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Differential Subordination and Superordination for Multivalent Functions Associated with Generalized Fox-Wright Functions

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Abstract

In this paper, we derive some subordination and superordination results for certain subclasses of p− valent analytic functions that defined by generalized Fox-wright functions using the principle of differential subordination, ----------producing best dominant univalent solutions. We have also derived inclusion relations and solved majorization problem.

Keywords: analytic function ,Univalent function , Multivalent function , Differential subordination, Fox-wright functions.

التابعية التفاضلية والتفاضلية العليا لدوال متعددة التكافؤ مرتبطة مع دوال Fox–Wright المعممة

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الخلاصة

فـي هذا البحث سوف نشتق بعض نتائج التابعية والتابعية العليا لاصناف جزئية حديثة لدوال تحليلية متعددة التكافؤ من النمط p معرفة بواسطـة دوال Fox– Wright المعممة باستخدام اساسيات التابعية التفاضلية ناتجة عن افضل الحلول المتكافئة المهيمنة .ايضا تم اشتقاق العلاقات الضمنية وحل مسالتها الخاصه.

1.Introduction

Let $\triangle = \{\omega\in\mathbb{C}:|\omega|< 1 \}$ be the open unit disk in the complex plane $\mathbb{C}$, and $\mu = \mu(\triangle)$ is the class of analytic functions defined in $\triangle$. For a positive integer n and a $\in\mathbb{C}$, let $\mu[a , n] = \{ f \in\mu : f(\omega) = a + a_n\omega^n + a_{n+1}\omega^{n+1}+ \cdots \}$, with $\mu_0 = \mu[0,1]$,$\mu = \mu[1,1]$.
Miller and Mocanu [1] assumed that f and g are functions of μ . The function f is said to be subordinate to g , written $f \prec g$ or $f(\omega) \prec g(\omega)$, if there exists a Schwarz function $w(\omega)$

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analytic in $\triangle$, with w(0)= 0 and $|w(\omega)| < 1$ such that $f(\omega) = g(w(\omega)), (\omega \in \triangle)$ .In particular , if the function g is univalent in $\triangle$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\triangle) \subset g(\triangle)$.
Let $\psi : \mathbb{C}^3 \times \triangle \longrightarrow \mathbb{C}$,and h is univalent in $\triangle$. If f is analytic in $\triangle$ and satisfies the (second –order) differential subordination

$$\psi(f(\omega), \omega f'(\omega), \omega^2 f''(\omega); \omega) \prec h(\omega) \quad (1.1)$$

then f is called a solution of the differential subordination . The univalent function q is called a dominant of the solutions of the differential subordination , or more simply dominant if $f \prec q$ for all f satisfying (1.1) A dominant $\hat{q}$ that satisfies $\hat{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1) .

Definition1.1: For the parameters $a_i, b_j \in \mathbb{C}$ and $A_i, B_j \in \mathcal{R} (i = 1,2,\dots,p;\ j = 1,2,\dots,q)$, with $\frac{a_i}{A_i} \neq 0,-1,-2,\dots,(i = 1,2,\dots,p)$ and $\frac{b_i}{B_j} \neq 0,-1,-2,\dots,(j = 1,2,\dots,q)$, the generalized Fox-Wright function ${}_p\Psi_q$ is defined by

$${}_p\Psi_q\left[\begin{matrix} (a_i;A_i)_{1,p} \\ (b_j,B_j)_{1,q}; \quad \omega \end{matrix}\right] := {}_p\Psi_q\left[\begin{matrix} (a_1,A_1),\dots\dots(a_p,A_q) \\ (B_1,b_1),\dots\dots,(b_q,B_q); \quad \omega \end{matrix}\right] =$$

$$\sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+nA_i)}{\prod_{j=1}^{q} \Gamma(b_j+nB_j)(1)_n} \omega^n, \quad (1.2)$$

for suitable bounded values of $\omega \in \mathbb{C}$, where Γ is the Gamma function. For more details see [2-5].

