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Coefficient Bounds for Certain Subclass of Analytic Functions Defined By Quasi-Subordination

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Abstract
In this paper, we define certain subclasses of analytic univalent function associated with quasi-subordination. Some results such as coefficient bounds and Fekete-Szego bounds for the functions belonging to these subclasses are derived.

Keywords: Analytic functions, Univalent function, Quasi-subordination, Subordination, Majorization.

قيود المعاملات لفئات جزئية من الدوال التحليلية المعرفة بواسطة شبه التابعية

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الخلاصة
في هذا البحث نعرف فئات جزئيه من فئة الدوال التحليليه الاحاديه المرفقه بشبه التابعيه. بعض النتائج لقيود المعاملات وقيود فيكيتي زيغوللدوال التي تنتمي لهذه الفئات الجزئيه اشتقت.

1.Introduction.

Let $\mathcal{A}$ be the class of analytic functions f (z) which are analytic in the open unit disk U={z:|z|<1}, normalized by f (0)=0 and f'(0)=1 of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \qquad (1.1)$$

Let f and g be two analytic functions in U. Then the function f is said to be subordinate to g, written as

$$f \prec g \;\text{ or } f(z) \prec g(z) \;\; (z \in U). \qquad (1.2)$$

if there exist Schwarz function w which is analytic in U, w(0)=0 and |w(z)|<1 such that $f(z) = g(w(z))$.Furthermore, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.For brief survey on the concept of subordination, see [1].
Robertson [2] introduced the concept of quasi-subordination defined as follows:
An analytic function f is quasi-subordination to analytic function g in the open unit disk is written

$$f(z) \prec_q g(z), \qquad (1.3)$$

if there exist analytic function φ and w, with |φ(z)|≤1, w(0) = 0 and $||w(z)||$<1 such that
$f(z) = \varphi(z) g(w(z))$.

Note, when $\varphi(z) = 1$, then $f(z) = g(w(z))$ so that $f(z) \prec g(z)$ in U. Furthermore if w(z)= z, then $f(z) = \varphi(z)g(z)$ and this case f is majorized to g , written $f(z) \ll g(z)$ in U. Hence it is

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obvious that quasi-subordination is generalization of subordination as well as majorization. For more information, see [3,4, 5] for works related to quasi-subordination.

Many authors have been investigated the bounds of Fekete-Szego coefficient for various classes (see [1,4,6-11]).

Now consider the following

$w(z)=\frac{1+k(z)}{1-k(z)}=1+w_1z+w_2z^2+w_3z^3+\cdots$,

then

$k(z)=\frac{1}{2}[w_1z+(w_2-½w_1^2)z^2+\cdots]$.

Throughout this paper it is assumed that ϕ is analytic in U with $\phi(0)=1$ and of the form

$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$, $B_1>0$.

Also,

$\varphi(z)=C_\circ+C_1z+C_2z^2+C_3z^3+\cdots$.

Now, we define the following subclasses of $\mathcal{A}$.

Definition (1.1).A function $f\in\mathcal{A}$ is said to be in the class $M_{\alpha,\gamma}^{q}(\phi)$ ($0\le\alpha<1,\gamma\in\mathbb{C}-\{0\}$), if it satisfies the following quasi-subordination

$\frac{1}{\gamma}\{zf''(z)+\alpha z^2f'''(z)\}\prec_q\phi(z)-1$

Definition (1.2).A function $f\in\mathcal{A}$ is said to be in the class $MH_{\alpha}^{q}(\phi)$ ($0<\alpha<1$), if it satisfies following quasi-subordination

$\alpha\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\}\prec_q\phi(z)-1$.

Definition (1.3). Let the class $MH^q(\alpha,\lambda,,\emptyset)$ consists of functions $f\in\mathcal{A}$ satisfying the quasi subordination

$(\frac{zf(z)'}{f(z)})^{\lambda}\alpha\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\}\prec_q\phi(z)-1$, $(\lambda\ge0)$

Definition (1.4). Let the class $MH^q(\alpha,\beta,\gamma,\phi)$consists of functions $f\in\mathcal{A}$ satisfying the quasi subordination

$$\frac{zf(z)'}{f(z)}(\frac{f(z)}{z})^{\beta}+\frac{1}{\gamma}\{zf''(z)+\alpha z^2f'''(z)\}\prec_q\phi(z)-1,\ (\beta\ge0).$$

To discuss main results we consider the following lemmas.

