Estimating the Reliability Function for Transmuted Pareto Distribution Using Simulation

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Abstract

In this work, the methods (Moments, Modified Moments, L-Moments, Percentile, Rank Set sampling and Maximum Likelihood) were used to estimate the reliability function and the two parameters (α, λ) of the Transmuted Pareto (TP) distribution. We use simulation to generate the required data from three cases this indicates sample size $n = 10,30,70,100$, and it replicates $N = 1000$ for the real value for parameters, for reliability times values we take $t_{(i)} = 0.1, 0.2, \ldots \leq x_{(n)}$. Results were compared by using mean square error (MSE), the result appears as follows :

The best methods are Modified Moments, Maximum likelihood and L-Moments in first case, second case and third case respectively.

Keywords: Transmuted Pareto Distribution, Estimation methods, Simulation, Mean square error.

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تقدير دالة المعولية لتوزيع باريتو المحول باستخدام المحاكاة

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قسم الرياضيات، كلية التربية، الجامعة المستنصرية، بغداد، العراق

الخلاصة

في هذا البحث ، تم استخدام طرائق (العزوم ، العزوم المطورة ، ل–العزوم ، النسبة المئوية ، رتبة مجموعة العينة و الامكان الاعظم) لتقدير دالة المعولية والمعلمتين (α, λ) لتوزيع باريتو المحول. استخدم اسلوب المحاكاة لإنشاء البيانات المطلوبة في ثلاث حالات من خلال حجم العينة $n = 10,30,70,100$ وتكرارها $N = 1000$ للقيم الحقيقية للمعلمات ، و لأوقات المعولية تم أخذ قيم $x_{(n)} \geq \ldots 0.1, 0.2 = t_{(i)}$.

تمت مقارنة النتائج باستخدام متوسط مربع الخطأ (MSE) ، وظهرت النتيجة على النحو التالي:

الافضل هي طرائق العزوم المطورة والامكان الاعظم و ل_العزوم في الحالة الاولى، الحالة الثانية والحالة الثالثة على الترتيب.

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1. Introduction

Vilfredo Pareto [1] suggested the Pareto distribution, the (pdf) and (cdf) of the Pareto distribution, respectively.

$$g(x; \alpha, x_0) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}},$$
$$G(x; \alpha, x_0) = 1 - \left(\frac{x_0}{x}\right)^\alpha,$$

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Where X , $\alpha > 0$, and $0 < x_0 < x_{(1)}$.

Shaw and Buckley [2] introduced the Quadratic Rank Transmutation Map (QRTM) that the relationship satisfies the cumulative distribution function (cdf) approach.

$F(x) = (1 + \lambda)G(x) - \lambda G^2(x), |\lambda| \leq 1,$

$f(x) = g(x)[1 + \lambda - 2\lambda G(x)]$

Where $G(x)$ is the (cdf), and $g(x)$ is the (pdf) of the base distribution.

Several authors used the quadratic rank transmutation map to find generalizations to some distributions. Some of them are Aryal and Tsokos [3] proposed the transmuted Weibull distribution. Aryal [4] proposed the transmuted log-logistic distribution. Haq [5] introduced transmuted exponentiated inverse Rayleigh distribution. Vardhan and Balaswamy [6] proposed a transmuted new modified Weibull distribution. Elgarhy et al. [7] proposed transmuted generalized quasi Lindley distribution.

Faton and Llukan [8] proposed a transmuted Pareto distribution based on a Pareto distribution and a quadratic transmutation map.

  In this paper, we will estimate the parameters and reliability function of the transmuted Pareto (TP) distribution through six methods which are not previously used in estimating by using simulation.

2. Transmuted Pareto distribution (TP):

Let X be a random variable of the (TP) distribution, then both of the (pdf) and (cdf) for the (TP) distribution are expressed respectively as [8].

$$f_{TP}(x; x_0, \alpha, \lambda) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}}\left[1 - \lambda + 2\lambda \left(\frac{x_0}{x}\right)^\alpha\right] \qquad \text{... (1)}$$

$$F_{TP}(x; x_0, \alpha, \lambda) = \left[1 - \left(\frac{x_0}{x}\right)^\alpha\right]\left[1 + \lambda \left(\frac{x_0}{x}\right)^\alpha\right] \qquad \text{... (2)}$$

Where:

x : is a value of random variable and $x > x_0$.

 . Shape parameter and $\alpha > 0$:α

Transmutation parameter $|\lambda| \leq 1$. : λ

x_0: The minimum possible value of x .

The reliability function is given by

$$R(t) = \left(\frac{x_0}{t}\right)^\alpha \left(\lambda \left(\frac{x_0}{t}\right)^\alpha - \lambda + 1\right) \qquad \text{... (3)}$$

 Inverse distribution for (TP) distribution is

$$x(p_i) = x_0 \left[\frac{\lambda - 1 + \sqrt{(1 - \lambda)^2 - 4\lambda(p_i - 1)}}{2\lambda}\right]^{\frac{-1}{\alpha}} \qquad \text{... (4)}$$

The rth order moment

$$\mu_r = \text{E}(x^r) = \frac{\alpha x_0^r (\alpha - r(1 + \lambda))}{(\alpha - r)(2\alpha - r)}, \qquad \alpha > r$$

When, $r = 1,2$, the variance is

$$\sigma^2 = var(x) = E(x^2) - (E(x))^2$$

$$= \frac{\alpha x_0^2 (\alpha - 1 - \lambda)}{(\alpha - 1)(\alpha - 2)} - \left(\frac{\alpha x_0 (2\alpha - 1 - \lambda)}{(\alpha - 1)(2\alpha - 1)}\right)^2$$

$$\sigma^2 = \frac{\alpha x_0^2}{\alpha - 1}\left[\frac{\alpha - 1 - \lambda}{\alpha - 2} - \frac{\alpha(2\alpha - 1 - \lambda)^2}{(\alpha - 1)(2\alpha - 1)^2}\right], \alpha > 2 \qquad \text{... (5)}$$

3. Estimation Methods:

   In this section, we will derive some estimation methods for Transmuted Pareto distribution (TP).

