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Bifurcation Diagram of $\mathcal{W}(u_j;\tau)$-Function with $(\wp,q)$-Parameters

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Abstract

This study aims to classify the critical points of functions with 4 variables and 8 parameters, we found the caustic for the certain function with the spreading of the critical points. Finally, as an application, we found the bifurcation solutions for the equation of sixth order with boundary conditions using the Lyapunov-Schmidt method in the variational case.

Keyword: Classification of critical points, caustic and Lyapunov-Schmidt method.

مخطط التفرع للدالة $\mathcal{W}(u_j;\tau)$ مع المعلمات $(\wp,q)$

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الخلاصة

تهدف هذه الدراسة الى تصنيف النقاط الحرجة لدالة ذات أربعة متغيرات وثمان معلمات. إضافة الى إيجاد الكاوستك للدالة مع انتشار نقاطها الحرجة. وكتطبيق على هذه الدراسة تم تقديم حلول التفرع لمعادلة من الرتبة السادسة بشروط حدية باستخدام طريقة Lyapunov–Schmidt في الحالة التغاير

1. Preface and elementary concepts.

It is possible to represent many nonlinear problems that appear in mathematics and physics by the operator equation below

$$\Omega(u,t)=\wp,\ u\in\mathcal{O}\subset\mathcal{C},\ \wp\in\mathcal{W},\ t\in\mathcal{R}^n. \tag{I}$$

Where $\mathcal{C}$ and $\mathcal{W}$ are Banach spaces and $\mathcal{O}$ is an open set. To solve this equation, the process of finite-dimensional reduction can be used. The technique is based on the reduction of Lyapunov-Schmidt to a reduced equation (1) to a finite-dimensional equation.

$$\mathcal{Q}(r,t)=\epsilon,\ r\in\hat{E},\ \epsilon\in\hat{K}. \tag{II}$$

Where $\hat{E}$ and $\hat{K}$ are smooth finite-dimensional manifold.

The solution of equations (I), and (II) is presented by Lyapunov [1] and Schmidt [2].

Furthermore, many researchers have discussed transform the equation (I) to equation (II) using the local method of Lyapunov–Schmidt with the conditions that equation (II) [3–5]. Abdul Hussain [6] introduced a study about theory of Fredholm functional on Banach manifolds and its applications.

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Critical points of smooth functions play an important role in the study of bifurcation solutions of BVPs. There are many articles of them in more than variable. see [7].
More than one study of classification of critical points, geometric description of caustic and find bifurcation solutions of nonlinear differential equations using Lyapunov-Schmidt method was introduced in ([8–14] and [15-18]).
In this work, we classify the critical points of functions with 4 variables and 8 parameters, and we found a caustic for the function. Finally, we can find the bifurcation solutions for the equation for the sixth order with boundary conditions using the Lyapunov-Schmidt method in the variational case.
Definition 1.1 [14]**.** Let h be a function from $\mathcal{R}^n$ into $\mathcal{R}$. then a function h has a critical point at u_0 if $\frac{\partial h}{\partial u}(u_0) = 0$.
Definition 1.2 [14]**.** Let Υ is an open set such that $\Upsilon \subset \mathcal{R}^n$, and u_0 is a critical point of h, the Hessian matrix of h at u_0 is the symmetric $m \times m$ matrix of second partial derivative
$\mathcal{H}_h(u_0) = (A_{ij}), \quad i, j = 1,2, \dots, n$.
Where $A_{ij} = \frac{\partial^2 h}{\partial u_i \partial u_j}$.
Definition 1.3 [15]**.** A critical point u_0 of h is non-degenerate if $det\big(\mathcal{H}_h(u_0)\big) \neq 0$.
Definition 1.4 [9]**.** Suppose that $\Upsilon \subset \mathcal{R}^n$ and $\Gamma \subset \mathcal{R}^m$ are open sets. A function $h: \Upsilon \to \Gamma$ is said to be diffeomorphism if

(I) h is a differentiable and bijective function;

(2) h^{-1} is a differentiable function.

Definition 1.5 [15]**.** functions $h, k: (\mathcal{R}^n, 0) \to (\mathcal{R}^m, 0)$ are said to be germ equivalent at $m \in \mathcal{R}^n$ if m is in both of their domains, and there is a neighborhood Υ of m such that for all $u \in \Upsilon, h(u) = k(u)$. A function germ at a point m is the equivalent class of germ equivalent function.
Definition 1.6 [8]**.** A continuous-linear operator of Banach spaces $\Omega: \mathcal{C} \to \mathcal{W}$ is called Fredholm operator if and only if $\dim(ker\Omega) \prec \infty$ and $\dim(Coker\Omega) \prec \infty$.
A Fredholm index $(ind(\Omega))$ is given by $dim(ker\Omega) - dim(Coker\Omega)$.
$Coker\Omega = \mathcal{W} \backslash rang\ (\mathcal{W})$.