Remark1.1:

1. According to [6]. If $\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1$, then the power series of (1.2) converges absolutely on C and the above defined function ${}_p\Psi_q$ is an entire function. If $\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i = -1$,then the power series of (1.2) converges absolutely on the disk $|\omega| < \frac{\prod_{j=1}^{q} |B_j|^{B_j}}{\prod_{i=1}^{p} |A_i|^{A_i}}$.

2.If $A_i = (i = 1,2,\dots\dots,p)$ and $B_j = 1 (j = 1,2,\dots.,q)$, one can find as the relationship [7]

$$\Omega\begin{pmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{pmatrix}.\ {}_p\Psi_q\left[\begin{matrix} (a_i,1)_{1,p} \\ (b_j,1)_{1,q}; \quad \omega \end{matrix}\right] = {}_p\mathcal{F}_q\begin{pmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q; \quad \omega \end{pmatrix},$$

where ${}_p\mathcal{F}_q\begin{pmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q; \quad \omega \end{pmatrix}$ is the generalized hypergeometric function [4] , and

$$\Omega = \Omega\begin{pmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{pmatrix} := \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} \quad (1.3)$$

3. *For* $p = 1, q = 1, A_1 = 1$ *and* $B_1 = 1$, the Fox-Wright defined function ${}_p\Psi_q$ reduces to

$${}_1\Psi_1\left[\begin{matrix} (a,1)_{1,1} \\ (b,1)_{1,1}; \quad \omega \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)(1)_n} \omega^n, \omega \in \triangle.$$

4. With the parameters values $a = 1$ and $b = a > 0$ $(b \geq a)$, the function ${}_1\Psi_1$ represents the classical Mittag-Leffler function for example see [8]. Moreover ${}_p\Psi_q$ is a special case of the Fox's $\mathfrak{H}$-function $\mathcal{H}_{k,l}^{m,n}$ [3].

Use the generalized hypergeometric function , the authors [9] introduced a linear operator which was subsequently extended in [7] by using the Fox-Wright generalized hypergeometric function. Let

$\Theta\begin{bmatrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{bmatrix}:\mathcal{D}_p\to\mathcal{D}_p$ be a linear operator which is defined by

$$\Theta\begin{bmatrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{bmatrix}f(\boldsymbol{\omega}) := \boldsymbol{\omega_p\phi_q}\begin{bmatrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}; \quad \omega\end{bmatrix} * f(\boldsymbol{\omega}).$$

Note that for $f(\omega)$ of the form (1.1)$and_p\phi_q = \Omega_p\Psi_p$, t one can obtain that

$$\Theta\begin{bmatrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{bmatrix}f(\boldsymbol{\omega}) := \boldsymbol{\omega} + \sum_{n=2}^{\infty}\boldsymbol{\delta_n(a_1)a_n\omega^n}, \qquad \textbf{(1.4)}$$

where $\boldsymbol{\delta_n(a_i)}$is given by

$$\boldsymbol{\delta_n(a_i)} = \frac{\Omega\Gamma(a_1+(n-1)A_1)\ldots..\Gamma(a_p+(n-1)A_p)}{\Gamma(b_1+(n-1)B_1)\ldots.\Gamma(b_q+(n-1)B_q)}$$

and Ω is given by (1.3).

For convenience propose the contracted notation $\Theta(a_i)f(\boldsymbol{\omega})$ can be represented as follows:

$$\Theta(a_i)f(\boldsymbol{\omega}) = \boldsymbol{\Theta}\begin{bmatrix}(a_1,A_1),\ldots\ldots(a_p,A_q)\\(B_1,b_1),\ldots\ldots,(b_q,B_q)\end{bmatrix}f(\boldsymbol{\omega}) \qquad (\mathbf{1.5})$$

From the equation (1.4) one can get the recursive relation that involves the operator $\Theta(a_i)f(\boldsymbol{\omega})$

$$A_i\omega\big(\Theta(a_i)f(\boldsymbol{\omega})\big)' = a_i\Theta(a_i+1)f(\boldsymbol{\omega}) - (a_i-A_i)\Theta(a_i)f(\boldsymbol{\omega}) \qquad (\mathbf{1.6})$$

Note that there are several interesting operators that are special cases of the linear operator (1.4). They have been extensively studied by researchers in [10-15], Hohlov operator and others. For more details see[16-18].

