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Z-Small Submodules and Z-Hollow Modules

Amina T. Hamad, Alaa A. Elewi
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

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Abstract
A submodule F of an R-module E is called small in E if whenever $F + W = E$, for some submodule W of E , implies $W = E$. In this paper , we introduce the notion of Z-small submodule , where a proper submodule F of an R-module E is said to be Z-small in E if $F + W = E$, such that $W \supseteq Z_2(E)$, then $W = E$, where $Z_2(E)$ is the second singular submodule of E . We give some properties of Z-small submodules . Moreover , by using this concept , we generalize the notions of hollow modules , supplement submodules, and supplemented modules into Z-hollow modules, Z-supplement submodules, and Z-supplemented modules. We study these concepts and provide some of their relations .
Keywords: small submodules , hollow modules ,supplement submodules ,essential ideals .

المقاسات الجزئية الصغيرة من النمط –Z و المقاسات المجوفة من النمط –Z

امنه طعمه حمد ، ألاء عباس عليوي
قسم الرياضيات ، كلية العلوم ، جامعة بغداد ، بغداد ، العراق

الخلاصة
يقال للمقاس الجزئي F من المقاس E متى ما كان W+F=E لمقاس جزئي ما W من E يؤدي الى W=E . في بحثنا هذا أعطينا تعريف المقاس الجزئي الصغير من النمط Z ، حيث المقاس الجزئي الفعلي F من المقاس E يقال له انه مقاس جزئي صغير من النمط Z في E اذا كان F+W=E و $Z_2(E) \subseteq W$ فأن W=E ، حيث $Z_2(E)$ هو المقاس الجزئي المنفرد الثاني في E . كذلك اعطينا تاعماما للمفاهيم : المقاس المجوف ، المقاسات الجزئية ، المقاس الجزئي المكمل والمقاسات المكملة الى المقاسات المجوفة من النمط Z و المقاسات الجزئية المكملة من النمط Z والمقاسات المكملة من النمط Z . تم دراسة هذه المفاهيم وقد أعطينا بعض العلاقات المتعلقة بهم.

1 . Introduction

In this paper , all rings are associative with identity and all modules are unital left R-modules, unless otherwise specified . Let R be a ring and E be a module . A proper submodule F of E is called small ($F \ll E$) , if $F + W = E$, where $W \leq E$ implies $W = E$ [1] , [2]. As a generalization of this concept , we introduce Z-small submodule , where a proper submodule F of E is said to be Z-small of E ($F \ll_z E$), if $F + W = E$ and $W \supseteq Z_2(E)$, then $W = E$, where $Z_2(E)$ is the second singular submodule of E (or Goldie torsion) defined by $Z(\frac{E}{Z(E)}) = \frac{Z_2(E)}{Z(E)}$, where $Z(E) = \{x \in E: xI = 0$, for some essential ideal I of R$\}$. In fact , $Z(E) = \{x \in E: ann(x) \leq_{ess} R\}$ where $ann(x) = \{r \in R: rx = 0\}$ [3].

In section two of this paper , we study this concept and provide some examples and basic properties of

*Email: amnatima92@gmail.com

these submodules and other related concepts .
In section three, we introduce and study the notions of Z-hollow modules and Z-supplement modules as generalizations of hollow modules and supplement modules, respectively , where a non zero R-module E is called Z-hollow module if every proper submodule is Z-small and a submodule F of an R-module E is called Z-supplement of a submodule W in E if $F + W = E$ and $F \cap W \ll_z F$. We give some examples and properties for these two concepts .

2 . Z-small submodules

In this section , we introduce the notion of Z-small submodule and give its basic properties .
Let E be an R-module. Recall that $Z(E) = \{x \in E: ann(x) \leq_{ess} R\}$ is called singular submodule of E , where $ann(x) = \{ r \in R : rx = 0\}$. If Z(E) = E , then E is called a singular module . If Z(E) = 0, then E is called a nonsingular module [3].