Definition 1.7 [8]**.** Let l be an open set. A nonlinear map $\rho: l \subset \mathcal{C} \to \mathcal{W}$ is called Fredholm if $\frac{\partial \rho}{\partial u}(u)$ is a Fredholm operator $u \in l$. (i.e. $ind(\rho) = ind(\frac{\partial \rho}{\partial u}(u))$).
Definition 1.8 [15]**. (Local-algebra)**
The quotient of the algebra of function germs is given by the gradient ideal of the function $h(u, \mu)$

$$\Sigma_h^u = \frac{\zeta_n}{(\frac{\partial h}{\partial u_1}, \frac{\partial h}{\partial u_2}, \dots, \frac{\partial h}{\partial u_n})}$$

where ζ_n the set of every smooth function germ at the origin on $\mathcal{R}^n$.
Then, Σ_h^u is called local-algebra of the singularity of h at the origin.
The multiplicity $\omega(h)$ of the critical point is the dimension of its local-algebra:
$\omega(h) = dim\ \Sigma_h^u$
If $dim\ \Sigma_h^u < \infty$, then a critical point is called an isolated point.
Theorem 1.9 [15]**.** The multiplicity $\omega(h)$ of the isolated critical point is equal to the number of Morse-critical points into which it decomposes under a generic deformation of the function.
Definition 1.10 [15]**.** Let $h, k: (\mathcal{R}^n, 0) \to (\mathcal{R}^m, 0)$ be two functions of germs. h, k are said to be contact equivalent (C-equivalent), if there exist;

(I) A diffeomorphism Ω of the source $(\mathcal{R}^n, 0)$;

(II) A matrix $\mathcal{M} \in GL_m(\zeta_n)$ when $GL_m(\zeta_n)$ is the set of invertible $m \times m$ matrices whose entries are in ζ_n, such that $h \circ \aleph(u) = \mathcal{M}(u)k(u)$, where $h(u)$ and $k(u)$ are written as column-vectors, and $h \circ \aleph(u)$ is the usual product of matrix times vector.

The previous idea of C-equivalent of two function h, k is that the solution sets $h^{-1}(0)$ and $k^{-1}(0)$ should be diffeomorphic.

Lemma 1.11 [15]**.** If h and k are C-equivalent then their o-sets are diffeomorphic.

Theorem 1.12 [15]**.** Let h, k be two function germs. Then the following (1), (2), and (3) are equivalent

(1) h and k are contact equivalent;

(2) The ideal P and S are induced isomorphic ideal in ζ_n, where P and S are generated by the components of h and k;

(3) $\frac{\zeta_n}{P}$ and $\frac{\zeta_n}{S}$ are induced isomorphic.

2. Main Results

This part includes finding the critical points and geometric description of the caustic set of the function that is defined by

$$\mathcal{W}(u_j;\tau) = \frac{1}{3}u_1^3 + \frac{1}{3}u_3^3 - u_1^2u_3 + u_1u_2^2 + u_3^2u_1 + u_4^2u_1 + u_3u_2^2 + u_3u_4^2 + u_1u_2u_4 + u_2u_3u_4 + \wp_1u_1^2 + \wp_2u_2^2 + \wp_3u_3^2 + \wp_4u_4^2 - q_1u_1 - q_2u_2 - q_3u_3 - q_4u_4 \qquad (2.1)$$

Where $u_j = u_{1,2,3,4}$, $\tau = \{\wp_{1,2,3,4}, q_{1,2,3,4}\}$ such that $\wp, q$ are parameters.

To study the bifurcation points of function (2.1) it is convenient to change variables into complex plane, so we assume that