3.1. Estimate Initial Value for Estimators:
Makki and Jaafer [9] proposed a method for finding initial values for estimating parameters of any distribution, as follows:
The median of (TP) distribution that is given in [8] as follows :

$$x_{med} = x_0 \left[\frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda} \right]^{\frac{-1}{\alpha}}$$

$$\frac{x_{med}}{x_0} = \left[\frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda} \right]^{\frac{-1}{\alpha}} \qquad \qquad ... (6)$$

Taking the natural logarithm for the equation (6), getting

$$ln\left(\frac{x_{med}}{x_0}\right) = \frac{-1}{\alpha} ln\left[\frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda} \right]$$

$$\hat{\alpha}_0 = \frac{-ln\left[\frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda} \right]}{ln\left(\frac{x_{med}}{x_0}\right)} \qquad \qquad ... (7)$$

From equation (6)

$$\left(\frac{x_{med}}{x_0}\right)^{-\alpha} = \frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda}$$

$$\hat{\lambda}_0 = \frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\left(\frac{x_{med}}{x_0}\right)^{-\alpha}} \qquad \qquad ... (8)$$

We can use $\hat{\alpha}_0$ and $\hat{\lambda}_0$ in equations (7) and (8) are the initial value to another parameter estimator formula. x_{med} can be obtained from the generating sample.

3.2. Moments Method (MOM):
The idea of this method is to equate the moments of the distribution with the moments of the sample.

$$\mu_r = \grave{\mu}_r \quad , r = 1,2 \qquad \qquad ... (9)$$

The first estimate can be get it when $(r = 1)$ in equation (9), then

$$\mu_1 = \grave{\mu}_1$$

Since,

$$E(x) = \frac{\alpha x_0(2\alpha - 1 - \lambda)}{(\alpha - 1)(2\alpha - 1)} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

$$2\alpha - 1 - \lambda = \frac{\bar{x}(\alpha - 1)(2\alpha - 1)}{\alpha x_0}$$

$$\hat{\lambda}_{MOM} = 2\alpha_0 - 1 - \frac{\bar{x}(\alpha_0 - 1)(2\alpha_0 - 1)}{\alpha_0 x_0} \qquad \qquad ... (10)$$

The second estimate can be get it when $(r = 2)$ in equation (9) then

$$\mu_2 = \grave{\mu}_2$$

Since

$$E(x^2) = \frac{\alpha x_0^2(\alpha - 1 - \lambda)}{(\alpha - 1)(\alpha - 2)} = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

$$(\alpha - 2) = \frac{\alpha x_0^2 (\alpha - 1 - \lambda)}{\frac{\sum_{i=1}^{n} x_i^2}{n} (\alpha - 1)}$$

$$\hat{\alpha}_{MOM} = \frac{\alpha_0 x_0^2 (\alpha_0 - 1 - \lambda_0)}{\frac{\sum_{i=1}^{n} x_i^2}{n} (\alpha_0 - 1)} + 2 \qquad \dots (11)$$

As for the estimator of the approximate reliability function for this distribution, we obtain it by substituting equations (10) and (11) into equation (3) as follows:

$$\hat{R}(t)_{MOM} = \left(\frac{x_0}{t}\right)^{\hat{\alpha}_{MOM}} \left(\hat{\lambda}_{MOM} \left(\frac{x_0}{t}\right)^{\hat{\alpha}_{MOM}} - \hat{\lambda}_{MOM} + 1\right) \qquad \dots (12)$$

3.3. Modified Moments Method (MM):

Cohen and Whitten [10] suggested a new modified, they used the following equation.

$$E\left(\hat{F}(x_{(i)})\right) = F(x_{(i)}) \qquad \dots (13)$$

Where i represents the order of observations that arrange in ascending orders. $\hat{F}(x_{(i)})$ is estimated unbiased for function distribution $F(x_{(i)})$, and $F(x_{(i)})$ is replaced by plotting position formula [11].

$$P_i = \frac{i}{n+1}, i = 1,2,\dots,n \qquad \dots (14)$$

$$F(x_i) = \frac{i}{n+1}, i = 1,2,\dots,n$$

Then

$$F(x_{(1)}) = \frac{1}{n+1} \qquad \dots (15)$$

From equations (2), and (15), we get:

$$\left[1 - \left(\frac{x_0}{x_{(1)}}\right)^\alpha\right]\left[1 + \lambda\left(\frac{x_0}{x_{(1)}}\right)^\alpha\right] = \frac{1}{n+1}$$

$$1 + \lambda\left(\frac{x_0}{x_{(1)}}\right)^\alpha - \left(\frac{x_0}{x_{(1)}}\right)^\alpha - \lambda\left(\frac{x_0}{x_{(1)}}\right)^{2\alpha} = \frac{1}{n+1}$$

$$\lambda\left(\frac{x_0}{x_{(1)}}\right)^\alpha - \left(\frac{x_0}{x_{(1)}}\right)^\alpha - \lambda\left(\frac{x_0}{x_{(1)}}\right)^{2\alpha} = \frac{-n}{n+1}$$

$$\lambda\left(\frac{x_0}{x_{(1)}}\right)^\alpha = \left(\frac{x_0}{x_{(1)}}\right)^\alpha + \lambda\left(\frac{x_0}{x_{(1)}}\right)^{2\alpha} - \frac{n}{n+1}$$

$$\hat{\lambda}_{MM} = \frac{\left(\frac{x_0}{x_{(1)}}\right)^{\alpha_0} + \lambda_0 \left(\frac{x_0}{x_{(1)}}\right)^{2\alpha_0} - \frac{n}{n+1}}{\left(\frac{x_0}{x_{(1)}}\right)^{\alpha_0}} \qquad \dots (16)$$

Since,

$$var(x) = E(x^2) - (E(x))^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} \qquad \dots (17)$$

Substitute equation (5) in equation (17), we get

$$\frac{\alpha x_0^2}{(\alpha - 1)}\left[\frac{\alpha - 1 - \lambda}{\alpha - 2} - \frac{\alpha(2\alpha - 1 - \lambda)^2}{(\alpha - 1)(2\alpha - 1)^2}\right] = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)}\frac{(\alpha - 1)}{\alpha x_0^2} = \frac{\alpha - 1 - \lambda}{\alpha - 2} - \frac{\alpha(2\alpha - 1 - \lambda)^2}{(\alpha - 1)(2\alpha - 1)^2}$$

$$(\alpha - 2)\left(\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)}\frac{(\alpha - 1)}{\alpha x_0^2} + \frac{\alpha(2\alpha - 1 - \lambda)^2}{(\alpha - 1)(2\alpha - 1)^2}\right) = \alpha - 1 - \lambda$$

$$\hat{\alpha}_{MM} = \frac{\alpha_0 - 1 - \lambda_0}{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)}\frac{(\alpha_0 - 1)}{\alpha_0 x_0^2} + \frac{\alpha_0(2\alpha_0 - 1 - \lambda_0)^2}{(\alpha_0 - 1)(2\alpha_0 - 1)^2}} + 2 \qquad \dots (18)$$