Definition (2.1) : Let E be an R-module and N be a submodule of E . N is said to be Z-small in E (briefly $N \ll_z E$) , if $N + B = E$, such that $B \supseteq Z_2(E)$, then $B = E$,
where $Z_2(E)$ is the second singular submodule defined by $Z(\frac{E}{Z(E)}) = \frac{Z_2(E)}{Z(E)}$ [3] .

Remarks and Examples (2.2)

1) $(0) \ll_z E$.

Proof : Suppose that $(0) + B = E$, $B \supseteq Z_2(E)$, hence $B = E$. Therefore, $(0) \ll_z E$.

2) It is clear that every small submodule is Z-small , but the converse is not true in general; for example: in the Z-module Z_6 , every proper submodule of Z_6 is Z-small since, for every proper submodule A of Z_6, there is a submodule B of Z_6 such that $A + B = Z_6$ with $B \not\supseteq Z_2(Z_6) = Z_6$ and $Z_6 \neq B$. But every proper submodule of Z_6 is not small in Z_6 .

3) In the Z-module, $N = 3Z$ is not Z-small , since if we take $B = 2Z$ as a submodule of Z, then $N + B = Z$ and $B \supseteq Z_2(Z) = 0$. But $B \neq Z$.

4) If $N \ll_z E$ and $A < N$, then $A \ll_z E$.

Proof : Assume that $A + B = E$ and $B \supseteq Z_2(E)$, hence $N + B = E$. Since $N \ll_z E$, then we have $B = E$. Therefore, $A \ll_z E$.

5) If X , Y are submodules of a module E such that $X \ll_z Y$, then $X \ll_z E$.

Proof : suppose that $X \neq 0$ and $X + B = E$, such that $B \supseteq Z_2(E)$, where $B < E$. Therefore, $(X + B = E) \cap Y$, thus $X + B \cap Y = Y$. Now , since $B \supseteq Z_2(E)$, then we have $B \cap Y \supseteq Z_2(E) \cap Y$. But $Z_2(E) \cap Y = Z_2(Y)$ [4] , hence $B \cap Y \supseteq Z_2(Y)$. Since $X \ll_z Y$, thus $B \cap Y = Y$, which implies that $Y \subset B$, so $X \subset B$. Therefore $X + B = B$. But $X + B = E$, thus $B = E$ and hence $X \ll_z E$.

6) If $K \leq L \leq E$ and $K \ll_z L$, then it is not necessarily that $L \ll_z E$, as the following example shows:
Consider the Z-module Z . If $K = 0$ and $L = 2Z$, then $K \ll_z Z$, but L is not Z-small in Z .

7) Let $A < E$ and $B < E$, where E is an R-module . Then $A \ll_z E$ and $B \ll_z E$ if and only if $A + B \ll_z E$.

Proof : (⟹) Suppose that $A \ll_z E$ and $B \ll_z E$ and suppose that $A + B + C = E$ with $C \supseteq Z_2(E)$. Thus $A + (B + C) = E$. Since $C \supseteq Z_2(E)$, then $C + B \supseteq Z_2(E)$. But $A \ll_z E$, thus $E = C + B$. Also , since $C \supseteq Z_2(E)$ and $B \ll_z E$, we have $C = E$.
(⟸) Now suppose that $A + B \ll_z E$. To prove that $A \ll_z E$, suppose that $A + C = E$ and $C \supseteq Z_2(E)$. Then $A + B + C = E$. But $A + B \ll_z E$, thus $E = C$ and hence $A \ll_z E$. Similarly, we can prove $B \ll_z E$.

8) Let A_i be proper submodule of an R-module E , i = 1 , 2 , 3 , …. , n . Then , $A_i \ll_z E$, for every i , if and only if $\sum_{i=1}^{n} A_i \ll_z E$.

Proof : It is clear by (7) .

Proposition (2.3):

Let $f : E \longrightarrow É$ be an R-homomorphism and $N \ll_z E$, then $f(N) \ll_z É$.