In order to prove the main results, we need the following definitions and theorem.

Definition 1.7.[19] The set of all function q hat, which is denoted by Q, are analytic and injective on $\overline{\triangle}\backslash E(q)$, where

$$E(q) = \{\xi \in \partial\triangle : \lim_{\omega\to\xi} q(\omega) = \infty\}, \qquad (1.7)$$

such that $q'(\xi) \neq 0$ for $\xi \in \partial\triangle\backslash E(q)$, Furthermore let (a) , $Q(0) \equiv Q_0$ and $Q(1) = Q_1$ such that $q(0) = a$.

Definition 1.8[19]. Let Ω be a subset of $\mathbb{C}$,$q \in Q$ and n is a positive integer. The class of admissible functions $\psi[\Omega, q]$ consists function $\psi:\mathbb{C}^3\times\triangle \to\mathbb{C}$ that satisfies the admissibility condition:

$$\psi(r, s, t; \omega, \xi) \notin \Omega,$$

whenever

$$r = q(\xi),\ s = k\xi q'(\xi), \text{ and}$$
$$\Re\left\{\frac{t}{s}+1\right\} \geq k\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\}, \qquad (1.8)$$

where $\omega \in \triangle, \xi \in \partial\triangle\backslash E(q)$ and $k \geq n$. We write $\psi_1[\Omega, q] = \psi[\Omega, q]$. In addition if $\omega \in\triangle, \xi \in \partial\triangle$ and $b \geq n \geq 1$, then in particular we write $\psi_1'[\Omega, q] = \psi'[\Omega, q]$.

Theorem 1.1[19].Let $\Psi \in \psi_n[\Omega, q]$, with $(0) = a$. If the analytic function$F \in \mu[a, n]$satisfies $\Psi\big(g(\omega), \omega g'(\omega), \omega^2 g''(\omega); \omega\big) \in \Omega$, then $F(\omega) \prec q(\omega)$.

1. Main Results

Definition 2.1: LetΩ be a subset of $\mathbb{C}$ with$q\in Q_0\cap\mathcal{H}_0$. The class of admissible functions $\Phi_k[\Omega, q]$ consists of those functions $\emptyset:\mathbb{C}^3\times\triangle\times\overline{\triangle}\to\mathbb{C}$which satisfies the admissibility condition:

$$\emptyset(u, v, w; \omega, \xi) \notin \Omega, \tag{2.1}$$

where

$$u = q(\xi), \quad v = \frac{kA_i\xi q'(\xi) - (A_i - a_i)q(\xi)}{a_i}, (a_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, a_i \neq 0),$$

and

$$\Re\left\{\frac{\omega a_i(a_i+1) - a_i(2a_i - 2A_i + 1)v + (A_i - a_i)^2 u}{A_i(a_i v + (A_i - a_i)u)}\right\} \geq k\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}, \tag{2.2}$$

$$\omega \in \triangle, \zeta \in \partial \triangle\backslash E(q), \xi \in \overline{\triangle} \text{ and } k \geq 1.$$

Theorem 2.1: Let $\emptyset \in \Phi_k[\Omega, q]$. If $f \in \mathcal{D}_p$ satisfies

$$\{\phi(\Theta(a_i)f(\omega)), \Theta(a_i+1)f(\omega), \Theta(a_i+2)f(\omega);\ \omega \in \triangle\} \subset \Omega \tag{2.3}$$

$$\Theta(a_i)f(\omega) \prec q(\omega).$$

Proof: From equation (1.6) we have

$$A_i\omega\big(\Theta(a_i)f(\boldsymbol{\omega})\big)' = a_i\Theta(a_i+1)f(\boldsymbol{\omega}) - (a_i - A_i)\Theta(a_i)f(\boldsymbol{\omega})$$