As for the estimator of the approximate reliability function for this distribution, we obtain it by substituting equations (16) and (18) into equation (3) as follows:

$$\hat{R}(t)_{MM} = \left(\frac{x_0}{t}\right)^{\hat{\alpha}_{MM}}\left(\hat{\lambda}_{MM}\left(\frac{x_0}{t}\right)^{\hat{\alpha}_{MM}} - \hat{\lambda}_{MM} + 1\right) \qquad \dots (19)$$

3.4. L-Moment Method (LM):

Hosking [12] proposed the L-Moments method. The L-Moments methods are anticipations of certain linear combinations of order statistics[13]. The distribution parameters can be estimated by the following

$$B_r = \int_{x_0}^{\infty} x(F^r)\ dF \qquad \dots (20)$$

Substitution equations (1) and (2) into equation (20), then we get

$$B_r = \int_{x_0}^{\infty} x\left[\left[1 - \left(\frac{x_0}{x}\right)^{\alpha}\right]\left[1 + \lambda\left(\frac{x_0}{x}\right)^{\alpha}\right]\right]^r \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}\left[1 - \lambda + 2\lambda\left(\frac{x_0}{x}\right)^{\alpha}\right]dx \qquad \dots (21)$$

If we substitute $r = 0$ in equation (21), then we get

$$B_0 = \alpha x_0^{\alpha}\int_{x_0}^{\infty}\frac{1}{x^{\alpha}}\left[1 - \lambda + 2\lambda x_0^{\alpha}\frac{1}{x^{\alpha}}\right]dx$$

$$= \alpha x_0^{\alpha}\left[\frac{-1}{(\alpha-1)x^{\alpha-1}} + \frac{\lambda}{(\alpha-1)x^{\alpha-1}} - \frac{2\lambda x_0^{\alpha}}{(2\alpha-1)x^{2\alpha-1}}\Big|_{x_0}^{\infty}\right]$$

$$= \frac{\alpha x_0^{\alpha}}{(\alpha-1)x_0^{\alpha-1}} - \frac{\alpha\lambda x_0^{\alpha}}{(\alpha-1)x_0^{\alpha-1}} + \frac{2\alpha\lambda x_0^{2\alpha}}{(2\alpha-1)x_0^{2\alpha-1}}$$

$$= \frac{\alpha x_0}{\alpha-1} - \frac{\alpha\lambda x_0}{\alpha-1} + \frac{2\alpha\lambda x_0}{2\alpha-1}$$

$$B_0 = \alpha x_0\left[\frac{1-\lambda}{\alpha-1} + \frac{2\lambda}{2\alpha-1}\right] \qquad \dots (22)$$

While if we substitute $r = 1$ in equation (21), we get

$$B_1 = \int_{x_0}^{\infty} x\left[\left[1 - \left(\frac{x_0}{x}\right)^{\alpha}\right]\left[1 + \lambda\left(\frac{x_0}{x}\right)^{\alpha}\right]\right]\frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}\left[1 - \lambda + 2\lambda\left(\frac{x_0}{x}\right)^{\alpha}\right]dx$$

$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty}\frac{1}{x^{\alpha}}\left[1 + \lambda x_0^{\alpha}\frac{1}{x^{\alpha}} - x_0^{\alpha}\frac{1}{x^{\alpha}} - \lambda x_0^{2\alpha}\frac{1}{x^{2\alpha}}\right]\left[1 - \lambda + 2\lambda x_0^{\alpha}\frac{1}{x^{\alpha}}\right]dx$$

$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty}\left[1 + \lambda x_0^{\alpha}\frac{1}{x^{\alpha}} - x_0^{\alpha}\frac{1}{x^{\alpha}} - \lambda x_0^{2\alpha}\frac{1}{x^{2\alpha}}\right]\left[\frac{1}{x^{\alpha}} - \frac{\lambda}{x^{\alpha}} + 2\lambda x_0^{\alpha}\frac{1}{x^{2\alpha}}\right]dx$$

$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty}\left[\frac{1}{x^{\alpha}} - \frac{\lambda}{x^{\alpha}} + \frac{4\lambda x_0^{\alpha}}{x^{2\alpha}} - \frac{\lambda^2 x_0^{\alpha}}{x^{2\alpha}} + \frac{3\lambda^2 x_0^{2\alpha}}{x^{3\alpha}} - \frac{x_0^{\alpha}}{x^{2\alpha}} - \frac{3\lambda x_0^{2\alpha}}{x^{3\alpha}} - \frac{2\lambda^2 x_0^{3\alpha}}{x^{4\alpha}}\right]dx$$

$$= \alpha x_0^\alpha \left[\frac{-1}{(\alpha-1)x^{\alpha-1}} + \frac{\lambda}{(\alpha-1)x^{\alpha-1}} - \frac{4\lambda x_0^\alpha}{(2\alpha-1)x^{2\alpha-1}} + \frac{\lambda^2 x_0^\alpha}{(2\alpha-1)x^{2\alpha-1}} \right.$$
$$\left. - \frac{3\lambda^2 x_0^{2\alpha}}{(3\alpha-1)x^{3\alpha-1}} + \frac{x_0^\alpha}{(2\alpha-1)x^{2\alpha-1}} + \frac{3\lambda x_0^{2\alpha}}{(3\alpha-1)x^{3\alpha-1}} + \frac{2\lambda^2 x_0^{3\alpha}}{(4\alpha-1)x^{4\alpha-1}} \Big|_{x_0}^\infty \right]$$

$$= \frac{\alpha x_0^\alpha}{(\alpha-1)x_0^{\alpha-1}} - \frac{\alpha\lambda x_0^\alpha}{(\alpha-1)x_0^{\alpha-1}} + \frac{4\alpha\lambda x_0^{2\alpha}}{(2\alpha-1)x_0^{2\alpha-1}} - \frac{\alpha\lambda^2 x_0^{2\alpha}}{(2\alpha-1)x_0^{2\alpha-1}} + \frac{3\alpha\lambda^2 x_0^{3\alpha}}{(3\alpha-1)x_0^{3\alpha-1}}$$
$$- \frac{\alpha x_0^{2\alpha}}{(2\alpha-1)x_0^{2\alpha-1}} - \frac{3\alpha\lambda x_0^{3\alpha}}{(3\alpha-1)x_0^{3\alpha-1}} - \frac{2\alpha\lambda^2 x_0^{4\alpha}}{(4\alpha-1)x_0^{4\alpha-1}}$$