Lemma(1.5) [12].Let w be analytic function in U, with $w(0)=0$,$|w(z)|<1$ and

$w(z)=w_1z+w_2z^2+w_3z^3+\cdots$.

Then

$|w_2-tw_1^2|\le\max\{1,|t|\}$, $t\in\mathbb{C}$

The result is sharp for the functions $w(z)=z^2$ or $w(z)=z$.

Lemma(1.6) [12]. Let $\varphi(z)$ be analytic function in U, with $|\varphi(z)|<1$ and let

$\varphi(z)=C_\circ+C_1z+C_2z^2+C_3z^3+\cdots$.

Then $|C_\circ|\le1$ and $|C_n|\le1-|C_\circ|^2$ for $n>0$.

2. Main Results.

Theorem (2.1). If f is given by (1.1) belong to $M_{\alpha,\gamma}^{q}(\phi)$**,** then

$$|a_2|\le\frac{|\gamma|B_1}{4},\ |a_3|\le\frac{|\gamma|}{12(1+\alpha)}\max\{B_1,\frac{1}{2}(B_1-|B_2|)\}. \quad (2.2)$$

$$|a_3-\mu a_2^2|\le\frac{|\gamma|}{12(1+\alpha)}\max\{B_1,\frac{1}{2}[B_1-|B_2|+\frac{3}{2}(1+\alpha)|\gamma||\mu|B_1^2]\}. \quad (2.2)$$

Proof. Let $f\in M_{\alpha,\gamma}^{q}$.Then there exist an analytic functions φ in U with $|\varphi(z)|\le1$ and $k:U\to U$, with $k(0)=0$ and $|k(z)|<1$ such that:

$$\frac{1}{\gamma}\{zf''(z)+\alpha z^2f'''(z)\}=\varphi(z)\phi(k(z))-1. \quad (2.3)$$

$$\frac{1}{\gamma}\{zf''(z)+\alpha z^2f'''(z)\}=\frac{2}{\gamma}a_2z+\frac{6}{\gamma}(1+\alpha)a_3z^2+\cdots \quad (2.4)$$

$$\varphi(z)\phi(k(z)-1=\frac{1}{2}B_1C_\circ w_1z+[\frac{1}{2}B_1(C_\circ w_2-\frac{1}{2}C_\circ w_1{}^2)+\frac{1}{4}B_2w_1{}^2]z^2\ldots \quad (2.5)$$

Putting (2.4) and (2.5) in (2.3) and equating coefficient both sides, we get

$a_2=\frac{\gamma}{4}B_1C_\circ w_1$ and $a_3=\frac{\gamma}{6(1+\alpha)}[\frac{1}{2}B_1(C_\circ w_2-\frac{1}{2}C_\circ w_1{}^2)+\frac{1}{2}B_2w_1{}^2]$.

Since φ(z) is analytic and bounded in U, we have $|C_n| \leq$ 1- $|C_\circ|^2 \leq 1$, n>0.Using this fact and well known inequality $|w_1| \leq 1$, we get

$|a_2| \leq \frac{|\gamma|}{4} B_1$, $|a_3| \leq \frac{|\gamma|}{12(1+\alpha)} \max\{B_1, B_1 + |B_2|\}$.

Also

$$a_3 - \mu a_2^2 = \frac{B_1 C_\circ \gamma}{12(1+\alpha)} [w_2 - \frac{1}{2}\{(1 - \frac{B_2}{B_1 C_\circ}) - \frac{3}{2}\mu\gamma C_\circ B_1(1+\alpha)\} {w_1}^2]. \quad (2.6)$$

Applying Lemma (1.5) and Lemma(1.6) for (2.6), we obtain

$|a_3 - \mu a_2^2| \leq \frac{B_1|\gamma|}{12(1+\alpha)} \max\{1, \frac{1}{2}[1 + \frac{|B_2|}{B_1} + \frac{3}{2}|\mu||\gamma|B_1(1+\alpha)]\}$

$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{12(1+\alpha)} \max\{B_1, \frac{1}{2}[B_1 + |B_2| + \frac{3}{2}|\mu||\gamma|B_1^2(1+\alpha)]\}$■

For α=0 in the Theorem (2.1), we get the following corollary.

