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The effect of the Coefficient Function on the Solution Behavior for the Second-Order Complex Differential Equation

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Abstract

The purpose of this research paper is to present the second-order homogeneous complex differential equation $f'' + \mathcal{H}(z)f = 0$, where $\mathcal{H}(z) = e^{p(z)}$, which is defined on the certain complex domain depends on solution behavior. In order to demonstrate the relationship between the solution of the second-order of the complex differential equation and its coefficient of function, by studying the solution in certain cases: a meromorphic function, a coefficient of function, and if the solution is considered to be a transformation with another complex solution. In addition, the solution has been provided as a power series with some applications.

Keywords: meromorphic function, odd univalent function, estimate solution.

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تأثير دالة المعامل على سلوك الحل للمعادلة التفاضلية المعقدة من الدرجة الثانية

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الخلاصه

الغرض الاساسي من البحث هو تقديم المعادلة التفاضلية المعقدة المتجانسة من الدرجة الثانية $f'' + \mathcal{H}(z)f = 0$ عندما $\mathcal{H}(z) = e^{p(z)}$ والتي تم تعريفها على مجال معقد معين يعتمد على سلوك الحل من أجل توضيح العلاقة بين حل الدرجة الثانية من المعادلة التفاضلية المعقدة ومعامل الوظيفة الخاص بها ، من خلال دراسة الحل في حالات معينة دالة ذات شكل محدد ، ومعامل دالة ، وإذا كان الحل هو يعتبر تحولا مع حل آخر معقد. بالإضافة إلى ذلك ، تم توفير الحل كمتسلسلة قوى مع بعض التطبيقات.

1. Introduction

Consider the second order complex differential equation in form

$$f'' + \mathcal{H}(z)f = 0 \qquad \text{... ... ... (1)},$$

where $\mathcal{H}(z) = e^{p(z)}$ is holomorphic function defined on the certain complex domain in $\mathbb{C}$ which can be written as follows

$$f'' + e^{p(z)}f = 0 \qquad \text{... ... ... (2)}$$

The relation between the complex coefficient function in the 2nd order complex differential equation and its solution has obtained great interest for researchers in this field. (cf. [1-11]).

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In this paper we have shown that the meromorphic solutions of the 2nd order of complex differential equations (CDEs) with the coefficient function of $\mathcal{H}(z)$ are connected to each other, and we are tend to follow the same behaviour. (cf. [8], [9]).

We also discuss that if the solution is a holomorphic function for the given equation (CDE) with some different conditions, the results and suggestions in this work would be useful.

In specific meaning, the results in this article would be identified the relation between the solution of the given equation and its coefficient of function $\mathcal{H}(z)$ based on what was stated in the results of Bank- Laine [2]. We also refer to the Bank- Laine functions [5] due to the solutions of the proposed equation in our research are known , when it is considered an entire function $\mathbb{E}$ and normalized under the condition if $\mathbb{E}$ (z) = 0, then $\mathbb{E}'(z) = \pm 1$, is Bank-Laine function that are closely linked to differential equations, and then let $\mathcal{H}$(z) be an entire function and consider the differential equation $f'' + \mathcal{H}(z)f = 0$ with linearly independent solutions f_1 and f_2. Then the product of these solutions $E = f_1 f_2$ will be Bank-Laine. The converse of this is also true.

In the following theorems some of the basic concepts set out in this paper that are to be helpful and more convenient for the reader.

Theorem (1.1) [Hurwitz' Theorem (Special case) [7]

Let f^*_n be a sequence of meromorphic functions on a Domain $\mathbb{D}^*$ which converges spherically uniformly on compact subsets to a function f (which may be identical to ∞). If each $f^*_n \neq 0$ on $\mathbb{D}^*$ then either $f \neq 0$ on $\mathbb{D}^*$; that is, f has no zeros on $\mathbb{D}^*$, or $f \equiv 0$.

Theorem (1.2) [Picard's Theorem] [10]

Let $a, b,$ and $c \in \hat{\mathbb{C}}$ be distinct points, and Let f be a meromrphic function which omits $a, b,$ and c on $\mathbb{C}$ .Then f is constant.