$$= \frac{\alpha x_0}{\alpha-1} - \frac{\alpha\lambda x_0}{\alpha-1} + \frac{4\alpha\lambda x_0}{2\alpha-1} - \frac{\alpha\lambda^2 x_0}{2\alpha-1} + \frac{3\alpha\lambda^2 x_0}{3\alpha-1} - \frac{\alpha x_0}{2\alpha-1} - \frac{3\alpha\lambda x_0}{3\alpha-1} - \frac{2\alpha\lambda^2 x_0}{4\alpha-1}$$

$$= \frac{\alpha x_0(1-\lambda)}{\alpha-1} + \frac{\alpha x_0(4\lambda-\lambda^2-1)}{2\alpha-1} + \frac{\alpha x_0(3\lambda^2-3\lambda)}{3\alpha-1} - \frac{2\alpha x_0\lambda^2}{4\alpha-1}$$

$$B_1 = \alpha x_0 \left[\frac{1-\lambda}{\alpha-1} + \frac{4\lambda-\lambda^2-1}{2\alpha-1} + \frac{3\lambda^2-3\lambda}{3\alpha-1} - \frac{2\lambda^2}{4\alpha-1} \right] \qquad \text{...(23)}$$

Now subtract equation (23) from equation (22), we have

$$B_0 - B_1 = \alpha x_0 \left[\frac{1-2\lambda+\lambda^2}{2\alpha-1} + \frac{3\lambda(\lambda-1)}{3\alpha-1} - \frac{2\lambda^2}{4\alpha-1} \right]$$

$$\hat{\alpha}_{LM} = \frac{B_0 - B_1}{x_0 \left[\frac{1-2\lambda_0+\lambda_0^2}{2\alpha_0-1} + \frac{3\lambda_0(\lambda_0-1)}{3\alpha_0-1} - \frac{2\lambda_0^2}{4\alpha_0-1} \right]} \qquad \text{...(24)}$$

From equation (22), we get

$$\frac{B_0}{\alpha x_0} - \frac{1-\lambda}{\alpha-1} = \frac{2\lambda}{2\alpha-1}$$

$$\hat{\lambda}_{LM} = \frac{(2\hat{\alpha}_{LM}-1)\left[\frac{B_0}{\hat{\alpha}_{LM}x_0} - \frac{1-\lambda_0}{\hat{\alpha}_{LM}-1} \right]}{2} \qquad \text{...(25)}$$

As for the estimator of the approximate reliability function for this distribution, we obtain it by substituting equations (24) and (25) into the equation (3) as follows:

$$\hat{R}(t)_{Lm} = \left(\frac{x_0}{t} \right)^{\hat{\alpha}_{LM}} \left(\hat{\lambda}_{LM} \left(\frac{x_0}{t} \right)^{\hat{\alpha}_{LM}} - \hat{\lambda}_{LM} + 1 \right) \qquad \text{...(26)}$$

3.5. Percentile Estimation Method (PER):

This method was originally explored by Kao [14]. This method depends on parameter estimation of any distribution on minimized of inverse distribution which represented from the equation (4) as follows:

$$\frac{x(p_i)}{x_0} = \left[\frac{\lambda-1+\sqrt{(1-\lambda)^2-4\lambda(p_i-1)}}{2\lambda} \right]^{\frac{-1}{\alpha}} \qquad \text{...(27)}$$

Taking the natural logarithm for equation (27), we get

$$ln\left(\frac{x(p_i)}{x_0} \right) = \frac{-1}{\alpha} ln\left[\frac{\lambda-1+\sqrt{(1-\lambda)^2-4\lambda(p_i-1)}}{2\lambda} \right] \qquad \text{...(28)}$$

We square the equation (28), and let it equal to zero, we get the following

$$\sum_{i=1}^n \left(ln\left(\frac{x(p_i)}{x_0} \right) + \frac{1}{\alpha} ln\left[\frac{\lambda-1+\sqrt{(1-\lambda)^2-4\lambda(p_i-1)}}{2\lambda} \right] \right)^2 = 0 \qquad \text{...(29)}$$

The partial derivative the equation (29) with respect to a is

$$\frac{\partial \delta}{\partial \alpha} = 2 \sum_{i=1}^{n} \left(\begin{pmatrix} ln\left(\frac{x(p_i)}{x_0}\right) + \frac{1}{\alpha} ln\left[\frac{\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}}{2\lambda} \right] \end{pmatrix} \\ \left(\frac{-1}{\alpha^2} ln\left[\frac{\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}}{2\lambda} \right] \right) \end{pmatrix} = 0$$

$$\frac{1}{\alpha^2} \sum_{i=1}^{n} ln\left[\frac{\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}}{2\lambda} \right] ln\left(\frac{x(p_i)}{x_0}\right)$$

$$= \frac{-1}{\alpha^3} \sum_{i=1}^{n} \left(ln\left[\frac{\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}}{2\lambda} \right] \right)^2$$

$$\hat{\alpha}_{PER} = \frac{- \sum_{i=1}^{n} \left(ln\left[\frac{\lambda_0 - 1 + \sqrt{(1-\lambda_0)^2 - 4\lambda_0(p_i - 1)}}{2\lambda_0} \right] \right)^2}{\sum_{i=1}^{n} ln\left[\frac{\lambda_0 - 1 + \sqrt{(1-\lambda_0)^2 - 4\lambda_0(p_i - 1)}}{2\lambda_0} \right] ln\left(\frac{x(p_i)}{x_0}\right)} \qquad \dots (30)$$

The partial derivative the equation (29) with respect to λ is

$$\frac{\partial \delta}{\partial \lambda}$$

$$= 2 \sum_{i=1}^{n} \left(\begin{pmatrix} ln\left(\frac{x(p_i)}{x_0}\right) + \frac{1}{\alpha} ln\left[\frac{\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}}{2\lambda} \right] \end{pmatrix} \\ \left(\frac{1}{\alpha} \left(\frac{2\lambda}{\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}} \right) \\ \left(\frac{2\lambda\left(1 + \frac{2(1-\lambda)(-1) - 4(p_i - 1)}{2\sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}}\right) - 2\left(\lambda - 1 + \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}\right)}{4\lambda^2} \right) \right) \end{pmatrix}$$

$$= 0$$

If we let $k_i = \sqrt{(1-\lambda)^2 - 4\lambda(p_i - 1)}$, then

$$\sum_{i=1}^{n} \left(\begin{pmatrix} ln\left(\frac{x(p_i)}{x_0}\right) + \frac{1}{\alpha} ln\left[\frac{\lambda - 1 + k_i}{2\lambda} \right] \end{pmatrix} \\ \left(\frac{1}{\alpha} \left(\frac{2\lambda}{\lambda - 1 + k_i} \right) \left(\frac{\lambda\left(1 + \frac{\lambda - 1 - 2(p_i - 1)}{k_i}\right) - (\lambda - 1 + k_i)}{\lambda^2} \right) \right) \end{pmatrix} = 0 \qquad \dots (31)$$