Corollary (2.2). If f given by (1.1) be in the class $M_{0,\gamma}^q(\phi)$**,** then

$a_2 \leq \frac{|\gamma|B_1}{4}$, $|a_3| \leq \frac{|\gamma|}{12} \max\{B_1, \frac{1}{2}(B_1 - |B_2|)\}$.

$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{12} \max\ \{B_1, \frac{1}{2}[B_1 - |B_2| + \frac{3}{2}|\gamma||\mu|B_1^2]\}$.

In next, if we are using the Schwarz function of the following form

k(z)= $w_1 z + w_2 z^2 + w_3 z^3 + \cdots$,

we get the following results.

Theorem (2.3). Let $f \in \mathcal{A}$ be of the form (1.1) belongs to the class $M_{\alpha,\gamma}^q(\phi)$.Then

$|a_2| \leq \frac{|\gamma|}{2} B_1$, $|a_3| \leq \frac{|\gamma|}{6(1+\alpha)} [B_1 + \max\{B_1, |B_2|\}$,

and for some μ∈ℂ:

$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{6(1+\alpha)} [B_1 + \max\{B_1, \frac{3}{2}(1+\alpha)|\mu|\,|\gamma|{B_1}^2 + |B_2|\}$.

Proof. If $f \in M_{\alpha,\gamma}^q(\phi)$, then there exist analytic functions φ in U with |φ(z)|≤1 and k:$U \to U$,with k(0)=0 and |k(z)|<1 such that:

$$\frac{1}{\gamma}\{z f''(z) + \alpha z^2 f'''(z)\} = \varphi(z)(\phi(k(z))-1) \quad (2.7)$$

We have

$\phi(k(z))- 1= B_1 w_1 z + (B_1 w_2 + B_2 {w_1}^2) z^2 + \cdots$, $B_1>0$

$$\varphi(z)(\phi(k(z))-1) = C_\circ B_1 w_1 z + [C_1 B_1 w_1 + C_\circ(B_1 w_2 + B_2 {w_1}^2)] z^2 \ldots \quad (2.8)$$

Putting (2.4) and (2.8) in (2.7) and equating coefficients in both sides, we get

$a_2 = \frac{\gamma}{2} B_1 C_\circ w_1$ and $a_3 = \frac{\gamma}{6(1+\alpha)} [C_1 B_1 w_1 + C_\circ(B_1 w_2 - B_2\, {w_1}^2)]$.

Also

$a_3 - \mu a_2^2 = \frac{\gamma}{6(1+\alpha)} [C_1 B_1 w_1 + C_\circ(B_1 w_2 - B_2\, {w_1}^2)] - \frac{1}{4}\mu\gamma^2 {C_\circ}^2 {B_1}^2 {w_1}^2$.

Since φ(z) is analytic and bounded in U, we have $|C_n| \leq$ 1- $|C_\circ|^2 \leq 1$, n>0.Using this fact and well known inequality $|w_1| \leq 1$, and applying Lemma (1.5),we obtain

$|a_2| \leq \frac{|\gamma|}{2} B_1$,

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{6(1+\alpha)} [B_1 + \max\{B_1, \frac{3}{2}(1+\alpha)|\mu|\,|\gamma|{B_1}^2 + |B_2|\}. \quad (2.9)$$

This is required result. Further setting μ=0 in (2.9) we get the bound on $|a_3|$■

Theorem (2.4). If $f \in \mathcal{A}$ satisfies

$\frac{1}{\gamma}\{z f''(z) + \alpha z^2 f'''(z)\} \ll (\phi(z) -1)$,

then the following inequalities hold

$|a_2| \leq \frac{|\gamma|}{2} B_1$, $|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{6(1+\alpha)} [B_1 + |B_2| + \frac{3}{2}|\mu||\gamma|{B_1}^2\}$,and $|a_3| \leq \frac{|\gamma|}{6(1+\alpha)} \{B_1 + |B_2|\}$.

Proof. The results follows by taking w(z)=z in the proof of Theorem (2.3) ■

For α=0 in the theorem (2.3), we get the following corollary.