Which is equivalent to

$$\Theta(a_i+1)f(\boldsymbol{\omega}) = \frac{A_i\omega\big(\Theta(a_i)f(\boldsymbol{\omega})\big)' + (a_i - A_i)\Theta(a_i)f(\boldsymbol{\omega})}{a_i}. \tag{2.4}$$

Now assume that $F(\omega) = \Theta(a_i)f(\boldsymbol{\omega})$, then

$$\Theta(a_i+1)f(\boldsymbol{\omega}) = \frac{A_i\omega F'(\omega) + (a_i - A_i)F(\omega)}{a_i}.$$

Therefore,

$$\Theta(a_i+2)f(\boldsymbol{\omega}) = \frac{A_i^2\omega^2F''(\omega) + A_i(2a_i - A_i + 1)\omega F'(\omega) + (a_i - A_i)(a_i - A_i - 1)\boldsymbol{F}(\boldsymbol{\omega})}{a_i(a_i+1)} \tag{2.5}$$

and from equation (2.4), we have

$$\big(\Theta(a_i+1)f(\boldsymbol{\omega})\big)' = \frac{A_i\omega F''(\omega) + a_iF'(\omega)}{a_i} \tag{2.6}$$

So that

$$\Theta(a_i+2)f(\boldsymbol{\omega}) = \frac{1}{a_i}\left[\frac{A_i\omega\left(A_i\omega F''(\omega) + a_iF'(\omega)\right)}{(a_i+1)} + (a_i + 1 - A_i)\left(\frac{A_i\omega F'(z) + (a_i - A_i)F(\omega)}{(a_i+1)}\right)\right]$$

$$= \frac{{A_i}^2\omega^2F''(\omega) + A_i(2a_i + 1 - A_i)\omega F'(z) + (A_i - a_i)(A_i - a_i - 1)F(\omega)}{a_i(a_i+1)}$$

Let $u = r$, $v = \frac{A_i s - (A_i - a_i)r}{a_i}$, $w = \frac{{A_i}^2 t + A_i(2a_i + 1 - A_i)s + (A_i - a_i)(A_i - a_i - 1)r}{a_i(a_i+1)}$

(2.7)

and

let $\psi(r, s, t; \omega, \xi) = \emptyset(u, v, w; \omega, \xi) =$

$$\emptyset\left(r, \frac{A_i s - (A_i - a_i)r}{a_i}, \frac{{A_i}^2 t + A_i(2a_i + 1 - A_i)s + (A_i - a_i)(A_i - a_i - 1)r}{a_i(a_i+1)}; \omega\right). \tag{2.8}$$

So that by equation (2.4) and (2.5) , we obtain

$$\psi(F(\omega), \omega F'(\omega), \omega^2F''(\omega); \omega, \xi) = \emptyset\left(\big(\Theta(a_i)f(\omega)\big), \Theta(a_i+1)f(\omega), \Theta(a_i+2)f(\omega); \omega\right). \tag{2.9}$$

by using equation (2.3), we get

$$\psi(F(\omega), \omega F^{'}(\omega), \omega^2 F^{''}(\omega); \omega, \xi) \in \Omega. \quad (2.10)$$

We also use the following equation

$$w = \frac{{A_i}^2 \omega^2 F^{''}(\omega) + A_i(2a_i + 1 - A_i)\omega F^{'}(z) + (A_i - a_i)(A_i - a_i - 1)F(\omega)}{a_i(a_i + 1)},$$

and by simple calculations we get

$$\frac{\omega a_i(a_i + 1) - a_i(2a_i - 2A_i + 1)v + (A_i - a_i)^2 \mathcal{u}}{A_i(a_i v + (A_i - a_i)\mathcal{u})} = \frac{t}{\mathcal{S}} + 1. \quad (2.11)$$

We note that the admissibility condition for $\emptyset \in \Phi_k[\Omega, q]$ is equivalent to the admissibility condition for ψ , then $F(\omega) \prec q(\omega)$.