We multiply equation (31) by α, then we have

$$\sum_{i=1}^{n} ln\left(\frac{x(p_i)}{x_0}\right) \left(\left(\frac{2\lambda}{\lambda - 1 + k_i} \right) \left(\frac{\lambda\left(1 + \frac{\lambda - 1 - 2(p_i - 1)}{k_i}\right) - (\lambda - 1 + k_i)}{\lambda^2} \right) \right)$$

$$= \frac{-1}{\alpha} \sum_{i=1}^{n} ln\left(\frac{\lambda - 1 + k_i}{2\lambda}\right) \left(\left(\frac{2\lambda}{\lambda - 1 + k_i} \right) \left(\frac{\lambda\left(1 + \frac{\lambda - 1 - 2(p_i - 1)}{k_i}\right) - (\lambda - 1 + k_i)}{\lambda^2} \right) \right)$$

$$\hat{\lambda}_{PER} = \left(-\alpha_0 \lambda_0 \sum_{i=1}^n ln\left(\frac{x(p_i)}{x_0}\right) \left(\left(\frac{2\lambda_0}{\lambda_0 - 1 + k_i}\right)\left(\lambda_0\left(1 + \frac{\lambda_0 - 1 - 2(p_i - 1)}{k_i}\right)\right) - (\lambda_0 - 1 + k_i)\right)\right) /$$

$$\left(\sum_{i=1}^n ln\left(\frac{\lambda_0 - 1 + k_i}{2\lambda_0}\right)\left(\left(\frac{2\lambda_0}{\lambda_0 - 1 + k_i}\right)\left(\lambda_0\left(1 + \frac{\lambda_0 - 1 - 2(p_i - 1)}{k_i}\right)\right) - (\lambda_0 - 1 + k_i)\right)\right) \quad \dots (32)$$

As for the estimator of the approximate reliability function for this distribution, we obtain it by substituting equations (30) and (32) into equation (3) as follows:

$$\hat{R}(t)_{PER} = \left(\frac{x_0}{t}\right)^{\hat{\alpha}_{PER}}\left(\hat{\lambda}_{PER}\left(\frac{x_0}{t}\right)^{\hat{\alpha}_{PER}} - \hat{\lambda}_{PER} + 1\right) \quad \dots (33)$$

3.6. Rank Set Sampling Method (RSS):

Let $(x_1, x_2, \dots, x_n)$ be a random sample. Assumed that $(x_{(1)}, x_{(2)}, \dots, x_{(n)})$ is the order statistics that obtained by ordering the sample in increasing order [15].
The (pdf) of $x_{(i)}$ is:

$$f(x_i) = \frac{n!}{(i-1)!\,(n-i)!}\left[F(x_{(i)})\right]^{i-1}\left[1 - F(x_{(i)})\right]^{n-i} f(x_{(i)}) \quad \dots (34)$$

By substitution equations (1) and (2) into equation (34), and let $Q_i = \frac{n!}{(i-1)!(n-i)!}$, getting

$$f(x_i) = Q_i \left[\left(1 - \left(\frac{x_0}{x_{(i)}}\right)^\alpha\right)\left(1 + \lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha\right)\right]^{i-1}\left[\left(\frac{x_0}{x_{(i)}}\right)^\alpha\left(\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha - \lambda + 1\right)\right]^{n-i}$$
$$\frac{\alpha x_0^\alpha}{x_{(i)}^{\alpha+1}}\left[1 - \lambda + 2\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha\right] \quad \dots (35)$$

The likelihood for equation (35) is

$$Lf = Q_i^n \prod_{i=1}^n \left[\left(1 - \left(\frac{x_0}{x_{(i)}}\right)^\alpha\right)\left(1 + \lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha\right)\right]^{i-1} \prod_{i=1}^n \left[\left(\frac{x_0}{x_{(i)}}\right)^\alpha\left(1 - \lambda\right.\right.$$
$$\left.\left. + \lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha\right)\right]^{n-i} \alpha^n x_0^{\alpha n} \prod_{i=1}^n \left[\frac{1 - \lambda + 2\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha}{x_{(i)}^{\alpha+1}}\right] \quad \dots (36)$$

Taking the natural logarithm for the equation (36), we get

$$ln\, Lf = nlnQ_i + \sum_{i=1}^n (i-1)ln\left[\left[1 - \left(\frac{x_0}{x_{(i)}}\right)^\alpha\right]\left[1 + \lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha\right]\right]$$
$$+ \sum_{i=1}^n (n-i)\, ln\left[\left(\frac{x_0}{x_{(i)}}\right)^\alpha\left(1 - \lambda + \lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha\right)\right] + nln\alpha + n\alpha lnx_0$$
$$+ \sum_{i=1}^n ln\left(1 - \lambda + 2\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha\right) - (\alpha + 1)\sum_{i=1}^n lnx_{(i)}$$

$$\ln Lf = n \ln Q_i + \sum_{i=1}^{n} (i-1)\ln\left(1 - \left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\right) + \sum_{i=1}^{n}(i-1)\ln\left(1 + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\right)$$

$$+ \alpha \sum_{i=1}^{n}(n-i)\ln\left(\frac{x_0}{x_{(i)}}\right) + \sum_{i=1}^{n}(n-i)\ln\left(1 - \lambda + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\right) + n\ln\alpha$$

$$+ n\alpha \ln x_0 - \alpha \sum_{i=1}^{n}\ln x_{(i)} - \sum_{i=1}^{n}\ln x_{(i)}$$

$$+ \sum_{i=1}^{n}\ln\left(1 - \lambda + 2\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\right) \cdots (37)$$