Corollary (2.5). If f given by (1.1) be in the class $M_{0,\gamma}^q(\phi)$**,** then

$|a_2| \leq \frac{|\gamma|}{2} B_1$, $|a_3| \leq \frac{|\gamma|}{6} [B_1 + \max\{B_1, |B_2|\}$.and for some μ∈ℂ

$|a_3 - \mu a_2^2| \leq \frac{|\gamma|}{6} [B_1 + \max\{B_1, \frac{3}{2}|\mu||\gamma|{B_1}^2 + |B_2|\}$.

Theorem (2.6). If $f \in\mathcal{A}$ (1.1) belong to the class $MH_\alpha^q(\phi)$, then
$|a_2|\le\frac{B_1}{2\alpha}$, $|a_3|\le \frac{1}{12\alpha}\max\{B_1,\frac{{B_1}^2}{\alpha}+|B_2|\}$, and for any $\mu\in$C
$\left|a_3-\mu a_2^2\right| \le \frac{1}{12\alpha}B_1+\frac{1}{12\alpha}\max\{B_1,\frac{3{B_1}^2}{\alpha}|\mu-\frac{1}{3}|+|B_2|\}$.
Proof. If $f\in MH_\alpha^q(\phi)$, then there exist analytic functions φ in U with |φ(z)|≤1 and k:$U\to U$,with k(0)=0 and |k (z)|<1 such that:

$$\alpha\left\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\right\}=\varphi(z)(\phi(k(z))-1) \quad (2.10)$$

$$\alpha\left\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\right\}=2\alpha a_2z+\alpha(12a_3-4{a_2}^2)z^2+\cdots \quad (2.11)$$

$$\varphi(z)(\phi(k(z))-1)=C_\circ B_1w_1z+[C_1B_1w_1+C_\circ(B_1w_2+B_2{w_1}^2)]z^2 \ldots \quad (2.12)$$

Putting (2.11) and (2.12) in (2.10) and equating coefficient both sides, we get
$a_2=\frac{1}{2\alpha}B_1C_\circ w_1$ and $a_3=\frac{1}{12\alpha}[C_1B_1w_1+C_\circ(B_1w_2+B_2{w_1}^2)]+\frac{1}{3}a_2^2$.

Since φ(z) is analytic and bounded in U, we have $|C_n|\le 1-|C_\circ|^2\le 1$, n>0.Using this fact and well known inequality $|w_1|\le 1$, we get $|a_2|\le\frac{1}{2\alpha}B_1$.
Also

$$a_3-\mu a_2^2=\frac{1}{12\alpha}[C_1B_1w_1+C_\circ(B_1w_2+B_2{w_1}^2)]+\frac{1}{3}a_2^2-\frac{1}{4\alpha^2}\mu {C_\circ}^2{B_1}^2{w_1}^2.$$
$$=\frac{1}{12\alpha}C_1B_1w_1+\frac{1}{12\alpha}[C_\circ(B_1w_2+B_2{w_1}^2)]+\frac{1}{3}\frac{{C_\circ}^2{B_1}^2{w_1}^2}{4\alpha^2}-\frac{1}{4\alpha^2}\mu {C_\circ}^2{B_1}^2{w_1}^2.$$
$$=\frac{1}{12\alpha}C_1B_1w_1+\frac{1}{12\alpha}C_\circ B_1[w_2-\{\frac{3}{\alpha}C_\circ B_1(\mu-\frac{1}{3})-\frac{B_2}{B_1}\}{w_1}^2]$$