Theorem (1.3) [Taylor's Theorem] ([6], [1])

Let f be the holomorphic function on a domain $\mathbb{D}$ and let $a \in \mathbb{D}$. Then $f(z) = \sum_{k=o}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$ for z near.

Theorem (1.4) [Weierstrass Theorem] [5]

Let f_n be a sequence of holomrphic functions on a domain $\mathbb{D}$ which converges uniformly on compact subsets of $\mathbb{D}$ to a function $f: \mathbb{D} \to \mathbb{C}$. Then f is holomorphic in $\mathbb{D}$ and, for $k \in \mathbb{N}$, the sequenc of derivatives $f_n^{(k)}$ convrges uniformly on compact subsets to $f^{(k)}$.

2. Main Results.

Theorem (2.1) Let f be a meromorphic univalent function in the domain$(|z| > 1)$, and let f be a solution of differential equation (1) where $\mathcal{H}(z) = e^{p(z)}$ and $p(z)$ are odd univalent function.

Then, if $f(z) = \frac{p(z)-p(\zeta)}{z-\zeta}$, then $e^{p(z)} = -\sum_{m=0}^{\infty} C(\zeta) \frac{z^{-m}}{-m}$.

Proof. Given f is a meromorphic univalent function on $\{z: |z| > 1\}$, and $p(z)$ is an odd univalent function which implies to

$$\mathcal{H}(z) = \frac{1}{f\left(\frac{1}{z}\right)}$$

$$= z - a_2 + (a_2{}^2 - a_3)z^{-1} + \cdots \quad , |z| > 1 ,$$

then $$f\left(\frac{1}{z}\right) = \frac{1}{\mathcal{H}(z)}$$

$$= z^{-1} + \sum_{n=2}^{\infty} a_n z^{-n} \ .$$

Define $$\mathcal{H}(z) = \frac{1}{\frac{p(z)-p(\zeta)}{z-\zeta}} = \frac{z-\zeta}{p(z)-p(\zeta)} \ . \qquad (3)$$

Derive equation (3) with respect to z as follows

$$\frac{d}{dz}[\mathcal{H}(z)] = \frac{d}{dz}\left[\frac{z-\zeta}{p(z)-p(\zeta)}\right] ,$$

$$\frac{d}{dz}\left(e^{p(z)}\right)=\frac{[p(z)-p(\zeta)]-(z-\zeta)p'(z)}{[p(z)-p(\zeta)]^2}\ ,$$
$$\frac{d}{dz}\left(e^{p(z)}\right)=\frac{1}{p(z)-p(\zeta)}-\frac{(z-\zeta)}{[p(z)-p(\zeta)]^2}\ p'(z)\ ,$$
$$\frac{d}{dz}\left(e^{p(z)}\right)=-\sum_{m=0}^{\infty}C(\zeta)z^{-m-1}\ ,$$
$$e^{p(z)}=-\sum_{m=1}^{\infty}C(\zeta)\int z^{-m-1}\ dz\ ,$$
$$e^{p(z)}=-\sum_{m=1}^{\infty}C(\zeta)\frac{z^{-m}}{-m}\ .$$

The proof is complete

Theorem (2.2).

Let $f(z)=e^{p(z)}$ be the solution of the given equation (2) such that $p(z)=b_1z+b_2z^2+\cdots,\quad |z|<1$ then $\| B_m \| \leq \| n(n-1)b_n \| + \| nb_n \|^2$ where B_m are the coefficients of power series of $e^{p(z)}$.

Proof. Given $p(z)=b_1z+b_2z^2+b_3z^3+\cdots \quad |z|<1$.

Let $e^{p(z)}=1+B_1z+B_2z^2+B_3z^3+\cdots$

be a power series convergent in unit disc with circle of convergence $R>0$.

