \varphi_3(t)dt + (y_3(0), v_3)\varphi_3(0) \quad (42)$$

Subtracting ((40)-(42)) from ((34)-(36)) resp., to get that ((13)b-(15)b) are held.

The strong convergence for $\vec{y}_n$ in $L^2(I, V)$: Substituting $v_i = y_i$, $\forall i = 1,2,3$ in ((13)a-(15)a) , and then we add them together, on the other hand substitute $v_i = y_{in}$, $\forall i = 1,2,3$ in ((19)a-(21)a) resp. and then we add them together, and integrat the three obtained equations from 0 to T , one has

$$\int_0^T \langle \vec{y}_t, \vec{y}\rangle\, dt + \int_0^T a(\vec{y}, \vec{y})dt = \int_0^T [(f_1(y_1, u_1), y_1) + (f_2(y_2, u_2), y_2) + f_3(y_3,\ u_3), y_3)]\, dt \quad (43a)$$

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle\, dt + \int_0^T a(\vec{y}_n, \vec{y}_n)dt = \int_0^T [(f_1(y_{1n}, u_1), y_{1n}) + (f_2(y_{2n}, u_2), y_{2n}) + f_3(y_{3n},\ u_3), y_{3n})]\, dt \quad (43b)$$

Where$a(\vec{y}, \vec{y}) = a_1(y_1, y_1) + a_2(y_2, y_2) + a_3(y_3, y_3)$, with $a_i(y_i, v_i) = (\nabla y_i, \nabla v_i) + (y_i, v_i)$.
Using Lemma 1.2 in [11] for the 1st terms in the L.H.S. of (43a&b), they become

$$\tfrac{1}{2}\|\vec{y}(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T a(\vec{y}, \vec{y})\, dt = \int_0^T [(f_1(y_1, u_1), y_1) + (f_2(y_2, u_2), y_2) + f_3(y_3, u_3), y_3)]\, dt \quad (44a)$$

$$\tfrac{1}{2}\|\vec{y}_n(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T a(\vec{y}_n, \vec{y}_n)dt = \int_0^T [(f_1(y_{1n}, u_1), y_{1n}) + (f_2(y_{2n}, u_2), y_{2n}) + f_3(y_{3n},\ u_3), y_{3n})] \quad (44b)$$

Since

$$\tfrac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 + \int_0^T a(\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt = \alpha - \beta - \zeta \quad (45)$$

where

$\alpha = \frac{1}{2}\|\vec{y}_n(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T a(\vec{y}_n(t), \vec{y}_n(t))dt$

$\beta = \frac{1}{2}(\vec{y}_n(T), \vec{y}(T)) - \frac{1}{2}(\vec{y}_n(0), \vec{y}(0)) + \int_0^T a(\vec{y}_n(t), \vec{y}(t))dt$ and

$\zeta = \frac{1}{2}(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)) - \frac{1}{2}(\vec{y}(0), \vec{y}_n(0) - y(0)) + \int_0^T a(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t))dt$

Since

$\vec{y}_n^0 = \vec{y}_n(0) \to \vec{y}^0 = \vec{y}(0)$ in $(L^2(\Omega))^3$ (46a)

$\vec{y}_n(T) \to \vec{y}(T)$ in $(L^2(\Omega))^3$ (46b)

Then

$\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)) \to 0$ and $(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)) \to 0$ (46c)

$\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 \to 0$ and $\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 \to 0$ (46d)

And since $\vec{y}_n \rightharpoonup \vec{y}$ in $(L^2(I,V))^3$, then

$\int_0^T a(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t))dt \rightharpoonup 0$ (46e)

Since $\int_0^T (f_i(y_{in}, u_i), y_{in})dt$ is continuous w.r.t. y_i and since $\vec{y}_n \to \vec{y}$ in $(L^2(Q))^3$, $\forall i = 1,2,3$, then

$\int_0^T [(f_1(y_{1n}, u_1), y_{1n}) + (f_2(y_{2n}, u_2), y_{2n}) + f_3(y_{3n}, u_3), y_{3n})]dt \to \int_0^T [(f_1(y_1, u_1), y_1) + (f_2(y_2, u_2), y_2) + f_3(y_3, u_3), y_3)]\, dt$ (46f)

Now, as $n \to \infty$ in (45), the following results are obtained:

(1) using (46d), the first two terms in the L.H.S. of (45) tend to zero

(2) Eq.α $\underset{(44b)}{\overset{\text{from}}{=}}$ $\int_0^T [(f_1(y_{1n}, u_1), y_{1n}) + (f_2(y_{2n}, u_2), y_{2n}) + f_3(y_{3n}, u_3), y_{3n})]$ $\underset{(46f)}{\overset{\text{from}}{\to}}$ $\int_0^T [(f_1(y_1, u_1), y_1) + (f_2(y_2, u_2), y_2) + f_3(y_3, u_3), y_3)]\, dt$

(3) Eq. $\beta \to$ L.H.S. of (44a) $= \int_0^T [\, (f_1(y_1, u_1), y_1) + (f_2(y_2, u_2), y_2) + (f_3(y_3, u_3), y_3)]dt$

(4) using (46c) and (46e) , all the terms in Eq.ζ tend to zero.