Proof: Let $f(N) + B = É$ with $B \supseteq Z_2(É)$. Now, for every $\in E$, $f(x) \in É$, so $f(x) = f(n) + b$, where $n \in N$ and $b \in B$. Thus $b = f(x) - f(n) = f(x - n) \in B$, then $(x - n) \in f^{-1}(B)$. Therefore $= n + (x - n) \in N + f^{-1}(B)$, hence $E = N + f^{-1}(B)$. Now , $B \supseteq Z_2(É)$ we have $f^{-1}(B) \supseteq f^{-1}(Z_2(E'))$. But $f^{-1}(Z_2(É)) \supseteq Z_2(E)$ (One can easily show this) . Therefore $f^{-1}(B) \supseteq Z_2(E)$, so $f^{-1}(B) = E$ (since $N \ll_z E$) . $ff^{-1}(B) = f(E) \cap B$, so that $f(E) = f(E) \cap B$, that is $f(E) \subseteq B$. Since $f(N) \subseteq f(E)$, then $f(N) \subseteq B$. On the other hand , we have $f(N) + B = É$, hence $B = É$ and hence

$f(\mathrm{N}) \ll_z \acute{\mathrm{E}}$.

Corollary (2.4) :

Let N , K be submodules of an R-module E , such that $\mathrm{K} \leq \mathrm{N}$ and $\mathrm{N} \ll_z \mathrm{E}$. Then $\frac{\mathrm{N}}{\mathrm{K}} \ll_z \frac{\mathrm{E}}{\mathrm{K}}$.

Proof : Let $\pi: \mathrm{E} \longrightarrow \frac{\mathrm{E}}{\mathrm{K}}$ be an R-homomorphism . Since $\mathrm{N} \ll_z \mathrm{E}$, then by proposition (2.3) , $\pi(\mathrm{N}) \ll_z \frac{\mathrm{E}}{\mathrm{K}}$, which implies that $\frac{\mathrm{N}}{\mathrm{K}} \ll_z \frac{\mathrm{E}}{\mathrm{K}}$.

Corollary (2.5) :

Let E be an R-module and $\mathrm{H} \leq \mathrm{N} \leq L \leq \mathrm{E}$, such that $\frac{L}{\mathrm{H}} \ll_z \frac{\mathrm{E}}{\mathrm{H}}$, then $\frac{L}{\mathrm{N}} \ll_z \frac{\mathrm{E}}{\mathrm{N}}$.

Proof : Let $f : \frac{\mathrm{E}}{\mathrm{H}} \longrightarrow \frac{\mathrm{E}}{\mathrm{N}}$ be a map defined by $f(x+\mathrm{H}) = x + \mathrm{N}, \forall x \in \mathrm{E}$. It is clear that f is an epimorphism . Since $\frac{\mathrm{L}}{\mathrm{H}} \ll_z \frac{\mathrm{E}}{\mathrm{H}}$, then $f(\frac{\mathrm{L}}{\mathrm{H}}) \ll_z \frac{\mathrm{E}}{\mathrm{H}}$, which implies that $\frac{L}{\mathrm{N}} \ll_z \frac{\mathrm{E}}{\mathrm{N}}$.

Remark (2.6) :

If H is a proper submodule of an R-module E and $\mathrm{K} \ll_z \mathrm{E}$, where $\mathrm{K} \leq \mathrm{H}$ and $\frac{\mathrm{H}}{\mathrm{K}} \ll_z \frac{\mathrm{E}}{\mathrm{K}}$, then it is not necessarily that $\mathrm{H} \ll_z \mathrm{E}$, as the following example shows :

Consider the Z-module Z . If $\mathrm{H} = <2>$, $\mathrm{K} = <4>$, $<4> + <2> = <2>$ with $<2> \supseteq \mathrm{Z}_2(\langle 2 \rangle)$, then $\mathrm{K} \ll_z \mathrm{H}$, $\frac{\mathrm{H}}{\mathrm{K}} = \frac{<2>}{<4>} \simeq \mathrm{Z}_2$, $\frac{\mathrm{E}}{\mathrm{K}} = \frac{\mathrm{Z}}{<4>} \simeq \mathrm{Z}_4$. One can easily show that $\frac{\mathrm{H}}{\mathrm{K}} \ll_z \frac{\mathrm{E}}{\mathrm{K}}$. But H is not Z-small of E , since $<2> + <3> = \mathrm{Z}$ and $<3> \supseteq \mathrm{Z}_2(\mathrm{Z}) = 0$, but $\mathrm{Z} \neq <3>$.