We take partial derivative for the equation (37) with respect to α, and put it equal to zero

$$\frac{\partial \ln lf}{\partial \alpha} = -\sum_{i=1}^{n}(i-1)\frac{\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} + \sum_{i=1}^{n}(i-1)\frac{\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}}$$

$$+ \sum_{i=1}^{n}(n-i)\ln\left(\frac{x_0}{x_{(i)}}\right) + \sum_{i=1}^{n}(n-i)\frac{\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \lambda + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} + \frac{n}{\alpha} + n\ln x_0$$

$$- \sum_{i=1}^{n}\ln x_{(i)} + \sum_{i=1}^{n}\frac{2\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \lambda + 2\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} = 0$$

$$-x_0^{\alpha}\sum_{i=1}^{n}(i-1)\frac{\frac{1}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} + x_0^{\alpha}\sum_{i=1}^{n}(i-1)\frac{\frac{\lambda}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} + \sum_{i=1}^{n}(n-i)\ln\left(\frac{x_0}{x_{(i)}}\right)$$

$$+ x_0^{\alpha}\sum_{i=1}^{n}(n-i)\frac{\frac{\lambda}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \lambda + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} + \frac{n}{\alpha} + n\ln x_0 - \sum_{i=1}^{n}\ln x_{(i)}$$

$$+ x_0^{\alpha}\sum_{i=1}^{n}\frac{\frac{2\lambda}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \lambda + 2\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} = 0$$

$$x_0^{\alpha}\left(-\sum_{i=1}^{n}(i-1)\frac{\frac{1}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} + \sum_{i=1}^{n}(i-1)\frac{\frac{\lambda}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}}\right.$$

$$\left. + \sum_{i=1}^{n}(n-i)\frac{\frac{\lambda}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \lambda + \lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}} + \sum_{i=1}^{n}\frac{\frac{2\lambda}{x^{\alpha}}\ln\left(\frac{x_0}{x_{(i)}}\right)}{1 - \lambda + 2\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}}\right)$$

$$+ \sum_{i=1}^{n}(n-i)\ln\left(\frac{x_0}{x_{(i)}}\right) + \frac{n}{\alpha} + n\ln x_0 - \sum_{i=1}^{n}\ln x_{(i)} = 0$$

$$x_0^\alpha = \cfrac{\sum_{i=1}^n lnx_{(i)} - \sum_{i=1}^n (n-i)ln\left(\frac{x_0}{x_{(i)}}\right) - \frac{n}{\alpha} - nlnx_0}{\left(\begin{array}{c} -\sum_{i=1}^n (i-1)\cfrac{\frac{1}{x_{(i)}^\alpha}ln\left(\frac{x_0}{x_{(i)}}\right)}{1-\left(\frac{x_0}{x_{(i)}}\right)^\alpha} + \sum_{i=1}^n (i-1)\cfrac{\frac{\lambda}{x_{(i)}^\alpha}ln\left(\frac{x_0}{x_{(i)}}\right)}{1+\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha} \\[4ex] +\sum_{i=1}^n (n-i)\cfrac{\frac{\lambda}{x_{(i)}^\alpha}ln\left(\frac{x_0}{x_{(i)}}\right)}{1-\lambda+\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha} + \sum_{i=1}^n \cfrac{\frac{2\lambda}{x_{(i)}^\alpha}ln\left(\frac{x_0}{x_{(i)}}\right)}{1-\lambda+2\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha} \end{array}\right)}$$

$$\hat{\alpha}_{RSS} = \cfrac{ln\left(\cfrac{\sum_{i=1}^n lnx_{(i)} - \sum_{i=1}^n (n-i)ln\left(\frac{x_0}{x_{(i)}}\right) - \frac{n}{\alpha_0} - nlnx_0}{\left(\begin{array}{c} -\sum_{i=1}^n (i-1)\cfrac{\frac{1}{x_{(i)}^{\alpha_0}}ln\left(\frac{x_0}{x_{(i)}}\right)}{1-\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}} + \sum_{i=1}^n (i-1)\cfrac{\frac{\lambda_0}{x_{(i)}^{\alpha_0}}ln\left(\frac{x_0}{x_{(i)}}\right)}{1+\lambda_0\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}} \\[4ex] +\sum_{i=1}^n (n-i)\cfrac{\frac{\lambda_0}{x_{(i)}^{\alpha_0}}ln\left(\frac{x_0}{x_{(i)}}\right)}{1-\lambda_0+\lambda_0\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}} + \sum_{i=1}^n \cfrac{\frac{2\lambda_0}{x_{(i)}^{\alpha_0}}ln\left(\frac{x_0}{x_{(i)}}\right)}{1-\lambda_0+2\lambda_0\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}} \end{array}\right)}\right)}{ln(x_0)} \quad \dots(38)$$

Now we take partial derivative for the equation (37) with respect to λ and put it equal to zero

$$\frac{\partial lnlf}{\partial \lambda} = \sum_{i=1}^n (i-1)\frac{\left(\frac{x_0}{x_{(i)}}\right)^\alpha}{1+\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha} + \sum_{i=1}^n (n-i)\frac{\left(\frac{x_0}{x_{(i)}}\right)^\alpha - 1}{1-\lambda+\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha}$$

$$+ \sum_{i=1}^n \frac{2\left(\frac{x_0}{x_i}\right)^\alpha - 1}{1-\lambda+2\lambda\left(\frac{x_0}{x_i}\right)^\alpha} = 0$$

$$-\sum_{i=1}^n (i-1)\frac{\left(\frac{x_0}{x_{(i)}}\right)^\alpha}{1+\lambda\left(\frac{x_0}{x_{(i)}}\right)^\alpha}$$

$$= \frac{1}{\lambda}\sum_{i=1}^n (n-i)\frac{\left(\frac{x_0}{x_{(i)}}\right)^\alpha - 1}{\frac{1}{\lambda}-1+\left(\frac{x_0}{x_{(i)}}\right)^\alpha} + \frac{1}{\lambda}\sum_{i=1}^n \frac{2\left(\frac{x_0}{x_{(i)}}\right)^\alpha - 1}{\frac{1}{\lambda}-1+2\left(\frac{x_0}{x_{(i)}}\right)^\alpha}$$

$$-\sum_{i=1}^{n}(i-1)\frac{\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}}{1+\lambda\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}}$$

$$=\frac{1}{\lambda}\left(\sum_{i=1}^{n}(n-i)\frac{\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}-1}{\frac{1}{\lambda}-1+\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}}+\sum_{i=1}^{n}\frac{2\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}-1}{\frac{1}{\lambda}-1+2\left(\frac{x_0}{x_{(i)}}\right)^{\alpha}}\right)$$

$$\hat{\lambda}_{RSS}=\frac{\sum_{i=1}^{n}(n-i)\dfrac{\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}-1}{\frac{1}{\lambda_0}-1+\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}}+\sum_{i=1}^{n}\dfrac{2\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}-1}{\frac{1}{\lambda_0}-1+2\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}}}{-\sum_{i=1}^{n}(i-1)\dfrac{\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}}{1+\lambda_0\left(\frac{x_0}{x_{(i)}}\right)^{\alpha_0}}} \qquad \dots(39)$$

As for the estimator of the approximate reliability function for this distribution, we obtain it by substituting equations (38) and (39) into equation (3) as follows:

$$\hat{R}(t)_{Rss}=\left(\frac{x_0}{t}\right)^{\hat{\alpha}_{RSS}}\left(\hat{\lambda}_{RSS}\left(\frac{x_0}{t}\right)^{\hat{\alpha}_{RSS}}-\hat{\lambda}_{RSS}+1\right) \qquad \dots(40)$$

3.7. Maximum Likelihood Estimation Method (MLE):

This method of estimation is a common and reliable method, and this method characterized by its accuracy compared to other methods, especially when increasing the sample size.