Example 2.1. Let the class of $\Phi_{kv}[\Omega, q]$ consists of those functions $\emptyset: \mathbb{C}^3 \times \triangle \rightarrow \mathbb{C}$ that satisfies the admissibility condition:

$$\mathcal{v} = \frac{A_i k \xi q^{'}(\xi) + (A_i - a_i)q(\xi)}{a_i} \notin \Omega$$

$\boldsymbol{\omega} \in \triangle, \zeta \in \partial\triangle \backslash Eq$, $and m \geq p. I \in \mathcal{D}_p$ satisfies $\mathcal{S}_{0,\omega}^{\lambda,\mu,\eta} f(\omega) \subset \boldsymbol{\Omega}$, then

$$\left(\Theta(a_i)f(\omega)\right) \prec q(\omega)$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. Now if we assume that $\Omega \neq \mathbb{C}$ be a simply connected , so that $\Omega = h(\triangle)$ for some conformal mapping h of $\triangle$onto Ω. The next result is an abrupt outcome of Theorem 2.1.

Theorem2.2 : Let $\emptyset \in \Phi_k[h, q]$.If $f \in \mathcal{D}_p$ satisfies

$$\emptyset\left(\left(\Theta(a_i)f(\omega)\right), \Theta(a_i + 1)f(\omega), \Theta(a_i + 2); \omega \in \triangle\right) \prec \mathrm{h}(\omega), \quad (2.12)$$

Then

$$\Theta(a_i)f(\omega) \prec q(\omega).$$

The next result is an extension of Theorem (2.1) to the case where the behavior of q on $\triangle$ is not known.

Corollary 2.1. Let $\Omega \subset \mathbb{C}$, q be univalent in $\triangle$ and $q(0)=0$.Let $\emptyset \in \Phi_k\left[\Omega, q_\rho\right]$ for some $\rho \in (0,1)$, where $q_\rho(\omega) = q(\rho\omega)$. If $f \in \mathcal{D}_p$satisfies

$$\emptyset\left(\left(\Theta(a_i)f(\omega)\right), \Theta(a_i + 1)f(\omega), \Theta(a_i + 2); \omega \in \triangle\right) \in \Omega, \quad (2.13)$$

Then

$$\left(\Theta(a_i)f(\omega)\right) \prec q(\omega).$$

Proof. From Theorem 2.1,we have $\left(\Theta(a_i)f(\omega)\right), \prec q_p(\omega)$.

Theorem 2.3. Let h and q are univalent .Also ,$q(0)=0$,$q_\rho(\omega) = q(\rho\omega)$ and $h_\rho(\omega) = h(\rho\omega)$. Let $\emptyset: \mathbb{C}^3 \times \triangle \times \overline{\triangle} \rightarrow \mathbb{C}$ satisfies one of the following conditions:

(1) $\emptyset \in \Phi_k\left[\Omega, q_\rho\right]$for some $\rho \in (0,1)$ or

(2) There exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k\left[h_\rho, q_\rho\right]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{D}\rho$ satisfies (2.12), then

$$\left(\Theta(a_i)f(\omega)\right) \prec q(\omega).$$

Proof:Case (1): By using Theorem (2.1) , we get $\left(\Theta(a_i)f(\omega)\right), \prec q_\rho(\omega)$. Since $q_\rho(\omega) \prec q(\omega)$, we deduce

$$\left(\Theta(a_i)f(\omega)\right) \prec q(\omega).$$

Case (2): Assume that $F(\omega)=\left(\Theta(a_i)f(\omega)\right)$, and $F_\rho(\omega) = F(\rho\omega)$. So that

$\emptyset\left(F_\rho(\omega), \omega F_\rho'(\omega), \omega^2 F_\rho''(\omega); \rho\omega\right) = \emptyset(F(\rho\omega), \rho\omega F'(\rho\omega), \rho^2\omega^2 F''(\rho\omega); \rho\omega) \in h_\rho(\triangle)$.

