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Stability for the Systems of Ordinary Differential Equations with Caputo Fractional Order Derivatives

Asad J. Taher [1] * , Fadhel S. Fadhel [2],Nabaa N. Hasan[1]
[1]*Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq*
[2]*Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Baghdad, Iraq*

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Abstract
Fractional calculus has **paid** much attention in recent years, because it plays an essential role in many fields of science and engineering, where the study of stability theory of fractional differential equations emerges to be very important. In this paper, the stability of fractional order ordinary differential equations will be studied and introduced the backstepping method. The Lyapunov function is easily found by this **method**. **This** method also gives a guarantee of stable solutions for the fractional order differential equations. **Furthermore it gives** asymptotically stable.

Keywords: Backstepping Method, Caputo Fractional Derivative, Fractional Differential Equations, Stability, Lyapunov Function.

الاستقرارية لنظام المعادلات التفاضلية الاعتيادية مع مشتقات كابوتو ذات الرتب الكسورية

اسعد جاسم طاهر [1*] , فاضل صبحي فاضل [2] , نبأ نجدي حسن [3]
[1]قسم الرياضيات, كلية العلوم , الجامعة المستنصرية , بغداد , العراق.
[2] قسم الرياضيات وعلوم الحاسوب , كلية العلوم , بغداد, العراق

الخلاصه
ان الحسابات الكسورية اخذت اهتمام كبير في السنوات الأخيرة، لأنها تلعب دورًا أساسيًا في العديد من مجالات العلوم وخاصة الهندسة، حيث تبرز أهمية دراسة نظرية الاستقرار في المعادلات التفاضلية الكسورية. في هذا البحث، سيتم دراسة استقرار المعادلات التفاضلية العادية ذات الرتبة الكسورية حيث تم تقديم طريقة الخطوة الراجعة.ومن خلال هذه الطريقة وجدنا بسهولة دالة ليابانوف. ان هذه الطريقة تعطي ضمانًا للحلول المستقرة للمعادلات التفاضلية ذات الرتبة الكسورية اضافة الى ذلك هي تعطي حلولا مستقرّة بشكل محاذي.

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1. Introduction:

The beginning of fractional calculus was in 1695 with classical calculus together, but it is widely developed in the twentieth century since the life became more complected and the researcher found that the fractional order differential equations are more accurate for

* Email: asadjassim1976@gmail.com

representing and modeling real life problems as in biology, bioengineering, astrum, physics, and engineering, etc. [1–3]. That reason made the fractional calculus very important subject of applied mathematics [4]. Stability analysis for all solutions of the fractional order ordinary differential equations (FODEs) are more complicated than the study of stability of ordinary differential equations (ODEs), because fractional derivatives are nonlocal and have weakly singular kernels [5–7]. Sstability is also an equivalent concept to the uniformly continuity for the solutions of the function of system with initial conditions on all point in neighborhood of the equilibrium point in time [8].
It is known that the dynamical system can be stable where the system is permitted to execute persistent small oscillations about the state of motion, or about the system equilibrium. There are many approaches in which this concept can be used to investigate stability of ODEs, first of them is to use the eigenvalues. During the second method (approach) was enshrined at the end of the 19^{th} century by Lyapunov which is called Lyapunov method, it is effectively applied to wholly new problems. The second method is also called the direct method [9]. This method can be applied straightforwardly to differential condition with no information about the solution. The idea beyond this technique is to create a scalar function say V which satisfies all the given conditions to check the stability of the system of ODEs [10].
The use of the trajectories to prove the asymptotic stability is not all time possible due to the complexity of the FODEs. For this reason we intend in this paper to find the Lyapunov characterization function for the asymptotic stability for the solution of the FODEs which is defined by adaptive backstepping method.
Matignon in 1994 was the first researcher who introduced in his Ph.D. thesis some stability results that relate to a restrictive modelling of FDEs. [6 , 11]. There are many important results that relate to linear system of FODEs with Caputo fractional derivative of order α, where $0 < \alpha \leq 1$, such as Qian et al. in 2012 [12], they investigated the linear FODEs with Riemann-Liouville fractional order derivative. After that, many researchers have been investigated the stability of nonlinear and linear FODEs with fractional order derivative α between 0 and 1, for more details see [13],
The backstepping technique methods are to be a good method to find controllers design for a large class of the nonlinear systems, which often used to nonlinear control technique to stabilize the system of ordinary differential equations which depends on the idea of the definition of a set of intermediate variables and the few steps are exactly given negative of Lyapunov functions derivative that leads to build a common control Lyapunov function for the system. Because of this nature, the backstepping technique is easy applied method to different classes of systems, that contain many engineering systems, physics, and bioengineering, etc. There are few studies for using the backstepping methods to find the controller function for dynamic systems of fractional order differential equations, we refer to these references [14–17].
In this paper, a backstepping stabilization method will modify and improve in order to be applicable for system of Caputo FODEs ,and to find solutions which are asymptotically stable. Some of theorems to connected between the concepts of ordinary differential equations, fractional order differential equations and Lyapunov functions.

2. Preliminaries:

In some real-life problems, the issues of stability, controlling, and solving certain systems are of great interest. Therefore, some basic concept seems to be necessary to understand are given in this section, we will also present some of them which are necessary for the rest of this paper.