Hence, (45) gives $\int_0^T \|\vec{y}_n - \vec{y}\|_1^2\, dt = \int_0^T a(\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt \to 0 \Rightarrow \vec{y}_n \to \vec{y}$ in $(L^2(I,V))^3$

Uniqueness of the solution: Let $\vec{y}, \vec{\bar{y}}$ be two TSVEs of ((13)a-(15)a), we subtract each equation from the other and then set $v_i = y_i - \bar{y}_i$, for $i = 1,2,3$, one obtains

$\langle (y_1 - \bar{y}_1)_t, y_1 - \bar{y}_1 \rangle + a_1(y_1 - \bar{y}_1, y_1 - \bar{y}_1) - (y_2 - \bar{y}_2, y_1 - \bar{y}_1) - (y_3 - \bar{y}_3, y_1 - \bar{y}_1)$
$= (f_1(y_1, u_1) - f_1(\bar{y}_1, u_1), y_1 - \bar{y}_1)$ (47)

$\langle (y_2 - \bar{y}_2)_t, y_2 - \bar{y}_2 \rangle + a_2(y_2 - \bar{y}_2, y_2 - \bar{y}_2) + (y_3 - \bar{y}_3, y_2 - \bar{y}_2) + (y_1 - \bar{y}_1, y_2 - \bar{y}_2)$
$= (f_2(y_2, u_2) - f_2(\bar{y}_2, u_2), y_2 - \bar{y}_2)$ (48)

$\langle (y_3 - \bar{y}_3)_t, y_3 - \bar{y}_3 \rangle + a_3(y_3 - \bar{y}_3, y_3 - \bar{y}_3) + (y_1 - \bar{y}_1, y_3 - \bar{y}_3) - (y_2 - \bar{y}_2, y_3 - \bar{y}_3)$
$= (f_3(y_3, u_3) - f_3(\bar{y}_3, u_3), y_3 - \bar{y}_3)$ (49)

Adding (47)-(49), the 2nd term of the L.H.S. is positive, applying Lemma 1.2 in [11] for the 1st term of L.H.S, it gives

$\frac{1}{2}\frac{d}{dt}\|\vec{y} - \vec{\bar{y}}\|_0^2 + \int_0^T \|\vec{y} - \vec{\bar{y}}\|_1^2 dt \leq f_1(y_1, u_1) - f_1(\bar{y}_1, u_1), y_1 - \bar{y}_1) +$
$(f_2(y_2, u_2) - f_2(\bar{y}_2, u_2), y_2 - \bar{y}_2) - (f_3(y_3, u_3) - f_3(\bar{y}_3, u_3), y_3 - \bar{y}_3)$ (50)

The 2nd term in the L.H.S of is positive, IBS w.r.t. *t* from 0 to *t*, by using Assumption (A-ii) of the R.H.S., one gets

$\int_0^t \frac{d}{dt}\|\vec{y} - \vec{\bar{y}}\|_0^2 dt \leq \int_0^t 2L\,\|\vec{y} - \vec{\bar{y}}\|_0^2 dt$, L=max$\{L_1, L_2, L_3\}$ $\Rightarrow$

$\|(\vec{y} - \vec{\bar{y}})(t)\|_0^2 \leq \int_0^t 2L\,\|\vec{y} - \vec{\bar{y}}\|_0^2 dt,$

The BGIN is applied to give that $\|(\vec{y} - \vec{\bar{y}})(t)\|_0^2 = 0, \forall t \in I.$

Again, IBS of (50) w.r.t. from *0* to *T*, Assumptions (A-ii) of the R.H.S., one has

$\int_0^T \frac{d}{dt}\|\vec{y}-\vec{\bar{y}}\|_0^2 dt + \int_0^T \|\vec{y}-\vec{\bar{y}}\|_1^2 dt \ \le L\int_0^T \|\vec{y}-\vec{\bar{y}}\|_0^2 dt$

$\Rightarrow \int_0^T \|\vec{y}-\vec{\bar{y}}\|_1^2 dt \ \le L\int_0^T \|\vec{y}-\vec{\bar{y}}\|_0^2 dt = 0 \Rightarrow \|\vec{y}-\vec{\bar{y}}\|_{L^2(I,V)} = 0 \Rightarrow \vec{y} = \vec{\bar{y}}$.