Proposition (2.7) :

Let $\mathrm{E} = \mathrm{E}_1 \oplus \mathrm{E}_2$ and $\mathrm{N} < \mathrm{E}$, such that $\mathrm{N} = \mathrm{H}_1 \oplus \mathrm{H}_2$, where $\mathrm{H}_1 \leq \mathrm{E}_1$ and $\mathrm{H}_2 \leq \mathrm{E}_2$. Then $\mathrm{N} \ll_z \mathrm{E}$ if and only if $\mathrm{H}_1 \ll_z \mathrm{E}_1$ and $\mathrm{H}_2 \ll_z \mathrm{E}_2$.

Proof : ($\Longrightarrow$) Let $\rho_1 : \mathrm{E}_1 \oplus \mathrm{E}_2 \longrightarrow \mathrm{E}_1$ be the natural projection and suppose that $\mathrm{N} \ll_z \mathrm{E}$. Then $\rho_1(\mathrm{N}) \ll_z \mathrm{E}_1$ (by proposition 2.3) . Thus, $\mathrm{H}_1 \ll_z \mathrm{E}_1$. Similarly, $\mathrm{H}_2 \ll_z \mathrm{E}_2$.

($\Longleftarrow$) Let $i_1: \mathrm{E}_1 \longrightarrow \mathrm{E}_1 \oplus \mathrm{E}_2$ be the inclusion map . Thus, $i_1(\mathrm{E}_1) \ll_z \mathrm{E}_1 \oplus \mathrm{E}_2$, which implies that $\mathrm{H}^{1 \ll_z}\mathrm{E}$, by proposition (2.3) . Also, $i_2: \mathrm{E}_2 \longrightarrow \mathrm{E}_1 \oplus \mathrm{E}_2$, $i_2(\mathrm{E}_2) = \mathrm{H}_2 \ll_z \mathrm{E}$ (by proposition 2.3) . Thus $\mathrm{H}_1 + \mathrm{H}_2 \ll_z \mathrm{E}$ (by remark and example 2.2 , 7) . Thus $\mathrm{N} \ll_z \mathrm{E}$.

Lemma (2.8) :

Let E be an R-module and $\mathrm{H} < \mathrm{N} < \mathrm{E}$. If $\mathrm{N} \ll_z \mathrm{E}$ and N is a direct summand of E , then $\mathrm{H} \ll_z \mathrm{N}$.

Proof : Since N is a direct summand of E , then $\mathrm{E} = \mathrm{N} \oplus \mathrm{L}$, for some $\mathrm{L} < \mathrm{E}$. But $\mathrm{H} \ll_z \mathrm{E}$ and $\mathrm{H} = \mathrm{H} \oplus (0)$, thus by proposition (2.7) , $\mathrm{H} \ll_z \mathrm{N}$.

Proposition (2.9) :

Let E be an R-module and $\mathrm{T} \leq \mathrm{H} \leq \mathrm{E}$, such that $\mathrm{T} \ll_z \mathrm{E}$, and H is a direct summand of E , then $\mathrm{T} \ll_z \mathrm{H}$.