Let $x_1, x_2, \dots, x_n$ be a sample of size n for (TP) distribution [8].

$$L(x_1, x_2, \dots, x_n; \alpha, x_0, \lambda)=\frac{\alpha^n x_0^{\alpha n}}{\prod_{i=1}^{n} x_i^{\alpha+1}}\prod_{i=1}^{n}\left(1-\lambda+2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha}\right) \qquad \dots(41)$$

Taking the natural logarithm for equation (41), we get

$$lnL=nln(\alpha)+n\alpha\, ln(x_0)-(\alpha+1)\sum_{i=1}^{n}ln(x_i)+\sum_{i=1}^{n}ln\left(1-\lambda+2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha}\right)$$

$$=nln(\alpha)+n\alpha\, ln(x_0)-\alpha\sum_{i=1}^{n}ln(x_i)-\sum_{i=1}^{n}ln(x_i)$$

$$+\sum_{i=1}^{n}ln\left(1-\lambda+2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha}\right) \qquad \dots(42)$$

By taking partial derivative for equation (42) with respect to α and equal to zero

$$\frac{\partial lnL}{\partial\alpha}=\frac{n}{\alpha}+n\, ln(x_0)-\sum_{i=1}^{n}ln(x_i)+\sum_{i=1}^{n}\frac{2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha}ln\left(\frac{x_0}{x_i}\right)}{1-\lambda+2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha}}=0$$

$$\frac{n}{\alpha}=\sum_{i=1}^{n}ln(x_i)-nln(x_0)-\sum_{i=1}^{n}\frac{2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha}ln\left(\frac{x_0}{x_i}\right)}{1-\lambda+2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha}}$$

$$\hat{\alpha}_{MLE} = \frac{n}{\sum_{i=1}^{n} ln(x_i) - nln(x_0) - \sum_{i=1}^{n} \frac{2\lambda_0 \left(\frac{x_0}{x_i}\right)^{\alpha_0} ln\left(\frac{x_0}{x_i}\right)}{1 - \lambda_0 + 2\lambda_0 \left(\frac{x_0}{x_i}\right)^{\alpha_0}}} \qquad \dots (43)$$

By taking partial derivative for equation (42) with respect to λ and equal to zero

$$\frac{\partial lnL}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\left(\frac{x_0}{x_i}\right)^{\alpha} - 1}{1 - \lambda + 2\lambda \left(\frac{x_0}{x_i}\right)^{\alpha}}$$

$$\sum_{i=1}^{n} \frac{2\left(\frac{x_0}{x_i}\right)^{\alpha}}{1 - \lambda + 2\lambda \left(\frac{x_0}{x_i}\right)^{\alpha}} - \sum_{i=1}^{n} \frac{1}{1 - \lambda + 2\lambda \left(\frac{x_0}{x_i}\right)^{\alpha}} = 0$$

$$\frac{1}{\lambda}\sum_{i=1}^{n} \frac{1}{\frac{1}{\lambda} - 1 + 2\left(\frac{x_0}{x_i}\right)^{\alpha}} = \sum_{i=1}^{n} \frac{2\left(\frac{x_0}{x_i}\right)^{\alpha}}{1 - \lambda + 2\lambda \left(\frac{x_0}{x_i}\right)^{\alpha}}$$

$$\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} \frac{1}{\frac{1}{\lambda_0} - 1 + 2\left(\frac{x_0}{x_i}\right)^{\alpha_0}}}{\sum_{i=1}^{n} \frac{2\left(\frac{x_0}{x_i}\right)^{\alpha_0}}{1 - \lambda_0 + 2\lambda_0 \left(\frac{x_0}{x_i}\right)^{\alpha_0}}} \qquad \dots (44)$$

As for the estimator of the reliability function for this distribution, we obtain it by substituting equations (43) and (44) into equation (3) as follows:

$$\hat{R}(t)_{MLE} = \left(\frac{x_0}{t}\right)^{\hat{\alpha}_{MLE}} \left(\hat{\lambda}_{MLE} \left(\frac{x_0}{t}\right)^{\hat{\alpha}_{MLE}} - \hat{\lambda}_{MLE} + 1\right) \qquad \dots (45)$$

4. Experiments and Results:

In this section, we review the simulation steps in terms of selecting sample sizes, real values of parameters and life time values that were used to estimate reliability:

1. The sample size is n=10, 30, 70, 100 , and the sample which is replicated N=1000.
2. Several values of the shape and the transmuted parameters (α, λ) are shown in Table (1) below:

Table 1-The default value for parameters

Cases		C_1	C_2	C_3
parameters	α	4	1.5	1
	λ	-1	0.1	0.5

3. In all cases $C_1, C_2, and\ C_3$ the life time for estimating reliability is chosen $x_0 \leq t \leq x_{(n)}$, such that, $t_{(i)} = 0.1, 0.2, \dots \leq x_{(n)}$.
4. At this stage, random data is generated of (TP) distribution by equation (4) and using MATLAB language version R2012b.
5. At this stage we find the value of parameter and reliability estimated according to the equations(10),(11),(12),(16),(18),(19),(24),(25),(26),(30),(32),(33),(38),(39),(40),(43),(44)and (45) .

6. Finally, comparison between the estimators is done by $MSE(\hat{\beta}) = \frac{\sum_{i=1}^{N}(\hat{\beta}_i - \beta)^2}{N}$, where$(\hat{\beta})$ is an estimator for parameter (β).

The next Tables 2,3 and 4 show the empirical value for estimation of parameters and reliability function.