Suppose that $f(z)=e^{p(z)}$, then

$$f'(z)=e^{p(z)}*p'(z),$$
$$f'(z)=p'(z)f(z)\ , \qquad (4)$$

and

$$f''(z)=f(z)\,p''(z)+f(z)\,p'^2(z)\ ,$$
$$f''(z)=[p''(z)+p'^2(z)]\,f(z)\ . \qquad (5)$$

Compare equation (5) with the given equation (2), we obtain

$$-[p''(z)+p'^2(z)]=e^{p(z)}\ .$$

It is known that $e^{p(z)}=\sum_{m=0}^{\infty}B_mz^m$, and $p(z)=\sum_{n=1}^{\infty}b_nz^n$ which implies to

$$-\left[\sum_{n=2}^{\infty}n(n-1)b_nz^{n-2}+\sum_{n=1}^{\infty}n^2{b_n}^2\,z^{2n-2}\right]=\sum_{m=0}^{\infty}B_m\,z^m\ ,$$
$$\left|\sum_{m=1}^{\infty}B_m\right|=|-1|\left|\sum_{n=1}^{\infty}n(n-1)b_n+\sum_{n=1}^{\infty}n^2{b_n}^2\right|$$
$$\leq\sum_{n=1}^{\infty}|n(n-1)|\,|b_n|+|n^2|\left|{b_n}^2\right|,$$

$$\left||B_m|\right|\leq\left||n(n-1)b_n|\right|+\left||n{b_n}^2|\right|^2\ ,$$

The proof is complete

Theorem (2.3).

Let $f=z+z^2+z^3+\cdots\ \in S$ be a solution to the given equation (2), and let $zF=f$ be a solution to the equation $F''+e^{P(z)}F=0$ ……… (6)

Then, the coefficient function of equation (2) is related with its equivalent in equation (6) by the form $e^{p(z)}=2-z^2e^{P(z)}$,

with singular point at $z=1$

Proof . Given $f=z+z^2+z^3+\cdots\ \in S$.

Then $f=z(1+z+z^2+\cdots),$

$$\frac{f}{z}=1+z+z^2+\cdots\ .$$

Let $F=1+z+z^2+\cdots$ such that $F=\frac{f}{z}$, Hence $zF=f$.

Suppose that

$F(z)=\frac{1}{(1-z)^{\beta}}$, $|z|<1$ and $\beta\in\mathbb{N}=\{1,2,3,\ldots\}$

Calculate $F',\ F''$ as follows

$$F'(z)=\frac{-\beta(1-z)^{\beta-1}}{(1-z)^{2\beta}}\ ,$$

$$F''(z) = \frac{\beta(\beta+1)}{(1-z)^{\beta+2}}.$$

Substitute F, F'' in the given equation

$$F'' + e^{P(z)}F = 0 \ ,$$

$$\frac{\beta(\beta+1)}{(1-z)^{\beta+2}} + e^{P(z)} \ \frac{1}{(1-z)^{\beta}} = 0 \ ,$$

$$e^{P(z)} = \frac{-\beta(\beta+1)}{(1-z)^{\beta+2}} / \frac{1}{(1-z)^{\beta}} \ ,$$

$$e^{P(z)} = \frac{-\beta(\beta+1)}{(1-z)^{2}}.$$

Now, Derive $F = \frac{\mathfrak{f}}{z}$ with respect to z as follows

$$F' = \frac{\mathfrak{f}'}{z} - \frac{\mathfrak{f}}{z^2} \ ;$$

$$F'' = \frac{\mathfrak{f}''}{z} - \frac{2\mathfrak{f}'}{z^2} + \frac{2\mathfrak{f}}{z^3}.$$

We can rewrite the equation (6) as follows

$$F'' - \left[\frac{\beta(\beta+1)}{(1-z)^2}\right] F = 0 \ ,$$

Hence

$$\frac{\mathfrak{f}''}{z} - \frac{2\mathfrak{f}'}{z^2} + \frac{2\mathfrak{f}}{z^3} - \frac{\beta(\beta+1)}{(1-z)^2} * \frac{\mathfrak{f}}{z} = 0 \ ,$$

$$z^2\mathfrak{f}'' - 2z\mathfrak{f}' + 2\mathfrak{f} - \frac{\beta(\beta+1)}{(1-z)^2} z^2 \mathfrak{f} = 0 \,. \tag{7}$$

Now, equating the coefficients of both sides for equations (7) and (2), we obtain the following

$$\mathfrak{f}: \quad e^{p(z)} = 2 - \frac{\beta(\beta+1)}{(1-z)^2} z^2 \ . \tag{8}$$

And,

$\mathfrak{f}''$: $z^2 = 1 \Rightarrow z = \pm 1 \ \Rightarrow \ z = 1$ is a singular point for the coefficient function $e^{p(z)}$ in (8).