Proof : Let $\mathrm{T} + \mathrm{B} = \mathrm{H}$ and $\mathrm{B} \supseteq \mathrm{Z}_2(\mathrm{H})$, where $\mathrm{B} \subseteq \mathrm{H}$. Since $\mathrm{H} \leq^{\oplus} \mathrm{E}$, then $\mathrm{H} \oplus \mathrm{C} = \mathrm{E}$. Hence, $E = (T + B) \oplus \mathrm{C} = T + (B \oplus \mathrm{C})$ and $(B \oplus \mathrm{C}) \supseteq Z_2(H) \oplus \mathrm{C}$. But $\mathrm{Z}_2(\mathrm{H}) \oplus \mathrm{C} \supseteq \mathrm{Z}_2(\mathrm{H}) \oplus \mathrm{Z}_2(\mathrm{C}) = \mathrm{Z}_2(\mathrm{E})$ [7 and 4 , proposition 2.2.13] , so $(B \oplus \mathrm{C}) \supseteq Z_2(E)$ and , since $\mathrm{T} \ll_z \mathrm{E}$, then $E = B \oplus \mathrm{C}$. But $H \oplus \mathrm{C} = E$, $B \subseteq \mathrm{H}$, therefore $\mathrm{B} = \mathrm{H}$ and $\mathrm{T} \ll_z \mathrm{H}$.

Proposition (2.10) :

Let A be a singular submodule of an R-module E . Then A is a Z-small in E .

Proof : Let $\mathrm{A} + \mathrm{B} = \mathrm{E}$ and $\mathrm{B} \supseteq \mathrm{Z}_2(\mathrm{E})$. Since A is singular , then $\mathrm{A} = \mathrm{Z}_2(\mathrm{A})$. But $\mathrm{Z}_2(\mathrm{A}) = \mathrm{Z}_2(\mathrm{E}) \cap \mathrm{A}$, so $\mathrm{A} \subseteq \mathrm{Z}_2(\mathrm{E})$. Hence $\mathrm{A} \subseteq \mathrm{B}$. Then $\mathrm{B} = \mathrm{E}$, therefore $\mathrm{A} \ll_z \mathrm{E}$.

We need the following definition for the following proposition .

Recall that " an R-module E is said to be prime if $\mathrm{ann}(x) = \mathrm{ann}(\mathrm{y})$, for every non-zero element x , y in E " [5] .

Definition (2.11) : [6]

A submodule H of E is said to be t-essential in E (denoted by $\mathrm{H} \leq_{\mathrm{tes}} \mathrm{E}$) , if for every submodule L of E , $\mathrm{H} \cap \mathrm{L} \leq \mathrm{Z}_2(\mathrm{E})$ implies that $\mathrm{L} \leq \mathrm{Z}_2(\mathrm{E})$.

Remark (2.12) : [7]

" $\mathrm{Z}_2(\mathrm{E}) = \{x \in \mathrm{E}: \mathrm{ann}_{\mathrm{R}} \leq_{\mathrm{tes}} \mathrm{R}\}$, where $\mathrm{ann}_{\mathrm{R}}(x) = \{\mathrm{r} \in \mathrm{R}: \mathrm{r}x = 0\}$ " .

Proposition (2.13) :

If E is prime and $\mathrm{Z}_2(\mathrm{E}) \neq 0$, then $\mathrm{A} \ll_z \mathrm{E}$, for every proper submodule A of E .

Proof : Let $\mathrm{A} + \mathrm{B} = \mathrm{E}$ and $\mathrm{B} \supseteq \mathrm{Z}_2(\mathrm{E})$. Since $\mathrm{Z}_2(\mathrm{E}) \neq 0$, then there exists $x \in \mathrm{Z}_2(\mathrm{E})$, hence $\mathrm{ann}(x) \leq_{\mathrm{tes}} \mathrm{E}$. But E is prime , so for every $a \in \mathrm{A}$ and $\mathrm{a} \in \mathrm{Z}_2(\mathrm{E})$, $\mathrm{ann}(a) = \mathrm{ann}(x)$, hence

$\text{ann}(a) \leq_{\text{tes}} E$ and $a \in Z_2(E)$. This implies that $A \subseteq Z_2(E)$, thus $A + B \subseteq Z_2(E) + B$. But $E = A + B$ and $B \supseteq Z_2(E)$, so that $B = E$. Hence $A \ll_z E$.