By using Theorem 2.1 with associated

$\emptyset(F(\omega), \omega F'(\omega), \omega^2 F''(\omega); w(\omega)) \in \Omega$, where w is any function mapping from

$\triangle$ onto $\triangle$, with $w(\omega)=\rho\omega$, we obtain $F_\rho(\omega) \prec q_\rho(\omega)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $F(\omega) \prec q(\omega)$.
Therefore

$$\left(\Theta(\alpha_i)f(\omega)\right) \prec q(\omega)$$

The next theorem gives the best dominant of the differential subordination (2.12).

Theorem2.4. Let h be univalent in $\triangle$ and let $\emptyset: \mathbb{C}^3 \times \triangle \times \overline{\triangle} \to \mathbb{C}$. Suppose that the differential equation

$$\emptyset\left(q(\omega), \frac{A_i\omega q'(\omega) + (A_i - \alpha_i)q(\omega)}{\alpha_i}, \frac{A_i{}^2\omega^2 q''(\omega) + A_i(2\alpha_i + 1 - A_i)\omega q'(\omega) + (A_i - \alpha_i)(A_i - \alpha_i - 1)q(\omega)}{\alpha_i(\alpha_i + 1)}; \omega\right) = h(\omega), \tag{2.14}$$

has a solution q with $q(0)=0$ and satisfies one of the following conditions:
(1) $q \in Q_0$ and $\emptyset \in \Phi_k[h, q]$.
(2) q is univalent in $\triangle$ and $\emptyset \in \Phi_k[h, q_\rho]$ for some $\rho \in (0,1)$, or
(3) q is univalent in $\triangle$ and there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, q_\rho]$, for all $\rho_0 \in (0,1)$. If $f \in \mathcal{D}\rho$ satisfies (2.12), then $\left(\Theta(\alpha_i)f(\omega)\right), \prec q(\omega)$ and q is the bestdominant.

Proof. By using Theorem 2.2 and Theorem 2.3, we get that q is a dominant of (2.12). Since q satisfies (2.14), it is also a solution of (2.12) and therefore q will be dominant by all dominants of (2.12). Hence q is the best dominant of (2.12).

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_k[\Omega, q]$ consists of functions $\emptyset: \mathbb{C}^3 \times \triangle \times \overline{\triangle} \to \mathbb{C}$ such that

$$\emptyset\left(Me^{i\theta}, \frac{(A_i k - (A_i - \alpha_i))Me^{i\theta}}{\alpha_i}, \frac{A_i{}^2 L + [A_i k(2\alpha_i - A_i + 1) + (A_i - \alpha_i)(A_i - \alpha_i - 1)]Me^{i\theta}}{\alpha_i(\alpha_i + 1)}; \omega\right) \notin \Omega, \tag{2.15}$$

Where $\lambda>0$, $\theta \in \mathcal{R}, \mathcal{R}\left(Le^{i\theta}\right) \geq k(k-1)M for\ all\ real\ \theta$, $k \geq 1$, $\omega \in \triangle$.

Corollary 2.2: Let $\emptyset \in \Phi_k[\Omega, M]$. If $f \in \mathcal{D}\rho$ satisfies that

$$\emptyset\left(\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega\right) \in \Omega, \text{then } \Theta(\alpha_i)f(\omega) \prec M\omega.$$

Corollary 2.3: Let $\emptyset \in \Phi_k[\Omega, M]$. If $f \in \mathcal{D}\rho$ satisfies that

$$\left|\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega\right| < M, \text{then } |\Theta(\alpha_i)f(\omega)| < M$$

Corollary 2.4: Let $M>0$, $\alpha_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, \mathcal{R}(\alpha_i) \geq 0$ and $f \in \mathcal{D}\rho$ satisfies the following inequality

$$|\Theta(\alpha_i + 1)f(\omega)| < M, \text{then } |\Theta(\alpha_i + 1)f(\omega)| < M, \boldsymbol{\omega} \in \triangle.$$