Hence, one can rewrite the equation (2) as follows

$$\mathfrak{f}'' + \left(2 - \frac{\beta(\beta+1)}{(1-z)^2} z^2\right)\mathfrak{f} = 0.$$

Since $e^{p(z)} > 0$. and this means $\frac{\beta(\beta+1)}{(1-z)^2} z^2 < 2$.

Finally, the result will draw out as follows

$$e^{p(z)} = 2 - z^2 e^{P(z)} \ .$$

The proof is complete. ■

Theorem (2.4). Let $\mathfrak{f}$ be a solution of (1). Then

$$\mathfrak{f}(z) = \sum_{m=0}^{n-1} \frac{\mathfrak{f}^{(m)}(\zeta)}{m!}(z-\zeta)^m + \frac{1}{(n-1)!}\int_{\gamma} (z-\eta)^{n-1}\mathcal{H}(\eta)\mathfrak{f}(\eta)d\eta.$$

Proof. Given equation (1),

$$\mathfrak{f}'' = -\mathcal{H}(z)\mathfrak{f} \ . \tag{9}$$

Now, we integrate both sides of (9) twice over a piecewise continuously differentiable curve $\gamma = [\zeta, z]$ in the unit disk $\mathbb{D}_r(0,1)$.

We obtain,

$$\iint_{\gamma} \mathfrak{f}^{(2)}(\eta)d\eta d\eta = \int_{\gamma} \mathfrak{f}^{(1)} d\eta$$

$$= [\,\mathfrak{f}(\eta)\,]_{\zeta}^{z}$$

$$= \mathfrak{f}(z) - \mathfrak{f}(\zeta)$$
$$= \mathfrak{f}(z) - \sum_{m=0}^{n-1} \frac{\mathfrak{f}^{(m)}(\zeta)}{m!}(z-\zeta)^m$$

On the other hand,

$$\iint_\gamma \mathcal{H}(\eta)\mathfrak{f}(\eta)d\eta d\eta = \frac{1}{(n-1)!}\int_\gamma (z-\eta)^{n-1}\mathcal{H}(\eta)\mathfrak{f}(\eta)d\eta$$

Then

$$\mathfrak{f}(z) - \sum_{m=0}^{n-1} \frac{\mathfrak{f}^{(m)}(\zeta)}{m!}(z-\zeta)^m = \frac{1}{(n-1)!}\int_\gamma (z-\eta)^{n-1}\mathcal{H}(\eta)\mathfrak{f}(\eta)d\eta$$
$$\mathfrak{f}(z) = \sum_{m=0}^{n-1} \frac{\mathfrak{f}^{(m)}(\zeta)}{m!}(z-\zeta)^m + \frac{1}{(n-1)!}\int_\gamma (z-\eta)^{n-1}\mathcal{H}(\eta)\mathfrak{f}(\eta)d\eta$$

The proof is complete

3. Some Applications.

The power series solution is dealt with just briefly in this part of our work. It explains how the functions are described as power series and how to find a series representations of the derivatives of such functions by differentiating the terms of the power series one by one. In such way this repetition relation allows to express each coefficient two terms faster in terms of the coefficient. This result is an expression for even values and a separate expression for odd values. It is therefore important to present some examples that explain the purpose of this work and learn more about it.

Example (3.1)

Consider $\mathfrak{f}'' + e^z\mathfrak{f} = 0$ ... ... (10) be a complex differential equation, and let $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$, and $\mathfrak{f} = \sum_{n=0}^{\infty} a_n z^n$. In order to show that $\mathfrak{f} = a_o\underbrace{\left[1 - \frac{z^2}{2} - \frac{z^3}{3.2} + \frac{z^5}{5.4.2} + \cdots\right]}_{z_1} + a_1\underbrace{\left[z - \frac{z^3}{3.2} - \frac{z^4}{4.3} - \frac{z^5}{5.4.3} + \cdots\right]}_{z_2}$, where a_0, a_1 are arbitrary.