Example (2.14) :

The last Proposition is not true for small submodules; for example : let $E = Z_2 \oplus Z_2$ as Z-module is prime and $Z_2(E) \neq 0$. All proper submodules of E are Z-small, but note that $N = Z_2 \oplus (0)$ is not small

In the following proposition, we show that the two concepts of small submodules and Z-small submodules are equivalent if the module is nonsingular.

Proposition (2.15) :

Let E be a nonsingular module and $H < E$, then $H \ll E$ if and only if H is Z-small submodule of E .

Proof : ($\Longrightarrow$) It is clear by remarks and examples (2.2 , 2) .

($\Longleftarrow$) Let $H + L = E$, $L \leq E$, $L \gneqq (0)$. Since E is nonsingular , then $Z_2(E) = 0$, hence $L > Z_2(E)$. But $H \ll_z E$, so that L = E , therefore $H \ll E$.

Note (2.16) :

If $f: E \longrightarrow \acute{E}$ is epimorphism , then it is not necessarily that $f^{-1}(Z_2(\acute{E})) = Z_2(E)$; for example : Let $\pi: Z \longrightarrow \frac{Z}{\langle 2 \rangle} \simeq Z_2$, where Z and Z_2 are Z-modules and π is epimorphism . Notice that $Z_2(Z) = 0$ and $Z_2(Z_2) = Z_2$, but $\pi^{-1}(Z_2) = Z \neq Z_2(Z) = 0$.

Hence , in general, if we have $f: E \longrightarrow \acute{E}$ being an epimorphism, such that E is nonsingular module, and É is singular module, then $f^{-1}(Z_2(\acute{E})) \neq Z_2(E)$.

Recall that " a module E is a multiplication, if for every submodule H of E, $H = (H: E)E$, where $(H: E) = \{r \in R: rE \subseteq H\}$ " [8].

Proposition (2.17) : [9 , p 18]

Let E be a finitely generated faithful multiplication module over a commutative ring R , and I , J be ideals of R, then

1) $E\, Z_2(R) = Z_2(E)$.
2) If $I \leq_{\text{tes}} R$, then $E\, I \leq_{\text{tes}} E$.
3) If $K \leq_{\text{tes}} E$ and $K = E\, I$, then $I \leq_{\text{tes}} R$.
4) If $I \leq_{\text{tes}} J$, then $E\, I \leq_{\text{tes}} E\, J$, and the converse is hold if R is regular .

Proposition (2.18) :

Let E be a finitely generated faithful multiplication R-module and $H \leq E$. Then $H \ll_z E$ if and only if $(H: E) \ll_z R$.

Proof : ($\Longrightarrow$) Suppose that $H \ll_z E$. To prove that $(H: E) \ll_z R$, suppose that $(H: E) + I = R$, where I is an ideal of R and $I \supseteq Z_2(R)$. Then $(H: E)E + IE = E$ and hence $H + IE = E$. Now , since $Z_2(R) \subseteq I$, then $Z_2(R)E \subseteq IE$. But by proposition (2.17) we have $Z_2(R)E = Z_2(E)$, so $Z_2(E) \subseteq IE$. Since $H \ll_z E$, then $E = IE$ and hence $R = I$, [8] . Thus $(H : E) \ll_z R$.

($\Longleftarrow$) suppose that $H + K = E$ and $K \supseteq Z_2(E)$. We want to prove that $K = E$. Since E is multiplication, then $H = (H: E)E$ and $K = (K: E)E$. Thus, $(H: E)E + (K: E)E = E$. Also, since E is finitely generated faithful multiplication module , then we have $(H: E) + (K: E) = R$ [8] and $Z_2(E) = Z_2(R)E$, by proposition (2.17). Thus, $Z_2(R)E \subseteq K$, which implies that $Z_2(R)E \subseteq (K: E)E$. Therefore $Z_2(R) \subseteq (K: E)$ and since $(H: E) \ll_z R$, then $R = (K: E)$. Hence $E = (K: E)E = K$ and therefore $H \ll_z E$.