$\zeta_1 = u_1 + iu_2, \zeta_2 = u_3 + iu_4$

In the complex plane, the function (2.1) has the form

$$\mathcal{W}(\zeta;\tau) = -\frac{\zeta_1^3}{12} - \frac{\zeta_2^3}{12} - \frac{\bar{\zeta}_1^3}{12} - \frac{\bar{\zeta}_2^3}{12} - \frac{\bar{\zeta}_1\bar{\zeta}_2^2}{8} - \frac{3\bar{\zeta}_1^2\bar{\zeta}_2}{8} - \frac{\zeta_1\zeta_2^2}{8} - \frac{3\zeta_2\zeta_1^2}{8} - \frac{\wp_2\zeta_1^2}{4} - \frac{\wp_4\zeta_2^2}{4} + \frac{\bar{\zeta}_1^2\zeta_1}{4} + \frac{\zeta_2^2\bar{\zeta}_2}{4} + \frac{\bar{\zeta}_2^2\zeta_2}{4} + \frac{\bar{\zeta}_1\zeta_1^2}{4} - \frac{\wp_2\bar{\zeta}_1^2}{4} - \frac{\bar{\zeta}_2\zeta_1^2}{8} - \frac{\wp_4\bar{\zeta}_2^2}{4} - \frac{\bar{\zeta}_1^2\zeta_2}{8} + \frac{\bar{\zeta}_1\zeta_2^2}{8} + \frac{\bar{\zeta}_2^2\zeta_1}{8} + \frac{\wp_3\zeta_2^2}{4} + \frac{\wp_3\bar{\zeta}_2^2}{4} + \frac{\wp_1\zeta_1^2}{4} + \frac{\wp_1\bar{\zeta}_1^2}{4} + \frac{\wp_1|\zeta_1|^2}{2} + \frac{\bar{\zeta}_1|\zeta_2|^2}{2} + \frac{|\zeta_2|^2\zeta_1}{2} + \frac{\wp_4|\zeta_2|^2}{2} + \frac{\wp_2|\zeta_1|^2}{2} + \frac{\wp_3|\zeta_2|^2}{2} - \frac{q_2\zeta_1}{2} - \frac{q_1\bar{\zeta}_1}{2} - \frac{q_3\zeta_2}{2} - \frac{q_3\bar{\zeta}_2}{2} + \frac{q_2\zeta_1}{2}i - \frac{q_2\bar{\zeta}_1}{2}i + \frac{q_4\zeta_2}{2}i - \frac{q_4\bar{\zeta}_2}{2}i \qquad (2.2)$$

Where $|\zeta_1|^2 = u_1^2 + u_2^2, |\zeta_2|^2 = u_3^2 + u_4^2$ and $\bar{\zeta}_1, \bar{\zeta}_2$ conjugates of ζ_1, ζ_2 respectively.

To study the function's behaviour (2.1) near the critical point, it is convenient to consider this function in polar coordination, $\zeta_1 = r_1e^{i\theta_1},\ \zeta_2 = r_2e^{i\theta_2} \ldots.(*)$

Since we are interested to find the set of all parameters values of the equation, which makes the equation having a real solution. So the real part of the equation $(*)$ is

$$\Gamma(r_{1,2};\vartheta_{1,2,3,4,5,6,7,8}) = \vartheta_1r_1^3 + \vartheta_2r_2^2r_1 + \vartheta_3r_1^2r_2 + \vartheta_4r_2^3 + \vartheta_5r_1^2 + \vartheta_6r_2^2 + \vartheta_7r_2 + \vartheta_8r_1 \qquad (2.3)$$

where, $\vartheta_1 = \left(\frac{cos\theta_1}{2} - \frac{cos3\theta_1}{6}\right), \vartheta_2 = \left(cos\theta_1 - \frac{cos(\theta_1+2\theta_2)+\cos(\theta_1-2\theta_2)}{4}\right)$,

$\vartheta_3 = \left(\frac{-3\cos(2\vartheta_1+\vartheta_2)-\cos(2\vartheta_1-\vartheta_2)}{4}\right)$, $\vartheta_4 = \left(\frac{cos\theta_2}{2} - \frac{cos3\theta_2}{6}\right)$, $\vartheta_5 = \left(\frac{\wp_1+\wp_2}{2} + \frac{(\wp_1-\wp_2)cos2\theta_1}{2}\right)$,

$\vartheta_6 = \left(\frac{\wp_3+\wp_4}{2} + \frac{(\wp_3-\wp_4)cos2\theta_2}{2}\right)$, $\vartheta_7 = (-q_3cos\theta_2 - q_4sin\theta_2)$

$$\vartheta_8 = (-q_1cos\theta_1 - q_2sin\theta_1) \qquad (2.4)$$

Now, suppose that $r_1 = \frac{(\varpi_1+\varpi_2)}{2\vartheta_1}$; $r_2 = \frac{(\varpi_1-\varpi_2)}{2\vartheta_4}$ we obtain the equation below, which is equivalent to the equation (2.3)

$$\Gamma(r_{1,2};\vartheta_{1,2,3,4,5,6,7,8}) = \frac{(\varpi_1+\varpi_2)^3}{8\vartheta_1{}^2} + \frac{\vartheta_2(\varpi_1+\varpi_2)(\varpi_1-\varpi_2)^2}{8\vartheta_4^2\vartheta_1} + \frac{\vartheta_3(\varpi_1-\varpi_2)(\varpi_1+\varpi_2)^2}{8\vartheta_4^2\vartheta_1} + \frac{(\varpi_1-\varpi_2)^3}{8\vartheta_4{}^2} + \frac{\vartheta_5(\varpi_1+\varpi_2)^2}{4\vartheta_1^2} + \frac{\vartheta_6(\varpi_1-\varpi_2)^2}{4\vartheta_4^2} + \frac{\vartheta_7(\varpi_1-\varpi_2)}{4\vartheta_4} + \frac{\vartheta_8(\varpi_1+\varpi_2)}{4\vartheta_1}. \qquad (2.5)$$