Applying Lemma (1.5) and Lemma (1.6), we get
$\left|a_3-\mu a_2^2\right|\le\frac{1}{12\alpha}B_1+\frac{1}{12\alpha}B_1\max\{1,\frac{3B_1}{\alpha}|\mu-\frac{1}{3}|+\frac{|B_2|}{B_1}\}$.
$\left|a_3-\mu a_2^2\right|\le\frac{1}{12\alpha}B_1+\frac{1}{12\alpha}\max\{B_1,\frac{3{B_1}^2}{\alpha}|\mu-\frac{1}{3}|+|B_2|\}$.
For μ=0, the above will reduce to $|a_3|$■
For α=½ in the Theorem (2.6), we get the following corollary.
Corollary (2.7). If f given by (1.1) be in the class $MH_{½}^q(\phi)$ (ϕ)**,** then
$|a_2|\le B_1$, $|a_3|\le\frac{1}{6}\max\{B_1,2{B_1}^2+|B_2|\}$,
and for any μ∈C
$\left|a_3-\mu a_2^2\right|\le\frac{1}{6}B_1+\frac{1}{6}\max\{B_1,6{B_1}^2|\mu-\frac{1}{3}|+|B_2|\}$.
Theorem (2.8). If $f\in\mathcal{A}$ satisfies
$\alpha\left\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\right\}\ll$ (ϕ(z)-1),
then the following inequalities hold
$|a_2|\le\frac{1}{2\alpha}B_1$, $|a_3|\le\frac{1}{12\alpha}\{\frac{1}{\alpha}{B_1}^2+B_1+|B_2|\}$,
and for any μ∈ℂ,
$|a_3-\mu a_2^2|\le\frac{1}{12\alpha}\{\frac{1}{\alpha}{B_1}^2+B_1+|B_2|+|\mu|\frac{{B_1}^2}{4\alpha}\}$
Proof. The result follows by taking k(z) =z in the proof of Theorem (2.6) ■
Theorem (2.9). Let λ≥0 , 0<α<1, if f∈$\mathcal{A}$ belong to $MH^q(\alpha,\lambda,,\emptyset)$, Then
$|a_2|\le\frac{B_1}{2\alpha}$, $|a_3|\le\frac{1}{12\alpha}[B_1+B_1\max\{1,\frac{\lambda-2}{2\alpha}B_1+\left|\frac{B_2}{B_1}\right|\}]$.
And for any complex number μ,
$|a_3-\mu a_2^2|\le\frac{1}{12\alpha}[B_1+B_1\max\{1,\frac{\lambda-2}{2\alpha}B_1+|\mu|\frac{B_1}{4\alpha^2}+\left|\frac{B_2}{B_1}\right|\}]$.
Proof. Let $f\in MH^q(\alpha,\lambda,,\emptyset)$, λ≥0 ,0<α<1.Then there exist analytic functions φ and k with |φ(z)|≤1 and k:$U\to U$,with k (0)=0 and |k (z)|<1 such that:

$$\left(\frac{zf(z)'}{f(z)}\right)^\lambda\alpha\left\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\right\}=\varphi(z)(\phi(k(z))-1). \quad (2.13)$$

Since
$\left(\frac{zf(z)'}{f(z)}\right)^\lambda=1+\lambda a_2z+\frac{1}{2}\left[(\lambda^2-3\lambda)a_2^2+4\lambda a_3\right]z^2+\ldots$, and

$$\alpha\left\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\right\}= 2\alpha a_2 z + \alpha(12a_3-4a_2^2)\, z^2+\ldots\ldots \quad (2.14)$$

Hence from (2.14), we have

$$(\frac{zf(z)'}{f(z)})^{\lambda}\alpha\left\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\right\} =2\alpha a_2 z + [12\alpha a_3 + 2\alpha(\lambda-2)a_2^2]\, z^2+\ldots\ldots \quad (2.15)$$

$\phi(k(z))$- 1= $B_1w_1z+(B_1w_2+B_2{w_1}^2)z^2+\cdots$, $B_1>0$

$$\varphi(z)(\phi(k(z))-1) = C_\circ B_1 w_1 z + [C_1B_1w_1 + C_\circ(B_1w_2+B_2{w_1}^2)]z^2 \ldots \quad (2.16)$$

Put (2.15) and (2.16) in (2.13) and equating coefficients in both sides, we get

$$a_2=\frac{1}{2\alpha}C_\circ B_1 w_1 \quad (2.17a)$$

$$a_3=\frac{1}{12\alpha}[C_1B_1w_1+C_\circ B_1\{w_2-\left(\frac{\lambda-2}{2\alpha}C_\circ B_1-\frac{B_2}{B_1}\right){w_1}^2\}] . \quad (2.17b)$$

By using this fact and well-known inequality, $|w_1|\le 1$, we get

$|a_2|\le\frac{1}{2\alpha}B_1$.

Further,

$a_3-\mu a_2^2=\frac{1}{12\alpha}[C_1B_1w_1+C_\circ B_1\{w_2-\left(\frac{\lambda-2}{2\alpha}C_\circ B_1+3\mu\frac{C_\circ B_1}{\alpha}-\frac{B_2}{B_1}\right){w_1}^2\}]$.