Remark (2.19) :

The condition that E is faithful cannot be dropped from the part ($\Longrightarrow$) of proposition (2.18) , as the following example shows : If E is the Z-module Z_{12} and $H = \langle \bar{4} \rangle$, then $H \ll_z E$, but $(H: E) = 4Z$ is not Z-small in Z .

3 . Z-Hollow modules

In this section, we introduce the Z-hollow module as a generalization of hollow module and study some of its basic properties .

Recall that an R-module $E \neq 0$ is called hollow module if every proper submodule of E is small in E [1] .

Now , we define the Z-hollow modules.

Definition (3.1) :

An R-module $E \neq 0$ is called Z-hollow if every proper submodule of E is Z-small.

Remarks and Examples (3.2) :

1) Z_6 as Z-module is Z-hollow since every proper submodule of Z_6 is Z-small .

Z as Z-module is not Z-hollow since 3Z is not Z-small in Z , as we show in remarks and examples (2.2, 3).

2) It is clear that every hollow module is Z-hollow , but the convers is not true; for example the Z-module Z_6 is Z-hollow but not hollow .

Proposition (3.3) :

The epimorphic image of Z-hollow module is Z-hollow module .

Proof : Let E be a Z-hollow module , É be a module , and f : E ⟶ É be an epimorphism. Suppose that Ḣ is a proper submodule of É, such that $H' + K' = E'$ and $K' \supseteq Z_2(E')$. Since f is an epimorphism, then $f(Z_2(E)) \subseteq Z_2(É)$. Now, notice that f^{-1} (Ḣ) < E because if f^{-1} (Ḣ) = E , then $f(f^{-1}(Ḣ)) = f(E) = É$ and hence Ḣ = E′, which is a contradiction . Thus f^{-1} (Ḣ) < E . Also, $Z_2(E) \subseteq f^{-1}(f(Z_2(E))) \subseteq f^{-1}(Z_2(É)) \subseteq f^{-1}$ (Ḱ) . Hence $f^{-1}(K') \supseteq Z_2(E)$. Now , since E is Z-hollow, thus $f^{-1}(H) \ll_z E$ and we get E = f^{-1} (Ḱ). Hence $f(f^{-1}(Ḱ)) = f(E) = É$, since f is epimorphism $f(f^{-1}(Ḱ))$ =Ḱ . Thus Ḱ = É and hence É is Z-hollow module .

Corollary (3.4) :

Let E be an R-module . If E is Z-hollow module , then $\frac{E}{H}$ is Z-hollow for every proper submodule H of E .

Corollary (3.5) :

A direct summand of a Z-hollow module is Z-hollow module .

Proof : Let E be a Z-hollow R-module and H be a direct summand of E . Hence $E = H \oplus K$, for some submodule K of E . Then by the second isomorphism theorem $\frac{E}{K} \simeq H$. By corollary (3.4), H is Z-hollow .

Proposition (3.6) :

Let E be a finitely generated faithful multiplication R-module . Then E is Z-hollow if and only if R is Z-hollow .

Proof: It follows by proposition (2.18) .

Proposition (3.7) :

Let E be an R-module $E \neq 0$. Then E is Z-hollow module if and only if there exists $H \ll_z E$ and $\frac{E}{H}$ is Z-hollow .

Proof : (⟹) It follows directly by taking H = 0 .

(⟸) To prove that E is Z-hollow , let $A < E$ and assume that $A + B = E$ with $B \supseteq Z_2(E)$. We must prove that $B = E$. Now, $\frac{E}{H} = \frac{A+H}{H} + \frac{B+H}{H}$, but $B + H \neq E$ (since $H \ll_z E$) and so $\frac{B+H}{H} \neq \frac{E}{H}$. Then $\frac{B+H}{H} \supseteq \frac{Z_2(E)+H}{H}$. But $\frac{Z_2(E)+H}{H} \supseteq Z_2(\frac{E}{H})$. To show that let $x + H \in Z_2(\frac{E}{H})$, so $ann(x + H) \leq_{tes} R$. But $x + (0) \subseteq x + H$, hence $ann(x + (0)) \supseteq ann(x + H)$[10] . Therefore $ann(x + (0)) = ann(x) \leq_{tes} R$, and hence $\in Z_2(E)$. Thus $+H \in \frac{Z_2(E)+H}{H}$, then $\frac{B+H}{H} \supseteq Z_2(\frac{E}{H})$, but $\frac{E}{H}$ is Z-hollow , so $\frac{B+H}{H} = \frac{E}{H}$. Hence B + H = E , but $H \ll_z E$, so B = E . Therefore E is Z-hollow module .