Which is equivalent to
$\wp = \sigma_1\varpi_1^3 + \sigma_2\varpi_1^2\varpi_2 + \sigma_3\varpi_1^2 + \sigma_4\varpi_1\varpi_2^2 + \sigma_5\varpi_1\varpi_2 + \sigma_6\varpi_3\varpi_2^3 + \sigma_3\varpi_2^2 + \sigma_7\varpi_1 + \sigma_8\varpi_2$
(2.6)
where, $\sigma_1 = \left(\frac{({\vartheta_1}^2+\vartheta_2\vartheta_1+\vartheta_4\vartheta_3+{\vartheta_4}^2)}{8{\vartheta_1}^2{\vartheta_4}^2}\right), \sigma_2 = \left(\frac{(-3{\vartheta_1}^2-\vartheta_2\vartheta_1+\vartheta_4\vartheta_3+3{\vartheta_4}^2)}{8{\vartheta_1}^2{\vartheta_4}^2}\right),$
$\sigma_3 = \left(\frac{2\vartheta_6\vartheta_1^2+2\vartheta_5\vartheta_4^2}{8{\vartheta_1}^2{\vartheta_4}^2}\right), \sigma_4 = \left(\frac{(3{\vartheta_1}^2-\vartheta_2\vartheta_1-\vartheta_4\vartheta_3+3{\vartheta_4}^2)}{8{\vartheta_1}^2{\vartheta_4}^2}\right), \sigma_5 = \left(\frac{-\vartheta_6\vartheta_1^2+\vartheta_5\vartheta_4^2}{2{\vartheta_1}^2{\vartheta_4}^2}\right)$
$\sigma_6 = \left(\frac{(-{\vartheta_1}^2+\vartheta_2\vartheta_1-\vartheta_4\vartheta_3+{\vartheta_4}^2)}{8{\vartheta_1}^2{\vartheta_4}^2}\right), \sigma_7 = \left(\frac{\vartheta_8\vartheta_1\vartheta_4^2+\vartheta_7\vartheta_4\vartheta_1^2}{2{\vartheta_1}^2{\vartheta_4}^2}\right), \ \sigma_8 = \left(\frac{\vartheta_8\vartheta_1\vartheta_4^2-\vartheta_7\vartheta_4\vartheta_1^2}{2{\vartheta_1}^2{\vartheta_4}^2}\right)$
Also, by change variables $\varpi_1 := \frac{1}{\sqrt[3]{\sigma_1}} \cdot \mathcal{S}_1, \varpi_2 := \frac{1}{\sqrt[3]{\sigma_6}} \cdot \mathcal{S}_2$; and substituting into (2.6), we get the equivalent equation
$\mathcal{O} = \mathcal{S}_1^3 + \mathcal{L}_1\mathcal{S}_1^2\mathcal{S}_2 + \mathcal{L}_2\mathcal{S}_1\mathcal{S}_2^2 + \mathcal{S}_2^3 + \mathcal{L}_3\mathcal{S}_1^2 + \mathcal{L}_4\mathcal{S}_2^2 + \mathcal{L}_5\mathcal{S}_1\mathcal{S}_2 + \mathcal{L}_6\mathcal{S}_1 + \mathcal{L}_7\mathcal{S}_2$ (2.7)
where
$\mathcal{L}_1 = \left(\frac{\sigma_2}{\sigma_1^{\frac{2}{3}}\sigma_6^{\frac{1}{3}}}\right); \mathcal{L}_2 = \left(\frac{\sigma_4}{\sigma_1^{\frac{1}{3}}\sigma_6^{\frac{2}{3}}}\right);$ $\mathcal{L}_3 = \left(\frac{\sigma_3}{\sigma_1^{\frac{2}{3}}}\right); \mathcal{L}_4 = \left(\frac{\sigma_3}{\sigma_6^{\frac{2}{3}}}\right);$ $\mathcal{L}_5 = \left(\frac{\sigma_5}{\sigma_1^{\frac{1}{3}}\sigma_6^{\frac{1}{3}}}\right);$ $\mathcal{L}_6 = \left(\frac{\sigma_7}{\sigma_1^{\frac{1}{3}}}\right);$ $\mathcal{L}_7 = \left(\frac{\sigma_8}{\sigma_6^{\frac{1}{3}}}\right).$