Applying $|C_n|\le 1, |w_1|\le 1$ and Lemma (1.5) , we get

$|a_3-\mu a_2^2|\le\frac{1}{12\alpha}\ [B_1+B_1\max\{1,\frac{\lambda-2}{2\alpha}B_1+|\mu|\frac{B_1}{4\alpha^2}+\left|\frac{B_2}{B_1}\right|\}]$.

If we put μ=0 in above inequality, we get desired estimate $|a_3|$ as following

$|a_3|\le\frac{1}{12\alpha}[B_1+B_1\max\{1,\frac{\lambda-2}{2\alpha}B_1+\left|\frac{B_2}{B_1}\right|\}]$■

Corollary (2.10). For φ(z)=1 and α=½, we get the estimate for $|a_2|$ and $|a_3|$ as

$|a_2|\le B_1$ and $|a_3|\le\frac{B_1}{6}\max\{1,(\lambda-2)B_1+\left|\frac{B_2}{B_1}\right|\}$.

Remark (2.11). When λ=0,Theorem (2.9) reduces to Theorem (2.6).

Theorem (2.12). Let λ≥0 , 0<α<1, if $f\in\mathcal{A}$ satisfies

$(\frac{zf(z)'}{f(z)})^{\lambda}\alpha\{\frac{zf''(z)}{f'(z)}+z^2f'''(z)\}\ll\phi(z)-1$.

Then the following inequalities hold:

$|a_2|\le\frac{B_1}{2\alpha}$, $|a_3|\le\frac{1}{12\alpha}[B_1+\frac{\lambda-2}{2\alpha}{B_1}^2+|B_2|\}]$.

And for any complex number μ,

$|a_3-\mu a_2^2|\le\frac{1}{12\alpha}\ [B_1+{B_1}^2(\frac{\lambda-2}{2\alpha}+3|\mu|)+|B_2|]$.

Proof. The results follows by taking w(z)=z in the proof of Theorem (2.9) ■

Theorem (2.13). Let β≥0 , if $f\in\mathcal{A}$ belong to $MH^q(\alpha,\beta,\gamma,\emptyset)$, then

$|a_2|\le\frac{B_1|\gamma|}{2+(1+\beta)|\gamma|}$, $|a_3|\le\frac{B_1|\gamma|}{6(1+\alpha)+(2+\beta)|\gamma|}[1+\max\{1,\frac{(\beta-1)(2+\beta)B_1|\gamma|^2}{2|2+\gamma(1+\beta)|^2}+\left|\frac{B_2}{B_1}\right|\}]$.

$|a_3-\mu a_2^2|\le\frac{B_1|\gamma|}{6(1+\alpha)+(2+\beta)|\gamma|}[1+\max\{1,\frac{B_1|\gamma|^2}{2|2+\gamma(1+\beta)|^2}((\beta-1)(2+\beta)+2|\mu|(6(1+\alpha)+(2+\beta)|\gamma|)\left|\frac{B_2}{B_1}\right|\}]$.

Proof. Let $f\in MH^q(\alpha,\beta,\gamma,\emptyset)$, for β≥0 ,.then there are analytic functions φ and k with |φ(z)|≤1 and $k:U\to U$,with k(0)=0 and $|k(z)|<1$ such that

$$\frac{zf(z)'}{f(z)}(\frac{f(z)}{z})^{\beta}+\frac{1}{\gamma}\{zf''(z)+\alpha z^2f'''(z)\}=\varphi(z)(\phi(k(z))-1). \quad (2.18)$$

A computation shows that

$$\frac{zf(z)'}{f(z)}(\frac{f(z)}{z})^{\beta}=1+a_2(1+\beta)z+\frac{(2+\beta)}{2}[2a_3+(\beta-1){a_2}^2]z^2+\ldots \quad (2.19)$$

$\frac{1}{\gamma}\{zf''(z)+\alpha z^2f'''(z)\}=\frac{2}{\gamma}a_2z+\frac{6}{\gamma}(1+\alpha)a_3z^2+\cdots$.

Put (2.8) and (2.19) in (2.18) and equating coefficients in both sides, we get

$$a_2=\frac{1}{1+\beta+\frac{2}{\gamma}}C_\circ B_1w_1, \quad (2.20)$$

$a_3=\frac{|\gamma|}{6(1+\alpha)+(2+\beta)|\gamma|}[B_1C_1w_1+\frac{(1-\beta)(2+\beta){C_\circ}^2{B_1}^2\gamma^2}{2|2+\gamma(1+\beta)|^2}{w_1}^2+C_\circ B_1w_2+C_\circ B_2{w_1}^2]$.