Table 2-Estimated values for (R) and (α, λ) using $(C_1: R = 0.56054)$

Methods	n	Mean Estimated Values			MSE		
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{R}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{R}$
MOM	10	3.72277	-0.24730	0.54652	0.07685	0.56654	0.00039
MM		2.86833	-1.81925	0.67598	1.28065	0.67118	0.02665
LM		2.48736	-2.11013	0.48752	2.28807	1.23239	0.13427
PER		3.37621	-1.43156	0.61044	0.38910	0.18624	0.00498
RSS		2.85868	-1.66261	0.66773	1.30260	0.43905	0.02298
MLE		5.23822	-1.66195	0.53467	1.53319	0.43817	0.00133
MOM	30	3.82981	-1.27414	0.29960	0.02896	0.07515	0.00028
MM		3.20095	-1.50389	0.62725	0.63847	0.25390	0.00889
LM		2.89237	-1.60037	0.32119	1.22684	0.36045	0.04004
PER		3.78210	-1.13547	0.57461	0.04747	0.01835	0.00039
RSS		3.98640	-1.22689	0.56780	0.00018	0.05148	0.00010
MLE		3.60794	-1.13183	0.58361	0.15370	0.01737	0.00106
MOM	70	3.85182	-1.20419	0.23873	0.02195	0.04169	0.00014
MM		4.06634	-1.03555	0.55890	0.00440	0.00126	$0.53783*10^{-5}$
LM		3.86621	-1.04340	0.13521	0.01789	0.00188	0.00019
PER		3.82503	-0.98321	0.29358	0.03061	0.00028	0.00005
RSS		4.00200	-1.14075	0.23341	$0.40290*10^{-5}$	0.01981	0.00001
MLE		4.35358	-0.9109	0.36907	0.12502	0.00793	0.00032
MOM	100	4.03570	-0.83677	0.22803	0.00127	0.02664	0.00003
MM		4.00239	-1.03502	0.23179	$0.57524*10^{-5}$	0.00122	$0.71667*10^{-6}$
LM		4.12611	-0.96316	0.06512	0.01590	0.00135	0.00007
PER		4.12690	-0.98711	0.37724	0.01610	0.00016	0.00004
RSS		3.99925	-0.92980	0.38024	$0.55311*10^{-6}$	0.00492	$0.57134*10^{-5}$
MLE		3.69776	-0.94278	0.29719	0.09134	0.00327	0.00015

Table 3-Estimated values for (R) and (α, λ) using $(C_2: R = 0.12193)$

Methods	n	Mean Estimated Values			MSE		
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{R}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{R}$
LM	10	0.73975	0.19392	0.30527	0.57797	0.00882	0.03445
PER		0.83693	0.17575	0.36077	0.43965	0.00573	0.01968
RSS		0.90193	0.15328	0.34411	0.35768	0.00283	0.01503
MLE		0.89462	0.18277	0.34149	0.36647	0.00685	0.01432
LM	30	0.93430	0.14772	0.22452	0.32001	0.00227	0.01363
PER		1.15235	0.12299	0.26205	0.12085	0.00052	0.00336
RSS		1.11096	0.10373	0.27312	0.15135	0.00001	0.00491
MLE		1.17519	0.13385	0.25623	0.10549	0.00114	0.00265
LM	70	1.42540	0.10343	0.14740	0.00556	0.00001	0.00015
PER		1.71230	0.08796	0.23527	0.04507	0.00014	0.00074
RSS		1.70284	0.10099	0.14499	0.04114	$0.98305*10^{-6}$	0.00049
MLE		1.68063	0.08734	0.05126	0.03263	0.00016	0.00012
LM	100	1.57141	0.09711	0.05897	0.00509	$0.83307*10^{-5}$	0.00005
PER		1.53628	0.10255	0.05695	0.00131	$0.65034*10^{-5}$	$0.75572*10^{-5}$
RSS		1.39296	0.09913	0.05558	0.01145	$0.74377*10^{-6}$	0.00006
MLE		1.58805	0.09919	0.11295	0.00775	$0.65335*10^{-6}$	0.00007

Table 4- Estimated values for (R) and (α, λ) using $(C_3 : R = 0.10352)$

Methods	n	Mean Estimated Values			MSE		
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{R}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{R}$
LM	10	0.47524	0.8560	0.21287	0.27536	0.12292	0.01304
PER		0.42657	0.93068	0.27774	0.32882	0.18548	0.01530
RSS		0.34417	0.87600	0.35307	0.43011	0.14138	0.04020
MLE		0.46512	0.80456	0.28086	0.28609	0.09276	0.01590
LM	30	0.61355	0.67701	0.16839	0.14933	0.03133	0.00604
PER		0.68005	0.60771	0.20627	0.10236	0.01160	0.00403
RSS		0.75106	0.55404	0.19361	0.06196	0.00292	0.00249
MLE		0.68149	0.62521	0.20287	0.10144	0.01567	0.00359
LM	70	0.94907	0.51307	0.12318	0.00259	0.00017	0.00008
PER		1.15729	0.42376	0.16639	0.02474	0.00581	0.00032
RSS		1.13740	0.48354	0.09385	0.01887	0.00027	0.00029
MLE		1.06863	0.47535	0.17407	0.00471	0.00060	0.00009
LM	100	1.04866	0.48895	0.04402	0.00236	$0.12191*10^{-3}$	0.00002
PER		1.03902	0.51961	0.02482	0.00152	0.00038	0.00001
RSS		0.93972	0.48880	0.02527	0.00363	$0.12521*10^{-3}$	0.00002
MLE		1.05076	0.47643	0.14971	0.00257	0.00055	0.00003

5. Conclusion:

   From the results in Tables 2, 3 and 4, we note the following:
•   In the second and third cases (C_2, C_3), we note that the parameters (α, λ) and the reliability function cannot be estimated for the two methods, namely MOM, and MM, because there is a condition in the estimate formulas, which is $\alpha > 2$.
•   In the first case (C_1), we note that RSS is the best in estimating (α) in all cases, while for $n = 10$ MOM is the best in estimating the (α),
and PER is the best in estimating (λ) in all cases, while for $n = 30$ MLE is the best in estimating the (λ).
For the reliability function of the distribution the MM is best when $n = 70,100,$ while if $n = 10$, then MOM is the best as well as if $n = 30$, then RSS is the best.
•   In the second case (C_2), in the estimation of (α), the RSS is best when $n = 10$, and if $n = 30$, then MLE is the best, while if $n = 70$, then the LM is the best as well as when $n = 100$ the PER is the best.
The RSS is the best in estimating (λ) in all cases, while for $n = 100$ the MLE is the best in estimating the (λ).
In estimating reliability the MLE is best in all cases, while for $n = 100$ the PER is the best.
•   In the third case (C_3), we note that the LM is the best in estimating (α) when $n = 10,70,$ however in $n = 30$ the RSS is the best, and when $n = 100$ the PER is the best. The LM is the best in estimating (λ) when $n = 70,100$, but when $n = 10$ the MLE is the best, and when $n = 30$ the RSS is the best.
In estimating reliability the LM is best when $n = 10,70,$ but when $n = 30$ the RSS is the best and when $n = 100$ the PER is the best.

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