Solution .

Given $\mathfrak{f} = \sum_{n=0}^{\infty} a_n z^n$. Deriving f twice times to obtain

$\mathfrak{f}' = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$\mathfrak{f}'' = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$.

Substitute e^z , $\mathfrak{f}$, and $\mathfrak{f}''$ in equation (10),we get

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \left[\sum_{n=0}^{\infty}\frac{z^n}{n!}\right]\left[\sum_{n=0}^{\infty} a_n z^n\right] = 0 \ ,$$
$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{a_{n-k}}{k!}\right] z^n = 0 \ ,$$
$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n + \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{a_{n-k}}{k!}\right] z^n = 0 \ ,$$
$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n}\frac{a_{n-k}}{k!}\right] z^n = 0 \ ,$$
$$(2a_2 + a_o) + (3.2a_3 + a_1 + a_0)z + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n}\frac{a_{n-k}}{k!}\right] z^n = 0 \ ,$$

$2a_2 + a_o = 0 \qquad \Rightarrow a_2 = -\frac{a_o}{2}$

$3.2a_3 + a_1 + a_0 = 0 \qquad \Rightarrow a_3 = -\frac{a_1}{3.2} - \frac{a_o}{3.2}$

And ,

$(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n} \frac{a_{n-k}}{k!} = 0$

$$a_{n+2} = -\frac{\sum_{k=0}^{n} \frac{a_{n-k}}{k!}}{(n+2)(n+1)} \qquad , n = 2,3,4, \dots$$

If $n = 2, \quad \Rightarrow a_4 = -\frac{a_1}{4.3}$

If $n = 3, \quad \Rightarrow a_5 = -\frac{a_1}{5.4.3} + \frac{a_o}{5.4.2}$

then, $f = \sum_{n=0}^{\infty} a_n z^n$,

$f = a_o + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots$,

$f = a_o + a_1 z - \frac{a_o}{2} z^2 - \frac{a_1}{3.2} z^3 - \frac{a_o}{3.2} z^3 - \frac{a_1}{4.3} z^4 - \frac{a_1}{5.4.3} z^5 + \frac{a_o}{5.4.2} z^5 + \cdots$,

$$f = a_o \underbrace{\left[1 - \frac{z^2}{2} - \frac{z^3}{3.2} + \frac{z^5}{5.4.2} + \cdots\right]}_{z_1} + a_1 \underbrace{\left[z - \frac{z^3}{3.2} - \frac{z^4}{4.3} - \frac{z^5}{5.4.3} + \cdots\right]}_{z_2},$$

where a_0, a_1 are arbitrary, and z_1, z_2 are two power series solutions that are holomorphic in the given domain.

Example (3.2)

Consider $f'' + e^z f = z \dots\dots\dots (11)$ be a complex differential equation, and let $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $f = \sum_{n=0}^{\infty} a_n z^n$, in order to show that

$$f = a_o \underbrace{\left[1 - \frac{z^2}{2} - \frac{z^3}{3.2} + \frac{z^5}{5.4.2} + ..\right]}_{z_1} + a_1 \underbrace{\left[z - \frac{z^3}{3.2} - \frac{z^4}{4.3} - \frac{z^5}{5.4.3} + ..\right]}_{z_2} + \underbrace{\frac{z^3}{3.2} - \frac{z^5}{5.4.3.2}}_{\text{particular solution}} + \dots,$$

where a_o, a_1 are arbitrary.

Solution.