The elements $\mathcal{S}_1^2\mathcal{S}_2, \mathcal{S}_1\mathcal{S}_2^2, \mathcal{S}_1^2, \mathcal{S}_2^2$, belongs to the tangent space generated by $\frac{\partial \mathcal{O}}{\partial \mathcal{S}_1}, \frac{\partial \mathcal{O}}{\partial \mathcal{S}_2}$ then, from theorem of germs we have that the function is equivalent to the following function
$U = \mathcal{L}_5\mathcal{S}_1\mathcal{S}_2 + \mathcal{S}_1^3 + \mathcal{S}_2^3 + \mathcal{L}_6\mathcal{S}_1 + \mathcal{L}_7\mathcal{S}_2$ (2.8)
The aim of the function (U) is to find the geometric description of the Caustic, and then classify the critical points of this function by determine the types of the critical points. The critical points of the function (U) are the solutions of the following system of nonlinear-algebraic equations,

$$\mathcal{L}_5\mathcal{S}_2 + 3\mathcal{S}_1^2 + \mathcal{L}_6 = 0;$$
$$\mathcal{L}_5\mathcal{S}_1 + 3\mathcal{S}_2^2 + \mathcal{L}_7 = 0$$

All the critical points of function U are degenerate on the surface given by the equation

$$-\mathcal{L}_5^2 + 36\mathcal{S}_1\mathcal{S}_2 = 0$$

Using program Maple17, we found that the caustic of the bifurcation set of equation (U) in the following
$\mathcal{L}_5(\mathcal{L}_5^8 - 96\mathcal{L}_5^4\mathcal{L}_6\mathcal{L}_7 - 256\mathcal{L}_5^2\mathcal{L}_6^3 - 256\mathcal{L}_5^2\mathcal{L}_7^3 - 768\mathcal{L}_6^2\mathcal{L}_7^2) = 0$ (2.9)
Using the Maple 17 program, we can to describe the Caustic set of equation (U) in the $\mathcal{L}_6\mathcal{L}_7$ −plane for some value $\mathcal{L}_5$.

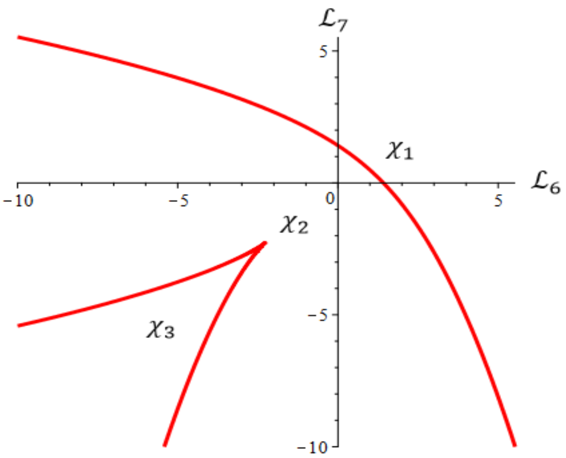

Figure 1-Describe Caustic when $\mathcal{L}_5 = 3$

The Caustic divided the parameter plane into (8) regions in the diagram above, each region having a fixed number of critical points. In the χ_1 region there is zero critical point, in the χ_2 region, there are two critical points ($1\ minimum\ and\ 1\ saddle$). In the χ_3 region, there are 4 critical points ($1\ minimum,\ 2\ saddle\ and\ 1\ maximum$).

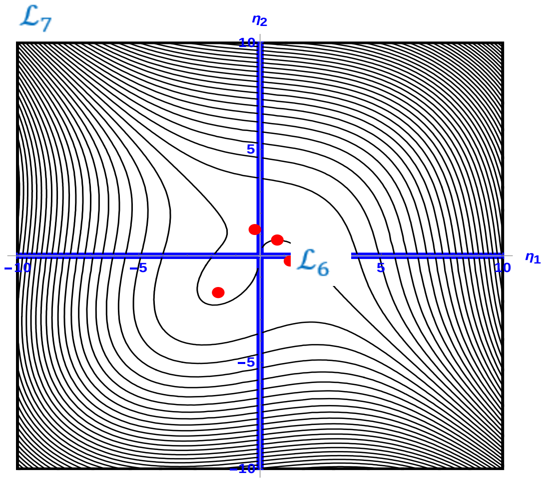

Figure 2- Contours and critical points in region χ_3 of caustic of function U

3. Application

An application of our results that obtained in section 2. We consider the following nonlinear differential equation.

$$\upsilon\left(\frac{d^6\varrho}{dg^6}\right)+\tau\left(\frac{d^4\varrho}{dg^4}\right)+\sigma\left(\frac{d^2\varrho}{dg^2}\right)+\Phi\varrho+\varrho^2=\mathcal{F} \tag{3.1}$$

$$\frac{d^4\varrho}{dg^4}(0)=\frac{d^2\varrho}{dg^2}(0)=\varrho(0)=\frac{d^4\varrho}{dg^4}(\pi)=\frac{d^2\varrho}{dg^2}(\pi)=\varrho(\pi)=0$$

where $\upsilon,\tau,\sigma,$ and Φ are the parameters of the problem, $\varrho=\varrho(g),\ g\in[0,\pi]$.