Theorem (3.2): In addition to Assumptions (A), if $\vec{y}$ and $\vec{y} + \overrightarrow{\delta y}$ are the TSVS corresponding to the CCTCV $\vec{u}$, $\vec{u} + \overrightarrow{\delta u} \in (L^2(Q))^3$, resp., then

$$\|\overrightarrow{\delta y}\|_{L^\infty(I,L^2(\Omega))} \le \mathrm{M}\,\|\overrightarrow{\delta u}\|_{Q},\ \|\overrightarrow{\delta y}\|_{L^2(Q)} \le \mathrm{M}\,\|\overrightarrow{\delta u}\|_{Q},\ \|\overrightarrow{\delta y}\|_{L^2(I,V)} \le \mathrm{M}\,\|\overrightarrow{\delta u}\|_{Q}.$$

Proof: For given $\vec{u} = (u_1,u_2,u_3)$, then by theorem (3.1)W.F (13-15) has a unique TSVS $\vec{y}$, also for given $\vec{\bar{u}} = (\bar{u}_1,\bar{u}_2,\bar{u}_3)$, then $\vec{\bar{y}} = (\bar{y}_1,\bar{y}_2,\bar{y}_3)$ is the solution of

$$\langle \bar{y}_{1t},v_1\rangle + (\nabla\bar{y}_1,\nabla v_1) + (\bar{y}_1,v_1) - (\bar{y}_2,v_1) - (\bar{y}_3,v_1) = (f_1(\bar{y}_1,\bar{u}_1),v_1) \quad (51a)$$

$$(\bar{y}_1(0),v_1) = (y_1^0,v_1) \quad (51b)$$

$$\langle \bar{y}_{2t},v_2\rangle + (\nabla\bar{y}_2,\nabla v_2) + (\bar{y}_2,v_2) + (\bar{y}_3,v_2) + (\bar{y}_1,v_2) = (f_2(\bar{y}_2,\bar{u}_2),v_2) \quad (52a)$$

$$(\bar{y}_2(0),v_2) = (y_2^0,v_2) \quad (52b)$$

$$\langle \bar{y}_{3t},v_3\rangle + (\nabla\bar{y}_3,\nabla v_3) + (\bar{y}_3,v_3) + (\bar{y}_1,v_3) - (\bar{y}_2,v_3) = (f_3(\bar{y}_3,\bar{u}_3),v_3) \quad (53a)$$

$$(\bar{y}_3(0),v_3) = (y_3^0,v_3) \quad (53b)$$

Subtracting ((13)-(15))from((51)-(53)), putting $\delta y_i = \bar{y}_i - y_i$,$\delta u_i = \bar{u}_i - u_i$, $\forall i = 1,2,3$, to get

$$\langle \delta y_{1t},v_1\rangle + (\nabla\delta y_1,\nabla v_1) + (\delta y_1,v_1) - (\delta y_2,v_1) - (\delta y_3,v_1) =$$
$$(f_1(y_1+\delta y_1,u_1+\delta u_1),v_1) - (f_1(y_1,u_1),v_1) \quad (54a)$$

$$(\delta y_1(0),v_1) = 0 \quad (54b)$$

$$\langle \delta y_{2t},v_2\rangle + (\nabla\delta y_2,\nabla v_2) + (\delta y_2,v_2) + (\delta y_3,v_2) + (\delta y_1,v_2) =$$
$$(f_2(y_2+\delta y_2,u_2+\delta u_2),v_2) - (f_2(y_2,u_2),v_2) \quad (55a)$$

$$(\delta y_2(0),v_2) = 0 \quad (55b)$$

$$\langle \delta y_{3t},v_3\rangle + (\nabla\delta y_2,\nabla v_3) + (\delta y_2,v_3) + (\delta y_3,v_3) - (\delta y_1,v_3) =$$
$$(f_3(y_3+\delta y_3,u_3+\delta u_3),v_3) - (f_3(y_3,u_3),v_3) \quad (56a)$$

$$(\delta y_3(0),v_3) = 0 \quad (56b)$$

By substituting $v_i = \delta y_i$, $\forall i = 1,2,3$ in ((54-56)a, and adding the result equations, we apply Lemma 1.2 in [11] for the 1^{st} term in the L.H.S. of, we get

$$\frac{1}{2}\frac{d}{dt}\|\overrightarrow{\delta y}\|_0^2 + \|\overrightarrow{\delta y}\|_1^2 = (f_1(y_1+\delta y_1,u_1+\delta u_1),v_1) - (f_1(y_1,u_1),\delta y_1)$$
$$+(f_2(y_2+\delta y_2,u_2+\delta u_2),v_2) - (f_2(y_2,u_2),\delta y_2)$$
$$+(f_3(y_3+\delta y_3,u_3+\delta u_3),v_3) - (f_3(y_3,u_3),\delta y_3) \quad (57)$$