Given $f = \sum_{n=0}^{\infty} a_n z^n$. Deriving f twice times to obtain

$f' = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$f'' = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$

Substitute each of e^z, f, f'' in equation (11)

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \left[\sum_{n=0}^{\infty} \frac{z^n}{n!}\right]\left[\sum_{n=0}^{\infty} a_n z^n\right] = z ,$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{a_{n-k}}{k!}\right] z^n = z ,$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n + \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{a_{n-k}}{k!}\right] z^n = z ,$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n} \frac{a_{n-k}}{k!}\right] z^n = z ,$$

$$(2a_2 + a_o) + (3.2a_3 + a_1 + a_0)z + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n} \frac{a_{n-k}}{k!}\right] z^n = z,$$

$2a_2 + a_o = 0 \qquad \Rightarrow a_2 = -\frac{a_o}{2}$,

$3.2a_3 + a_1 + a_0 = 1 \qquad \Rightarrow a_3 = \frac{1}{3.2} - \frac{a_1}{3.2} - \frac{a_o}{3.2}$

And ,

$(n+2)(n+1)a_{n+2} + \sum_{k=0}^{n} \frac{a_{n-k}}{k!} = 0$,
such that

$$a_{n+2} = -\frac{\sum_{k=0}^{n} \frac{a_{n-k}}{k!}}{(n+2)(n+1)} \qquad , n = 2,3,4, \ldots$$

If $n = 2$, $\Rightarrow a_4 = -\frac{a_1}{4.3}$

If $n = 3$, $\Rightarrow a_5 = -\frac{1}{5.4.3.2} - \frac{a_1}{5.4.3} + \frac{a_0}{5.4.2}$,

then $f = \sum_{n=0}^{\infty} a_n z^n$,

$$f = a_o + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots ,$$

$$f = a_o + a_1 z^1 - \frac{a_o}{2} z^2 + \frac{1}{3.2} z^3 - \frac{a_1}{3.2} z^3 - \frac{a_o}{3.2} z^3 - \frac{a_1}{4.3} z^4 - \frac{1}{5.4.3.2} z^5 - \frac{a_1}{5.4.3} z^5 + \frac{a_0}{5.4.2} z^5 + \cdots ,$$

$$f = a_o \underbrace{\left[1 - \frac{z^2}{2} - \frac{z^3}{3.2} + \frac{z^5}{5.4.2} + ..\right]}_{z_1} + a_1 \underbrace{\left[z - \frac{z^3}{3.2} - \frac{z^4}{4.3} - \frac{z^5}{5.4.3} + ..\right]}_{z_2} + \underbrace{\frac{z^3}{3.2} - \frac{z^5}{5.4.3.2}}_{\text{particular solution}} + \ldots$$

where a_0, a_1 are arbitrary, and z_1, z_2 are two power series solutions that are holomorphic in the given domain. It should also be remembered that the term of the non-homogenous component reflects a particular solution.

Example (3.3)

Consider a complex differential equation (2), and let

$e^{p(z)} = 2 - \frac{z^2 \beta(\beta+1)}{(1-z)^2}$ as a result that is obtained by means of a proof of theorem (2.3), and let $f = \sum_{n=0}^{\infty} a_n z^n$. Then

$$f = a_o \underbrace{\left[1 - z^2 + \frac{z^4}{3.2} + \frac{\beta(\beta+1)z^4}{4.3} + \frac{\beta(\beta+1)z^5}{5.2} + \cdots\right]}_{z_1} + a_1 \underbrace{\left[z - \frac{z^3}{3} + + \frac{z^5}{5.3.2} + \frac{\beta(\beta+1)z^5}{5.4} + \cdots\right]}_{z_2};$$

$\beta \in \mathbb{N} = \{1,2,3, \ldots\}$, where a_0, a_1 are arbitrary.

Solution.

Derive f twice times as follows

$f' = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$f'' = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$, and then

substitute $e^{p(z)}, f, f''$ in equation (2)

$$\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \left[2 - \frac{z^2 \beta(\beta+1)}{(1-z)^2}\right] \left[\sum_{n=0}^{\infty} a_n z^n\right] = 0 ,$$

$$\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=0}^{\infty} 2 a_n z^n - \sum_{n=0}^{\infty} \beta(\beta+1) a_n \frac{z^{n+2}}{(1-z)^2} = 0 ,$$

$$(1-z)^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + (1-z)^2 \sum_{n=0}^{\infty} 2 a_n z^n - \sum_{n=0}^{\infty} \beta(\beta+1) a_n z^{n+2} = 0 ,$$