Suppose that $\widetilde{\mathcal{B}}:\mathcal{G}\to\mathcal{H}$ is Fredholm operator of index zero, where $\mathcal{G}=C^6([0,\pi],\mathcal{R})$ and $\mathcal{H}=C^0([0,1],\mathcal{R})$ is defined by the operator equation,

$\widetilde{\mathcal{B}}(\varrho,l)=\upsilon\left(\frac{d^6\varrho}{dg^6}\right)+\tau\left(\frac{d^4\varrho}{dg^4}\right)+\sigma\left(\frac{d^2\varrho}{dg^2}\right)+\Phi\varrho+\varrho^2,\ \ l=(\upsilon,\tau,\sigma,\Phi)$

Every solution of the equation in (3.1) is a solution of the operator equations,

$$\widetilde{\mathcal{B}}(\varrho,l)=\mathcal{F},\ \ \mathcal{F}\in\mathcal{H} \tag{3.2}$$

Since the operator $\widetilde{\mathcal{B}}$ have variational property, then there exists functional $\tilde{\mathcal{A}}$ such that $\widetilde{\mathcal{B}}(\varrho,l)=\nabla_H\tilde{\mathcal{A}}(\varrho,l,0)$ or equivalently,

$$\frac{\partial\tilde{\mathcal{A}}}{\partial\varrho}(\varrho,l)k=\langle\mathcal{H}(\varrho,l),k\rangle\quad\forall\varrho\in\mathcal{C},\qquad k\in\mathcal{G}$$

and then every solution of equation (3.2) is a critical point of the functional $\tilde{\mathcal{A}}$ where,

$$\tilde{\mathcal{A}}(\varrho,l,\mathcal{F})=\int_0^\pi(\upsilon\,(\frac{(\varrho'''(g))^2}{2})+\tau\left(\frac{(\varrho''(g))^2}{2}\right)+\sigma\left(\frac{(\varrho'(g))^2}{2}\right)+\frac{1}{2}\Phi\varrho^2+\frac{1}{3}\varrho^3-\mathcal{F}\varrho)dg$$

Thus, the study of equation (3.2) is equivalent to the study extremely problem,

$\tilde{\mathcal{A}}(\varrho,l,\mathcal{F})\to extr,\ \ \varrho\in\mathcal{G}$.

The analysis of bifurcation can be found by using the local method of Lyapunov-Schmidt to reduce it into finite-dimensional space and by localized parameters

$\upsilon = \tilde{\upsilon} + \sigma_1,\ \tau = \tilde{\tau} + \sigma_2,\ \sigma = \tilde{\sigma} + \sigma_3$, and $\Phi = \tilde{\Phi} + \sigma_4$, where $\sigma_1, \sigma_2, \sigma_3$, and σ_4 are small parameters.
The reduction lead to the function in four variables,

$$\Xi(\eta, \sigma) = \inf_{g^{\pm}(\varrho, r_j) = \eta_j \forall j} \tilde{\mathcal{A}}(\varrho, \sigma), \quad \eta = (\eta_1, \eta_2, \eta_3, \eta_4)$$

$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$.
It is well known that in the reduction of Lyapunov-Schmidt the function, $\Xi(\eta, \sigma)$ is smooth. This function has all the topological and analytical properties of functional $\tilde{\mathcal{A}}$ [7]. In particular, for small σ there is 1-1 corresponding between the critical points of functional $\tilde{\mathcal{A}}$ and smooth function Ξ, preserving the type of critical points (multiplicity, index Morse, etc.) [7]. By the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (3.2) has the form:

$$\upsilon\, \tilde{y}'''''' + \tau \tilde{y}'''' + \sigma\, \tilde{y}'' + \Phi \tilde{y} = 0, \ \ \tilde{y} \in \mathcal{H}.$$

$$\frac{d^4 \tilde{y}}{dg^4}(0) = \frac{d^2 \tilde{y}}{dg^2}(0) = \tilde{y}(0) = \frac{d^4 \tilde{y}}{dg^4}(\pi) = \frac{d^2 \tilde{y}}{dg^2}(\pi) = \tilde{y}(\pi) = 0$$

The point $(\upsilon, \tau, \sigma, \Phi) = (0,0,0,0)$ is a bifurcation point [6]. Localized parameters $\upsilon, \tau, \sigma, \Phi$ as follow $\upsilon = 0 + \sigma_1,\ \tau = 0 + \sigma_2,\ \sigma = 0 + \sigma_3$, and $\Phi = 0 + \sigma_4$, lead to bifurcation along the modes $r_j(g) = c_j \sin(jg),\ j = 1,2,3,4$.

Where $\|r_j\| = 1$ and $c_j = \sqrt{\frac{2}{\pi}}$.