The 2^{nd} term of L.H.S. is positive, IBS w.r.t. t from 0 to t, by taking the absolute value, and using Assumptions (A-ii), it gives $\forall t \in [0,T]$

$$\|\overrightarrow{\delta y}(t)\|_0^2 \le 2L_1\int_0^t \|\delta y_1\|_0^2 dt + \bar{L}_1\int_0^T \|\delta u_1\|_0^2 dt + \bar{L}_1\int_0^t \|\delta y_1\|_0^2 dt +$$
$$2L_2\int_0^t \|\delta y_2\|_0^2 dt + \bar{L}_2(\int_0^T \|\delta u_2\|_0^2 dt + \int_0^t \|\delta y_2\|_0^2 dt) +$$
$$2L_3\int_0^t \|\delta y_3\|_0^2 dt + \bar{L}_3(\int_0^T \|\delta u_3\|_0^2 dt + \int_0^t \|\delta y_3\|_0^2)dt$$

$$\Rightarrow \|\overrightarrow{\delta y}(t)\|_0^2 \le \tilde{L}_1\|\overrightarrow{\delta u}\|_Q^2 + \tilde{L}_2\int_0^t \|\overrightarrow{\delta y}\|_0^2 dt,$$

Where $\tilde{L}_1$ =max $\{\bar{L}_1,\bar{L}_2,\bar{L}_3\}$, $\tilde{L}_2$ =max $\{2L_1+\bar{L}_1, 2L_2+\bar{L}_2, 2L_3+\bar{L}_3\}$.

Applying BGIN, to give

$$\|\overrightarrow{\delta y}(t)\|_0^2 \le \mathrm{M}^2\|\overrightarrow{\delta u}\|_Q^2 \Rightarrow \|\overrightarrow{\delta y}(t)\|_0 \le M\|\overrightarrow{\delta u}\|_Q,\ t\in[0,T] \Rightarrow \|\overrightarrow{\delta y}\|_{L^\infty(I,L^2(\Omega))} \le M\|\overrightarrow{\delta u}\|_Q$$

then, $\|\overrightarrow{\delta y}\|_{L^2(Q)}^2 \le M\|\overrightarrow{\delta u}\|_Q$, where M denotes to various constants.

The same previous way can be used for the R.H.S. of (57) with $t = T$, to get

$$\frac{1}{2}\int_0^T \frac{d}{dt}\|\overrightarrow{\delta y}\|_0^2 + \int_0^T \|\overrightarrow{\delta y}\|_1^2 dt \le \tilde{L}_1\|\overrightarrow{\delta u}\|_Q^2 + \tilde{L}_2\int_0^T \|\overrightarrow{\delta y}\|_0^2 dt$$

$\Rightarrow \int_0^T \|\overrightarrow{\delta y}\|_1^2 \, dt \leq (\tilde{L}_1 + \tilde{L}_2 \mathrm{M}^2) \, \|\overrightarrow{\delta u}\|_{\boldsymbol{Q}}^2$

$\Rightarrow \|\overrightarrow{\delta y}\|_{\boldsymbol{L^2(I,V)}}^2 \leq \mathrm{M}^2 \|\overrightarrow{\delta u}\|_{\boldsymbol{Q}}$, where M denotes to various constants.

Theorem (3.3): With Assumptions (A) the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is continuous from $(L^2(Q))^3$ in to $(L^\infty(I, L^2(\Omega)))^3$,or into $(L^2(I,V))^3$ is continuous

Proof: let $\overrightarrow{\delta u} = \vec{\vec{u}} - \vec{u}$ and $\overrightarrow{\delta y} = \vec{\vec{y}} - \vec{y}$ where $\vec{\vec{y}}$ and $\vec{y}$ are the correspond TSVS to the CCTCV $\vec{\vec{u}}$ and $\vec{u}$ resp., using the first results in (theorem 3.1), one has $\|\vec{\vec{y}} - \vec{y}\|_{\boldsymbol{L^\infty(I,L^2(\Omega)}} \leq \|\vec{\vec{u}} - \vec{u}\|_{\boldsymbol{Q}}$, now if $\vec{\vec{u}} \xrightarrow[(L^2(Q))^3]{} \vec{u}$ then $\vec{\vec{y}} \xrightarrow[(L^\infty(I.V^2(\Omega))^3]{} \vec{y}$, thus the operator $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is Lip continuous from $(L^2(Q))^3$in to $(L^\infty(I, L^2(\Omega)))^3$. One can easily obtained the other results.