Applying $|C_n|\le 1, |w_1|\le 1$ in (2.20), we get the value of $|a_2|$

Also,
$a_3-\mu a_2^2=\frac{\gamma}{6(1+\alpha)+(2+\beta)\gamma}[B_1C_1w_{1+}C_\circ B_1\{w_2-(\frac{C_\circ B_1\gamma^2}{2[2+\gamma(1+\beta)]^2}((\beta-1)(2+\beta)+2\mu(6(1+\alpha)+(2+\beta)\gamma))-\frac{B_2}{B_1})w_1{}^2\}].$
By using $|c_n|\leq 1$ and $|w_1|\leq 1$, we obtain
$\left|a_3-\mu a_2^2\right|=\frac{|\gamma|}{6(1+\alpha)+(2+\beta)|\gamma|}[B_1+B_1\left|\{w_2-(\frac{C_\circ B_1\gamma^2}{2[2+\gamma(1+\beta)]^2}((\beta-1)(2+\beta)+2\mu(6(1+\alpha)+(2+\beta)\gamma))-\frac{B_2}{B_1})w_1{}^2\}\right|].$
Now we shall use Lemma (1.5) to

$$\left|\{w_2-(\frac{C_\circ B_1\gamma^2}{2[2+\gamma(1+\beta)]^2}((\beta-1)(2+\beta)+2\mu(6(1+\alpha)+(2+\beta)\gamma))-\frac{B_2}{B_1})w_1{}^2\}\right|$$

yields
$|a_3-\mu a_2^2|\leq\frac{B_1|\gamma|}{6(1+\alpha)+(2+\beta)|\gamma|}$ [1+max{1,| $\frac{B_1\gamma^2}{2[2+\gamma(1+\beta)]^2}$ $((\beta-1)(2+\beta)+2\mu(6(1+\alpha)+(2+\beta)\gamma))-\frac{B_2}{B_1})|$,
and hence we can conclude that
$|a_3-\mu a_2^2|\leq\frac{B_1|\gamma|}{6(1+\alpha)+(2+\beta)|\gamma|}$[1+max{1, $\frac{B_1|\gamma|^2}{2|2+\gamma(1+\beta)|^2}$ $((\beta-1)(2+\beta)+2|\mu|(6(1+\alpha)+(2+\beta)|\gamma|)\left|\frac{B_2}{B_1}\right|\}].$
If we put μ=0 in above inequality, we get desired estimate | a_3|■
Corollary (2.14). For β =1 and α=0, the coefficient estimates becomes
$|a_2|\leq\frac{B_1|\gamma|}{2+|\gamma|}$,$|a_3|\leq\frac{B_1|\gamma|}{6+2|\gamma|}[1+\max\{1,\frac{2B_1|\gamma|^2}{2|2+\gamma|^2}+\left|\frac{B_2}{B_1}\right|\}].$
Theorem (2.15). If $f\in\mathcal{A}$ satisfies
$\frac{zf(z)'}{f(z)}(\frac{f(z)}{z})^{\beta}+\frac{1}{\gamma}\{zf''(z)+\alpha z^2f'''(z)\}\ll\phi(z)-1,$
then the following inequalities hold
$|a_2|\leq\frac{B_1|\gamma|}{|2+(1+\beta)\gamma|}$,
$|a_3|\leq\frac{B_1|\gamma|}{|6(1+\alpha)+(2+\beta)\gamma|}[1+|\frac{(\beta-1)(2+\beta)B_1\gamma^2}{2(2+\gamma(1+\beta))^2}|+\left|\frac{B_2}{B_1}\right|],$
and for any complex number μ
$|a_3-\mu a_2^2|\leq\frac{B_1|\gamma|}{|6(1+\alpha)+(2+\beta)\gamma|}[1+\frac{B_1|\gamma|^2}{2|2+\gamma(1+\beta)|^2}|(1-\beta)(2+\beta)-2\mu\left(\frac{6}{\gamma}(1+\alpha))|+(2+\beta)\right)|+\left|\frac{B_2}{B_1}\right|].$
Proof. The results follows by taking w(z)=z in the proof of Theorem (2.13) ■

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