Let $S_d = Ker(N^*) = span\,\{r_1, r_2, r_3, r_4\}$,
where $N^* = d\tilde{\mathcal{B}}(0, l) = \upsilon \left(\frac{d^6 \varrho}{dg^6}\right) + \tau \left(\frac{d^4 \varrho}{dg^4}\right) + \sigma \left(\frac{d^2 \varrho}{dg^2}\right) + \Phi$
Then the space $\mathcal{G}$ can be decomposed in direct sum of two subspaces, $ker(N^*)$ and the orthogonal complement to $ker(N^*)$,

$$\mathcal{G} = \ker(N^*) \oplus \mathcal{G}^{\infty-4},\ \mathcal{G}^{\infty-4} = \ker(N^*)^{\perp} \cap \mathcal{G} = \{s \in \mathcal{G} : s \perp \ker(N^*)\}$$

Similarly, the space $\mathcal{H}$ can be decomposed in direct sum of two subspaces, $ker(N^*)$ and the orthogonal complement to $ker(N^*)$,

$$\mathcal{H} = ker(N^*) \oplus \mathcal{H}^{\infty-4}, \mathcal{H}^{\infty-4} = ker(N^*)^{\perp} \cap \mathcal{H} = \{a \in \mathcal{H} : a \perp ker(N^*)\}$$

Hence every vector $\varrho \in \mathcal{G}$, can be written in the form,
$\varrho = t + s,\ t = \sum_{j=1}^{4} r_j \eta_j \in ker(N^*), ker(N^*) \perp s \in \mathcal{G}^{\infty-4},\ \eta_j = \langle v, r_j \rangle$
Similarly,

$$\tilde{\mathcal{B}}(\varrho, l) = \tilde{\mathcal{B}}^{(4)}(\varrho, l) + \tilde{\mathcal{B}}^{(\infty-4)}(\varrho, l)$$

$$\tilde{\mathcal{B}}^{(4)}(\varrho, l) = \sum_{j=1}^{4} s_j(\varrho, l) r_j \in ker(N^*),\ \ \tilde{\mathcal{B}}^{(\infty-4)}(\varrho, l) \in \mathcal{H}^{\infty-4},$$

$$s_j(\varrho, l) = \langle \tilde{\mathcal{B}}(\varrho, l), r_j \rangle.$$

Where $\tilde{\mathcal{B}}^{(4)}(\varrho, l)$ is the projection of the space $\mathcal{H}$ on $ker(N^*)$ and $\tilde{\mathcal{B}}^{(\infty-4)}(\varrho, l)$ is the projection of the space $\mathcal{H}$ on $\mathcal{H}^{\infty-4}$.
Since $\mathcal{F} \in \mathcal{H}$ implies that
$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2,\ \ \mathcal{F}_1 = \sum_{j=1}^{4} r_j \eta_j \in N^*, \mathcal{F}_2 \in \mathcal{H}^{\infty-4}$.

According to equation (3.2), can be written in the form,

$$\tilde{\mathcal{B}}^{(4)}(\varrho, l) = \mathcal{F}_1$$

$$\tilde{\mathcal{B}}^{(\infty-4)}(\varrho, l) = \mathcal{F}_2.$$

By the implicit function theorem, there exist smooth function $\mathcal{C}^* : ker(N^*) \to \mathcal{G}^{\infty-4}$, such that $\Xi(\eta, \sigma, \mathcal{F}) = \tilde{\mathcal{A}}(\mathcal{C}^*(\eta, \sigma, \mathcal{F}), \sigma, \mathcal{F})$ and the key-function can be written in the form

$$\Xi(\eta,\sigma) = \tilde{\mathcal{A}}(r_1\eta_1 + r_2\eta_2 + r_3\eta_3 + r_4\eta_4 + \mathcal{C}^*(r_1\eta_1 + r_2\eta_2 + r_3\eta_3 + r_4\eta_4,\sigma),\sigma)$$
$$= P^*(\eta,\sigma) + o(|\eta|^4) + O(|\eta|^4)O(\sigma)$$

The function $P^*(\eta,\sigma)$ can be found as follows, substitute the value of ϱ in the previous Functional, we obtain