Proof. From Corollary (2.2) one can take $\emptyset(u, v, w, \omega,) = v = \frac{(A_i k - (A_i - \alpha_i))Me^{i\theta}}{\alpha_i}$

Corollary 2.5. If $M>0$, $\alpha_i \in \mathbb{C}\backslash\mathbb{Z}_0^-$, If $f \in \mathcal{D}\rho$ satisfies the following inequality

$$\left|\Theta(\alpha_i + 1)f(\omega) - \left(\frac{A_i}{\alpha_i} - 1\right)\Theta(\alpha_i)f(\omega)\right| < \frac{MA_i}{\alpha_i}$$

then $|\Theta(\alpha_i)f(\omega)| < M, , \boldsymbol{\omega} \in \triangle$.

Proof. Let $\emptyset(u, v, w, \omega,) = v + \left(\frac{A_i}{\alpha_i} - 1\right)u$ and $\Omega = h(\triangle)$ where $h(\boldsymbol{\omega}) = \frac{MA_i}{\alpha_i}\boldsymbol{\omega}, M > 0$.

From the corollary (1.3), it is enough to show that $\phi \in \Phi_k[\Omega, q]$, that means the admissibility condition (1.6) is satisfied. Hence,

$$\left|\phi\left(Me^{i\theta}, \frac{(A_i k - (A_i - \alpha_i))Me^{i\theta}}{\alpha_i}, \frac{A_i{}^2 L + [A_i k(2\alpha_i - A_i + 1) + (A_i - \alpha_i)(A_i - \alpha_i - 1)]Me^{i\theta}}{\alpha_i(\alpha_i + 1)}; \omega\right)\right| \geq \left|\frac{A_i Me^{i\theta}}{\alpha_i}\right| \geq \frac{MA_i}{|\alpha_i|}$$

Whenever $\boldsymbol{\omega} \in \triangle$. $\theta \in \boldsymbol{\mathcal{R}}$, $\alpha_i \in \mathbb{C}\backslash\mathbb{Z}_0^-, \alpha_i \neq -1$ and $k \geq 1$.

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{k,1}[\Omega, q]$ consists of $\emptyset: \mathbb{C}^3 \times \triangle \to \mathbb{C}$ that satisfies

$$\emptyset(u, v, w, \omega, \xi) \notin \Omega,$$

whenever

$$u = q(\xi), v = \frac{1}{\alpha_i + 1}\left(\frac{kA\xi q'(\xi)}{q(\xi)} + \alpha_i q(\xi) + 1\right), \alpha_i \in \mathbb{C}\backslash\mathbb{Z}_0^-$$

and

$$\Re\left\{\frac{v(\alpha_i + 1)\left((\alpha_i + 1)(w - v) + w - 1\right)}{A_i(v(\alpha_i + 1) - \alpha_i u - 1)} + \frac{(v(\alpha_i + 1) - \alpha_i u(A_i + 1) - 1)}{A_i}\right\}$$
$$\geq k\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$\omega \in \triangle, \zeta \in \partial\triangle\backslash E(q), k \geq 1$ *and* $\alpha_i \neq -1$.

Theorem 2.5. Let $\emptyset \in \Phi'_k[h, q]$. If $f \in \mathcal{D}\rho, \Theta(\alpha_i)f(\omega) \in Q_0$ and

$$\emptyset\left(\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega\right)$$

is univalent in$\triangle$,then

$\Omega \subset \left\{\emptyset\left(\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega \in \triangle, \xi \in \overline{\triangle}\right)\right\}$ which implies that

$$q(z) \prec \Theta(\alpha_i)f(\omega).$$

Proof: By (11) and $\Omega \subset \{\emptyset\left(\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega \in \triangle, \xi \in \overline{\triangle}\right)$,
we have$\Omega \subset \{\psi(F(\omega), \omega F'(\omega), \omega^2 F''(\omega); \omega, \xi); \omega \in \triangle, \xi \in \overline{\triangle})\}$.
from