4. Existence of a CCTOCV

To study the existence of a CCTOCV we need the following assumptions and lemma.

Assumptions **(B):** Consider g_{0i} ($\forall\, i = 1,2,3$) is of CAT on $Q \times (\mathbb{R} \times \mathbb{R})$ which satisfies*:*

$|g_{0i}\,(x,t,y_i,u_i)| \leq \eta_{0i}(x,t) + c_{0i1}(y_i)^2 + c_{0i2}(u_i)^2$, where $y_i, u_i \in \mathbb{R}$ with $\eta_{0i} \in L^1(Q)$

Lemma (4.1): If assumptions (B) are held, then $\vec{u} \mapsto G_0(\vec{u})$ is continuous functional on $(L^2(Q))^3$.

Proof: By employing assumptions (B) on $g_{0i}\,(x,t,y_i,u_i)$, then we apply Lemma 1.2 in [11], to get $\int_\Omega g_{0i}\,(x,t,y_i,u_i)dxdt$ which is continuous on $L^2(Q)$ for each $i = 1,2,3$.

Theorem (4.1): Consider the set $\overrightarrow{\mathrm{W}}_{\mathrm{A}} \neq \emptyset$, the functions f_i, $\forall i = 1,2,3$, have the form

$$f_i(x,t,y_i,u_i) = f_{i1}(x,t,y_i) + f_{i2}(x,t)u_i$$

With $|f_{i1}\,(x,t,y_i)| \leq \eta_i(x,t) + c_i|\,y_i|$ where $\eta_i \in L^2(Q)$ and $|f_{i2}(x,t)| \leq k_i$,

If $\forall i = 1,2,3$, g_{0i} is convex w.r.t. u_i for fixed (x,t,y_i). Then there exists a CCTOCV.

Proof: Since W_i is convex, closed and bounded for each $i = 1,2,3$ then $W_1 \times W_2 \times W_3$ is convex, closed and bounded and then it is weakly compact. Because of $\overrightarrow{W}_A \neq \emptyset$, then there exist $\vec{\vec{u}} \in \overrightarrow{W}_A$ and a minimum sequence $\{\vec{u}_k\}$ with $\vec{u}_k \in \overrightarrow{W}_A$, $\forall k$. s.t

$$lim_{k\to\infty} G_0(\vec{u}_k) = inf_{\vec{\vec{u}} \in \overrightarrow{W}_A}\, G_0(\vec{\vec{u}}).$$

Since $\vec{u}_k \in \overrightarrow{W}_A$, $\forall\, k$, there is a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$, s.t

$$\vec{u}_k \rightharpoonup \vec{u} \in \overrightarrow{W} \text{ in } (L^2(Q))^3, \text{ and } \|\,\vec{u}_k\|_Q \leq c,\ \forall\, k$$

From theorem (3.1), for each CCTCV $\vec{u}_k$, the W.F of the TSVEs has a unique solution $\vec{y}_k = \vec{y}_{\vec{u}_k}$ with $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ and $\|\vec{y}_k\|_{L^2(I,V)}$ are bounded, then by theorem 2.2 there is subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ s.t. $\vec{y}_k \rightharpoonup \vec{y}$ in $L^\infty(I, L^2(\Omega))^3$, $(L^2(Q))^3$, and in $(L^2(I,V))^3$. Also, from theorem (3.1), $\|\vec{y}_k\|_{L^2(I,\,V^*)}$is bounded and since

$(L^2(I,V))^3 \subset (L^2(Q))^3 \cong ((L^2(Q))^*)^3 \subset (L^2(I,V^*))^3$

So, by the COMTH in [11] a subsequence of $\{\vec{y}_k\}$ can be found say again $\{\vec{y}_k\}$ s.t $\vec{y}_k \to \vec{y}$ in $(L^2(Q))^3$. Since $\forall k$, $\vec{y}_k$ is the TSVS of the W.F corresponding to the CCTCV $\vec{u}_k$, then

$$\langle y_{1kt}, v_1\rangle + (\nabla y_{1k}, \nabla v_1) + (y_{1k}, v_1) - (y_{2k}, v_1) - (y_{3k}, v_1) = (f_{11}(x,t,y_{1k}) + (f_{12}(x,t)\,u_{1k}, v_1) \tag{58}$$

$$\langle y_{2kt}, v_2\rangle + (\nabla y_{2k}, \nabla v_2) + (y_{2k}, v_2) + (y_{3k}, v_2) + (y_{1k}, v_2) = (f_{21}(x,t,y_{2k}) + (f_{22}(x,t)\,u_{2k}, v_2) \tag{59}$$

$$\langle y_{3kt}, v_3\rangle + (\nabla y_{3k}, \nabla v_3) + (y_{3k}, v_3) + (y_{1k}, v_3) - (y_{2k}, v_3) = (f_{31}(x,t,y_{3k}) + (f_{32}(x,t)\,u_{3k}, v_3) \tag{60}$$