$$\upsilon\int_0^{\pi}(\frac{\left(\varrho'''(g)\right)^2}{2}dg = \upsilon\cdot\left(32\eta_2^2 + 2048\eta_4^2 + \frac{729\eta_3^2}{2} + \frac{\eta_1^2}{2}\right)$$
$$\tau\int_0^{\pi}\frac{\left(\varrho''(g)\right)^2}{2}dg = \tau\cdot\left(\frac{81\eta_3^2}{2} + 128\eta_4^2 + 8\eta_2^2 + \frac{\eta_1^2}{2}\right)$$
$$\sigma\int_0^{\pi}\frac{\left(\varrho'(g)\right)^2}{2}dg = \sigma\cdot\left(\frac{9\eta_3^2}{2} + 8\eta_4^2 + 2\eta_2^2 + \frac{\eta_1^2}{2}\right)$$
$$\Phi\int_0^{\pi}\frac{1}{2}\varrho^2 dg = \Phi.\left(\frac{\eta_3^2}{2} + \frac{1}{2}\eta_4^2 + \frac{1}{2}\eta_2^2 + \frac{\eta_1^2}{2}\right)$$
$$\int_0^{\pi}\frac{1}{3}\varrho^3 dg = \sqrt{2}\left(\frac{1}{\pi}\right)^{3/2}\left(\frac{4}{9}\eta_3^3 + \frac{4}{3}\eta_1^3 + \frac{16}{5}\eta_1\eta_2^2 + \frac{108}{35}\eta_1\eta_3^2 + \frac{64}{21}\eta_1\eta_4^2 - \frac{4}{5}\eta_1^2\eta_3 + \frac{16}{7}\eta_2^2\eta_3 + \frac{64}{55}\eta_3\eta_4^2 - \frac{64}{35}\eta_1\eta_2\eta_4 + \frac{64}{15}\eta_2\eta_3\eta_4\right)$$
$$\int_0^{\pi}\mathcal{F}_1\varrho dg = \tilde{q}_1\cdot\eta_1 + \tilde{q}_2\cdot\eta_2 + \tilde{q}_3\cdot\eta_3 + \tilde{q}_4\cdot\eta_4$$

then we get,

$$P^*(\eta,\sigma) = \frac{\upsilon(\eta_1^2 + 64\eta_2^2 + 729\eta_3^2 + 4096\eta_4^2)}{2} + \frac{\tau(\eta_1^2 + 16\eta_2^2 + 81\eta_3^2 + 256\eta_4^2)}{2} + \frac{\sigma\,(\eta_1^2 + 4\eta_2^2 + 9\eta_3^2 + 16\eta_4^2)}{2} + \frac{\Phi(\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2)}{4}$$
$$+\frac{1}{10395\pi}\left(8\sqrt{2}\sqrt{\frac{1}{\pi}}(1155\eta_1^3 - 693\eta_1^2\eta_3 + 2772\eta_1\eta_2^2 - 1584\eta_1\eta_2\eta_4 + 2673\eta_1\eta_3^2 + 2640\eta_1\eta_4^2 + 1980\eta_2^2\eta_3 + 3696\eta_2\eta_3\eta_4 + 385\eta_3^3 + 1008\eta_3\eta_4^2)\right)$$
$$-\tilde{q}_1\cdot\eta_1 - \tilde{q}_2\cdot\eta_2 - \tilde{q}_3\cdot\eta_3 - \tilde{q}_4\cdot\eta_4.$$

Hence

$$P^*(\eta,\sigma) = \frac{8\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_1^3}{9\pi} + \left(\frac{\upsilon}{2} + \frac{\tau}{2} + \frac{\sigma}{2} + \frac{\Phi}{4}\right)\eta_1^2 - \frac{8\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_3\eta_1^2}{15\pi} + \frac{32\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_1\eta_2^2}{15\pi} - \frac{128\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_4\eta_1\eta_2}{105\pi} + \frac{72\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_1\eta_3^2}{35\pi} - \tilde{q}_1\eta_1 + \frac{128\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_4^2\eta_1}{63\pi} + \left(32\,\upsilon + 8\tau + 2\sigma + \frac{\Phi}{4}\right)\eta_2^2 + \frac{32\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_3\eta_2^2}{21\pi} - \tilde{q}_2\eta_2 + \frac{128\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_3\eta_4\eta_2}{45\pi} + \frac{8\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_3^3}{27\pi}$$
$$+\left(\frac{729\alpha}{2} + \frac{81\sigma\pi}{2} + \frac{9\sigma}{2} + \frac{\Phi}{4}\right)\eta_1^2 - \tilde{q}_3\eta_3 + \frac{128\sqrt{2}\sqrt{\frac{1}{\pi}}\eta_4^2\eta_3}{25\pi} - \tilde{q}_4\eta_4 + \left(2048\upsilon + 128\tau + 8\sigma + \frac{\Phi}{4}\right)\eta_4^2.$$

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating for the functions Ξ are completely determined by its principal part P^*. By changing variables in the function P^* as follows

$$\eta_j = u_j$$

we have the function P^* is equivalent to the function $\mathcal{W}(u_j;\tau)$.

Hence the caustic of the function P^* coincides with the caustic of the function $\mathcal{W}(u_j;\tau)$.
The functions $\mathcal{W}(u_i;\tau)$ have all the topological and analytical properties of functional $\tilde{\mathcal{A}}$, so the study of bifurcation analysis of the equation (3.2) is equivalent to the study of bifurcation analysis of the function $\mathcal{W}(u_j;\tau)$. This shows that the study of bifurcation of extremals of the functional $\tilde{\mathcal{A}}$ is reduced to the study of bifurcation of extremals of the function $\mathcal{W}(u_j;\tau)$.

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