$$(1 - 2z + z^2) \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + (1 - 2z + z^2) \sum_{n=0}^{\infty} 2 a_n z^n - \sum_{n=0}^{\infty} \beta(\beta+1) a_n z^{n+2} = 0 ,$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} - \sum_{n=2}^{\infty} 2n(n-1)a_n z^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n z^n + \sum_{n=0}^{\infty} 2a_n z^n - \sum_{n=0}^{\infty} 4a_n z^{n+1} + \sum_{n=0}^{\infty} 2a_n z^{n+2} - \sum_{n=0}^{\infty} \beta(\beta+1)a_n z^{n+2} = 0 \ ,$$

$$2a_2 + 6a_3 z + \sum_{n=4}^{\infty} n(n-1)a_n z^{n-2} - 4a_2 z - \sum_{n=4}^{\infty} 2(n-1)(n-2)a_{n-1} z^{n-2} + \sum_{n=4}^{\infty} (n-2)(n-3)a_{n-2} z^{n-2} + 2a_o + 2a_1 z + \sum_{n=4}^{\infty} 2a_{n-2} z^{n-2} - 4a_o z - \sum_{n=4}^{\infty} 4a_{n-3} z^{n-2} + \sum_{n=4}^{\infty} 2a_{n-4} z^{n-2} - \sum_{n=4}^{\infty} \beta(\beta+1)a_{n-4} z^{n-2} = 0 \ ,$$

$$2a_2 + 2a_o + (6a_3 - 4a_2 + 2a_1 - 4a_o)z + \sum_{n=4}^{\infty} [n(n-1)a_n - 2(n-1)(n-2)a_{n-1} + (n-2)(n-3)a_{n-2} + 2a_{n-2} - 4a_{n-3} + 2a_{n-4} - \beta(\beta+1)a_{n-4}]\, z^{n-2} = 0 \ ,$$

$$2a_2 + 2a_o = 0 \qquad \Longrightarrow a_2 = -a_o$$

$$6a_3 - 4a_2 + 2a_1 - 4a_o \quad \Longrightarrow \ a_3 = -\frac{a_1}{3}$$

And ,

$$n(n-1)a_n - 2(n-1)(n-2)a_{n-1} + (n-2)(n-3)a_{n-2} + 2a_{n-2} - 4a_{n-3} + 2a_{n-4} - \beta(\beta+1)a_{n-4} = 0 \qquad , n = 4,5,\dots$$

If $n = 4$, $\Longrightarrow a_4 = \frac{a_o}{3.2} + \frac{\beta(\beta+1)a_o}{4.3}$

If $n = 5$, $\Longrightarrow a_5 = \frac{\beta(\beta+1)a_o}{5.2} + \frac{a_1}{5.3.2} + \frac{\beta(\beta+1)a_1}{5.4}$

then $\qquad \mathfrak{f} = \sum_{n=0}^{\infty} a_n z^n$,

$$\mathfrak{f} = a_o + a_1 z^1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots \ ,$$

$$\mathfrak{f} = a_o + a_1 z^1 - a_o z^2 - \frac{a_1}{3} z^3 + \frac{a_o}{3.2} z^4 + \frac{\beta(\beta+1)a_o}{4.3} z^4 + \frac{\beta(\beta+1)a_o}{5.2} z^5 + \frac{a_1}{5.3.2} z^5 + \frac{\beta(\beta+1)a_1}{5.4} z^5 + \cdots \ ,$$

$$\mathfrak{f} = a_o \underbrace{\left[1 - z^2 + \frac{z^4}{3.2} + \frac{\beta(\beta+1)z^4}{4.3} + \frac{\beta(\beta+1)z^5}{5.2} + \cdots\right]}_{z_1} + \ a_1 \underbrace{\left[z - \frac{z^3}{3} + + \frac{z^5}{5.3.2} + \ \frac{\beta(\beta+1)z^5}{5.4} + \cdots\right]}_{z_2};$$

$\beta \in \mathbb{N} = \{1,2,3,\dots\}$,
where a_0, a_1 are arbitrary, and z_1, z_2 are two power series solutions that are holomorphic in the given domain.

Conclusion

In this work, the meromorphic univalent function theorem has been shown to demonstrate the depth of the relationship between the second order solution of the complex differential equation and its coefficient of function.

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