Let $\varphi_i \in C^1[I]$, then MBS of ((58)-(60)) by $\varphi_i(t)$ ($\forall i = 1,2,3$) resp. with $\varphi_i(T) = 0$, then IBS w.r.t t from 0 to T and then using IBPS for the 1^{st} term in the L.H.S., one gets

$$-\int_0^T (y_{1k}, v_1)\,\acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1k}, \nabla v_1)\varphi_1(t) + (y_{1k}, v_1)\varphi_1(t) - (y_{2k}, v_1)\varphi_1(t) - (y_{3k}, v_1)\varphi_1(t)]dt = \int_0^T (\,f_{11}(x,\mathrm{t},y_{1k}), v_1\,)\varphi_1(t)dt + \int_0^T (\,f_{12}(x,\mathrm{t})\,u_{1k}, v_1\varphi_1(t))\,dt + (y_{1k}(0), v_1)\,\varphi_1(0) \tag{61}$$

$$-\int_0^T (y_{2k}, v_2)\, \acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2k}, \nabla v_2)\varphi_2(t) + (y_{2k}, v_2)\varphi_2(t) + (y_{3k}, v_2)\varphi_2(t) + (y_{1k}, v_2)\varphi_2(t)]dt = \int_0^T (f_{21}(x,t,y_{2k}), v_2)\varphi_2(t)dt + \int_0^T (f_{22}(x,\mathrm{t})\, u_{2k}, v_2\varphi_2(t))\, dt + (y_{2k}(0), v_2)\, \varphi_2(0) \quad (62)$$

$$-\int_0^T (y_{3k}, v_3)\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3k}, \nabla v_3)\varphi_3(t) + (y_{3k}, v_3)\varphi_3(t) + (y_{1k}, v_3)\varphi_3(t) - (y_{2k}, v_3)\varphi_3(t)]dt = \int_0^T (f_{31}(x,t,y_{3k}), v_3)\varphi_3(t)dt + \int_0^T (f_{32}(x,\mathrm{t})\, u_{3k}, v_3\varphi_3(t))\, dt + (y_{3k}(0), v_3)\, \varphi_3(0) \quad (63)$$

Since $\vec{y}_k \rightharpoonup \vec{y}$ in $(L^2(Q))^3$, and in $(L^2(I,V))^3$, then

$$-\int_0^T (y_{1k}, v_1)\, \acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1k}, \nabla v_1)\varphi_1(t) + (y_{1k}, v_1)\varphi_1(t) - (y_{2k}, v_1)\varphi_1(t) - (y_{3k}, v_1)\varphi_1(t)]dt \rightharpoonup -\int_0^T (y_1, v_1)\, \acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\, \varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) - (y_3, v_1)\varphi_1(t)]dt \quad (64)$$

$$-\int_0^T (y_{2k}, v_2)\, \acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2k}, \nabla v_2)\varphi_2(t) + (y_{2k}, v_2)\varphi_2(t) + (y_{3k}, v_2)\varphi_2(t) + (y_{1k}, v_2)\varphi_2(t)]dt \rightharpoonup -\int_0^T (y_2, v_2)\, \acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt \quad (65)$$

$$-\int_0^T (y_{3k}, v_3)\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3k}, \nabla v_3)\varphi_3(t) + (y_{3k}, v_3)\varphi_3(t) + (y_{1k}, v_3)\varphi_3(t) - (y_{2k}, v_3)\varphi_3(t)]dt \rightharpoonup -\int_0^T (y_3, v_3)\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3)\, \varphi_3(t) + (y_3, v_3)\varphi_3(t) + (y_1, v_3)\varphi_3(t) - (y_2, v_3)\varphi_3(t)]dt \quad (66)$$

Since $\vec{y}_k(0)$ are bounded in $(L^2(\Omega))^3$ and from theorem 2.1 one has

$$(y_{ik}(0), v_i)\, \varphi_i(0) \longrightarrow (y_i^0, v_i)\varphi_i(0), \text{ for } i = 1,2,3 \quad (67)$$