Recall that " a submodule H of E is called fully invariant if for each endomorphism from E to E , f (H) ⊆ H " , [3 , p.4] .

" An R-module E is called duo if every submodule of E is fully invariant " [11].

Proposition (3.8) :

Let E_1 and E_2 be R-modules , $E = E_1 \oplus E_2$, such that E is a duo module . Then E is Z-hollow if and only if E_1 and E_2 are Z-hollow modules , provided that $H \cap E_i \neq E_i$ for each i = 1,2,H < E .

Proof : (⟹) It follows directly by (3.5) .

(⟸) Let $H < E$. Since H is fully invariant , then $H = (H \cap E_1) \oplus (H \cap E_2)$ by [11 , lemma 3.1] . Now, $(H \cap E_1)$ and $(H \cap E_2)$ are proper submodules of E_1 and E_2 respectively . But E_1 and E_2 are Z-hollow modules , thus $H \cap E_1 \ll_z E_1$ and $H \cap E_2 \ll_z E_2$. Then by proposition (2.7), $H = (H \cap E_1) \oplus (H \cap E_2) \ll_z E$. Thus E is a Z-hollow module .

Recall that " an R-module E is called distributive if for all $H, K, L \leq E$, $H \cap (L + K) = (H \cap L) + (H \cap K)$ " [12] .

Proposition (3.9) :

Let $E = E_1 \oplus E_2$ be a distributive R-module such that $E_1 , E_2 \leq E$. Then E is a Z-hollow if and only if

E_1 and E_2 are Z-hollow, provided that for each $H < E$, $H \cap E_i \neq E_i$, $\forall$ i=1 , 2.

Proof : ($\Longrightarrow$) It follows by corollary (3.5) .

($\Longleftarrow$) Let $H < E$. Since E is distributive , then $H = (H \cap E_1) \oplus (H \cap E_2)$ and then by the same proof of proposition (3.8), $H \ll_z E$. Thus E is Z-hollow .

Recall that a submodule K of an R-module E is called supplement of V if $E = K + V$ and V is a minimal element in the set of submodules H, where $H \leq E$ with $V + H = E$. Equivalently , a submodule K of E is called supplement of V if $K + V = E$ and $K \cap V \ll K$ [13, 14].

An R-module E is called supplemented if every submodule of E is supplement .

Let E be an R-module , then E is called amply supplemented module if, for any two submodules H and F of E with $H + F = E$, F has asupplement of H in E .

Definition (3.10) :

A submodule K of an R-module E is called Z-supplement of V If $K + V = E$ and $K \cap V \ll_z K$.

An R-module E is called Z-supplemented if every submodule of E is Z-supplement .

Let E be an R-module , then E is called amply Z-supplemented module if, for any two submodule H and F of E with $H + F = E$, F has a Z-supplement of H in E .

It is clear that every supplement submodule is Z-supplement , but the converse is not true; for example : $K =< 2 >$, $V =< 3 >$ in the Z-module Z_{12} . $K + V = Z_{12}$, $K \cap V =< 6 >$ and is Z-small in K , but it is not small in K .

An R-module E is called Z-lifting if for any submodule N of E , there exist submodules K , H of E such that $E = K \oplus H$ with $K \leq N$ and $N \cap H \ll_z N$ [15] .

Proposition (3.11) :

Every Z-hollow module is amply Z-supplemented .

Proof : Let E be a Z-hollow , then E is Z-lifting [15] . Also, since every Z-lifting is Z-amply supplemented , thus E is Z-amply supplemented.

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