$$u = r \quad , v = \frac{A_i S - (A_i - \alpha_i)r}{\alpha_i}, \quad w = \frac{{A_i}^2 t + A_i(2\alpha_i + 1 - A_i)S + (A_i - \alpha_i)(A_i - \alpha_i - 1)r}{\alpha_i(\alpha_i + 1)},$$

we see that the admissibility for $\emptyset \in \Phi'_k[\Omega, q]$ is equivalent to admissibility condition for ψ. Hence , $\psi \in \Psi'[\Omega, q]$ and so we have $q(z) \prec \Theta(\alpha_i)f(\omega)$.

The following Theorem is immediately consequence of Theorem(2.5).

Theorem2.6. Let $q \in \mu[0, p], h$ be analytic in $\triangle$ and$\emptyset \in \Phi'_k[h, q]$. If $f(\omega) \in \mathcal{D}\rho$,

$$\Theta(\alpha_i)f(\omega) \in Q_0$$

and

$$\{\emptyset\left(\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega\right\}$$

It is univalent in $\triangle$, then

$$h(\omega) \prec \{\emptyset\left(\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega\right\}, \tag{2.18}$$

which implies that $q(\omega) \prec \Theta(\alpha_i)f(\omega)$.

Theorem 2.7. Let h be analytic in $\triangle$ and $\emptyset: \mathbb{C}^3 \times \triangle \times \overline{\triangle} \to \mathbb{C}$.Suppose that

$$\emptyset\left(q(\omega), \frac{A_i\omega q'(\omega) + (A_i - \alpha_i)q(\omega)}{\alpha_i}, \frac{{A_i}^2\omega^2 q''(\omega) + A_i(2\alpha_i + 1 - A_i)\omega q'(\omega) + (A_i - \alpha_i)(A_i - \alpha_i - 1)q(\omega)}{\alpha_i(\alpha_i + 1)}; \omega\right) = h(\omega),$$

has a solution $q \in Q_0$. If $\emptyset \in \Phi'_k[h, q]$, $f \in \mathcal{D}\rho, \Theta(\alpha_i)f(\omega) \in Q_0$ and
$\emptyset\left\{\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega\right\}$ is univalent in $\triangle$, then

$$h(z) \prec \emptyset\left(\left(\Theta(\alpha_i)f(\omega)\right), \Theta(\alpha_i + 1)f(\omega), \Theta(\alpha_i + 2); \omega\right\}, \tag{2.19}$$

implies that $q(z) \prec \Theta(\alpha_i)f(\omega)$, and q is the best dominant.

Proof: The proof of this Theorem is the same as the proof Theorem (2.4).

Theorem (2.2) and Theorem (2.6), we obtained the following Theorem.

Theorem 2.8 Suppose that $h_1$and q_1 are analytic functions in $\triangle$, and h_2 is a univalent functions in $\triangle$,$q_2 \in Q_0$ with $q_1(0) = q_2(0) = 0$ $with\ \emptyset \in \Phi_k[h_2, q_2] \cap \Phi'_k[h_1, q_1]$.

If $f \in \mathcal{D}\rho$,$\Theta(\mathrm{a}_i)f(\omega) \in \mu[0,p] \cap Q_0$ and

$\{\emptyset\left(\left(\Theta(\mathrm{a}_i)f(\omega)\right), \Theta(\mathrm{a}_i+1)f(\omega), \Theta(\mathrm{a}_i+2); \omega\right\}$ is univalent in $\triangle$, then

$$h_1(\omega) \prec \emptyset\left(\left(\Theta(\mathrm{a}_i)f(\omega)\right), \Theta(\mathrm{a}_i+1)f(\omega), \Theta(\mathrm{a}_i+2); \omega\right\} \prec h_2, \quad (2.20)$$

And this implies that $q_1(\omega) \prec \Theta(\mathrm{a}_i)f(\omega) \prec q_2(\omega)$.

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