Let $w_i = v_i\varphi_i$, then for fixed $(x,t) \in Q$, $f_{i1}(x,t,y_i)w_i$ is continuous w.r.t. y_{ik}, from Assumptions on f_{i1} and then by applying Proposition 3.1in [12], for $i = 1,2,3$ the integral $\int_Q f_{i1}(y_{ik})w_i dxdt$ is continuous w.r.t. y_{1k} but $y_{ik} \to y_i$ in $L^2(Q)$, then

$$\int_Q f_{i1}(y_{ik})w_i dxdt \to \int_Q f_{i1}(y_i)w_1 dxdt,\ \forall w_1 \in C[\bar{Q}] \quad (68)$$

Since $u_{ik} \rightharpoonup u_i$ in$(L^2(Q)$, with $|f_{i2}| \le k_i$, then

$$\int_Q (f_{i2}(x,t)u_{ik}, v_i)\, dxdt \rightharpoonup \int_Q (f_{i2}(x,t)u_i, v_i)\, dxdt \quad (69)$$

Finally, we use ((64)-(66)), ((67)-(69)) in ((61) -(63)), to obtain that

$$-\int_0^T (y_1, v_1)\, \acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) - (y_3, v_1)\varphi_1(t)]dt = \int_0^T (f_{11}(x,t,y_1), v_1)\varphi_1(t)dt + \int_0^T (f_{12}(x,t)u_1, v_1)\varphi_1(t)dt + +(y_1^0, v_1)\varphi_1(0) \quad (70)$$

$$-\int_0^T (y_2, v_2)\, \acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt = \int_0^T (f_{21}(x,t,y_2), v_2)\varphi_2(t)dt + \int_0^T (f_{22}(x,t)u_2, v_2)\varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0) \quad (71)$$

$$-\int_0^T (y_3, v_3)\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3k}, \nabla v_3)\varphi_3(t) + (y_3, v_3)\varphi_3(t) + (y_1, v_3)\varphi_3(t) - (y_2, v_3)\varphi_3(t)]dt = \int_0^T (f_{31}(x,t,y_3), v_3)\varphi_3(t)dt + \int_0^T (f_{32}(x,t)\, u_3, v_3)\varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0) \quad (72)$$

Equations (70-72) are held for each $v_i \in V, \forall i = 1,2,3$, since $C[\bar{\Omega}]$is dense in V.

Case1: Choose $\varphi_i \in D[I]$for $i = 1,2,3$ s.t. $\varphi_i(T) = 0$ and $\varphi_i(0) = 0$, by using IBPs for the 1^{st} term in the L.H.S. of (70- 72) , we get

$$\int_0^T \langle y_{1t}, v_1\rangle\, \varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) - (y_3, v_1)\varphi_1(t)]dt = \int_0^T (f_{11}(x,\mathrm{t}, y_1), v_1)\varphi_1(t)dt + \int_0^T (f_{12}(x,\mathrm{t})\, u_1, v_1)\varphi_1(t))dt \quad (73)$$

$\int_0^T \langle y_{2t}, v_2\rangle\, \varphi_2(t)dt +$
$\int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t)(y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) + (y_{1k}, v_2)\varphi_2(t)]dt =$
$\int_0^T (f_{21}(x, \mathrm{t}, y_2), v_2\,)\varphi_2(t)d + \int_0^T (f_{22}(x, \mathrm{t})\, u_2, v_2)\varphi_2(t))\, dt$ (74)
$\int_0^T \langle y_{3t}, v_3\rangle\, \varphi_3(t)dt +$
$\int_0^T [(\nabla y_3, \nabla v_3)\varphi_3(t) + (y_3, v_3)\varphi_3(t) + (y_1, v_3)\varphi_3(t) - (y_2, v_3)\varphi_3(t)]dt =$
$\int_0^T (f_{31}(x, \mathrm{t}, y_3), v_3\,)\varphi_3(t)dt + \int_0^T (f_{32}(x, \mathrm{t})\, u_3, v_3)\varphi_3(t))\, dt$ (75)
$\Rightarrow$
$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) - (y_3, v_1)$
$= (f_{11}(x, t\,, y_1), v_1)\, + (f_{12}(x, t)\, u_1, v_1)$
$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_3, v_2) + (y_1, v_2)$
$= (f_{21}(x, t, y_2), v_2)\, + (f_{22}(x, t)\, u_2, v_2)$
$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) + (y_3, v_3) + (y_1, v_3) - (y_2, v_3)$
$= (f_{31}(x, t, y_3), v_3)\, + (f_{32}(x, t)\, u_3, v_3)$
Which means that $\vec{y}$ satisfies the W.F of the TSVEs.
Case2: Choose $\varphi_i \in C^1[I]$, for $i = 1,2,3$ s.t. $\varphi_i(T) = 0$ and $\varphi_i(0) \neq 0$, also using IBPs for the 1^{st} term in the L.H.S. of (73-.75), it gives
$-\int_0^T (y_1, v_1)\, \acute{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t) -$
$(y_3, v_1)\varphi_1(t)]dt\, = \int_0^T (f_{11}(x, t, y_1), v_1)\varphi_1(t)dt + \int_0^T (f_{12}(x, t)\, u_1, v_1\varphi_1(t)dt +$
$(y_1(0), v_1)\varphi_1(0)$ (76)
$-\int_0^T (y_2, v_2)\, \acute{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_3, v_2)\varphi_2(t) +$
$(y_1, v_2)\varphi_2(t)]dt\, = \int_0^T (f_{21}(x, t, y_2), v_2)\varphi_2(t)dt + \int_0^T (f_{22}(x, t)\, u_2, v_2\varphi_2(t)dt$
$+(y_2(0)\,, v_2)\varphi_2(0)$ (77)
$-\int_0^T (y_3, v_3)\, \acute{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3k}, \nabla v_3)\varphi_3(t) + (y_3, v_3)\varphi_3(t) + (y_1, v_3)\varphi_3(t) -$
$(y_2, v_3)\varphi_3(t)]dt\, = \int_0^T (f_{31}(x, t, y_3), v_3)\varphi_3(t)dt + \int_0^T (f_{32}(x, t)\, u_3, v_3\varphi_3(t)dt +$
$(y_3(0)\; v_3)\varphi_3(0)$ (78)
By subtracting ((76)-(78)) from ((70) - (72)) one obtains that

$$(y_i(0), i)\varphi_i(0) = (y_i^0, v_i)\varphi_i(0) \Rightarrow\; y_i^0 = y_i(0),\quad \forall i = 1,2,3,$$

Then $\vec{y}$ is a solution of the TSVEs .
Now, for each $\forall i = 1,2,3$, we have from Lemma 4.1 $\int_\Omega g_{0i}\,(x, t, y_i, u_i)dxdt$ is continuous on $L^2(Q)$ for each $i = 1,2,3$, but $u_i(x, t) \in U_i$ a.e. in Q and U_i is compact, then by Lemma 4.2 in [12],

$$\int_Q g_{0i}(x, t, y_{ik}, u_{ik})\, dxdt \longrightarrow \int_Q g_{0i}(x, t, y_i, u_{ik})\, dxdt \tag{79}$$

But $g_{0i}(x, t, y_i, u_i)$ is convex and continuous w.r.t. u_i. Therefore

$$\begin{aligned}\int_Q g_{0i}(x, t, y_i, u_i)\, dxdt &\leq \lim_{k\to\infty} \inf \int_Q g_{0i}(x, t, y_i, u_{ik})\, dxdt\\ &= \lim_{k\to\infty} \inf \int_Q g_{0i}(x, t, y_i, u_{ik})\, dxdt - g_{0i}(x, t, y_{ik}, u_{ik}))dxdt\\ &\quad + \lim_{k\to\infty} \inf \int_Q g_{0i}(x, t, y_{ik}, u_{ik})\, dxdt\quad ,\qquad (\text{for } i = 1,2,3)\end{aligned}$$

Then by (79), one obtains that
$\int_Q g_{0i}(x, t, y_i, u_i)\, dxdt \leq \lim_{k\to\infty} \inf \int_Q g_{0i}(x, t, y_{ik}, u_{ik})\, dxdt$
i.e. $G_0(\vec{u})$ is W.L.S.C. w.r.t. $(\vec{y}, \vec{u})$,
Since $G_0(\vec{u}) \leq\; \lim_{k\to\infty} \inf G_0(\vec{u}_k) = \lim_{k\to\infty} G_0\, (\vec{u}_k) = \inf_{\vec{\vec{u}}\in \vec{W}_A} G_0(\vec{\vec{u}}_k)$
$\Rightarrow G_0(\vec{u}) = \min_{\vec{\vec{u}}\in \vec{W}_A} G_0(\vec{\vec{u}}_k) \Rightarrow \vec{u}$ is a CCTOCV.
Conclusions: Under suitable conditions and for fixed CCTCV, the MGA with the COMTH

are successfully used to prove the existence and the uniqueness theorem of the TSVs for the TNPPDEs. The continuity of the Lipschitz operator between the CCTCV and the corresponding TSVEs is proved. Under a suitable conditions the existence theorem of a CCTOCV for the continuous classical optimal control governing by the TNPPDEs, is also proved.

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