2.1. Fractional calculus:

We start with the most elementary and most useful definitions in fractional calculus which is the Caputo derivative.

Definition 1: [5 , 18] The Caputo left–handed and the right- handed fractional derivatives of order $\alpha \in R^{+}, \mathrm{n}-1<\alpha \leq \mathrm{n}, \mathrm{n} \in N$ of a function f are defined as follows:

$$ {}_{a}^{C}D_{x}^{\alpha} f(x)=\frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{1}{(x-y)^{\alpha-n}} f^{(n)}(y) d y \quad (1)$$

and

$$ {}_{x}^{C}D_{b}^{\alpha} f(x)=\frac{(-1)^{\alpha}}{\Gamma(n-\alpha)} \int_{x}^{b} \frac{1}{(y-x)^{\alpha-n}} f^{(n)}(y) d y \quad (2)$$

The statement of the problem is established as a system of FODEs which has to be stabilized and solved. In the next section we use the backstepping method that takes the form of Caputo fractional order derivative of nonautonomous system

$$ {}_{t_0}^{C}D_{x}^{\alpha} x(t)=f(t, x(t)), x(t_0)=x_0 \quad (3)$$

where $\alpha \in (0,1]$, $f:[t_0, \infty] \times \Omega \rightarrow R^{n}$ is a piecewise continuous function in t and locally Lipschitz in x and $\Omega \in R^{n}$ is a certain domain that contains the origin. The constant x_0 is an equilibrium point of the dynamic system (3), if and only if $f(t, x_0)=0$.

The equilibrium state not always be single [19]. for example, the position of the two balls is in the equilibrium position, and the two points are all the equilibrium points of the system, where one of which governs the drift of the system's equilibrium position and is identifies as the dynamic relaxation–retardation operations shows in the system, [18–22]

2.2. Lyapunov Function:

Among the basic tools in these backstepping method is based on constructing the Lyapunov function, which must guarantee the asymptotic stability of this system. Lyapunov A.M. in 1890 [25] considered the stability of a model of the dynamical systems described by nonlinear ODEs. Lyapunov functions, if there exist, and when exist not unique, are scalar functions used to prove and establish stability of solutions or an equilibrium point of a system of ODEs. Recall that a continuous function $V(x)>0$ for all $x \neq 0$ is called positive definite, however if $V(x)<0, \forall x \neq 0$, then V is called negative definite. The stability of any solution is given in the next theorem.

Theorem1 : [23–26] Consider $V(x)$ be a scalar continuous real valued function of the state variables $x_1, x_2, \ldots, x_n$ and if:

i. $V(x)$ is positive definite i $that\ means\ V(x)>0, x \in \Omega$, and $\dot{V}(x)$ is negative semi-definite i.e. $\dot{V}(x) \leq 0, x \in \Omega$ on some region Ω containing the origin, then the zero solution is stable.

ii. $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite i.e. $(V(x)<0, x \in \Omega)$ on some region Ω containing the origin, then the zero solution is asymptotically stable.

iii. $V(x)$ is positive definite and $\dot{V}(x)$ is positive definite on some region Ω, then the zero solution is unstable.

We will give in the following some theorems that we need to link differential equations with fractional orders with some concepts and methods of solutions that are used in ordinary differential equations, which will allow to use many theorems that are related to ordinary differential equations.

Theorem2:[15] Let the Lyapunov function which is given by $V=\frac{1}{2} z^{2}$, where the variable of interest is z. If $z z^{(\beta)}<0$, where $0<\beta \leq 1$ is ensured, then $z \dot{z}<0$ is satisfied.

The mean result for the previous theorem that it makes relations between ordinary and fractional differential equations. Next theorem it gives the connection between Lyapunov function and Caputo fractional order derivative.

Theorem3: [26] Let $V(t) \in R$ is a derivable function. Then

$$\frac{1}{2}\, {}_{0}^{C}D_{t}^{\beta} V^{2}(t) \leq V(t)\, {}_{0}^{C}D_{t}^{\beta} V(t) \text{ where } 0<\alpha<1 \quad (4)$$

for any instant $t>0$.

Theorem4:

Suppose that the Caputo fractional order equation

$$ {}_0^C D_t^{\beta} x(t) = f\big(x(t)\big) \qquad (5)$$

where $0 < \beta \leq 1,\ x(t) \in R^n$ and x= 0 is an equilibrium point. Then

a) If $x(t){}_0^C D_t^{\beta} x(t) \leq 0$, then the equilibrium point is stable.

b) If $x(t){}_0^C D_t^{\beta} x(t) < 0$, then the equilibrium point is asymptotically stable.

Proof:

If we suppose the positive definite Lyapunov function $V\big(x(t)\big) = \frac{1}{2}x^2(t)$, then by using theorem (4) we get the result

$$\frac{1}{2}{}_0^C D_t^{\beta} V^2\big(x(t)\big) = \frac{1}{8}{}_0^C D_t^{\beta} x^4(t) = \frac{\Gamma(5)}{8\Gamma(5-\beta)} x^{4-\beta} \leq x(t)\frac{\Gamma(2)}{\Gamma(2-\beta)} x^{1-\beta}(t) = x(t){}_0^C D_t^{\beta} x(t) \qquad (6)$$

in case that $x(t){}_0^C D_t^{\beta} x(t) \leq 0$ that means equation(6) is negative semi-definite since $V\big(x(t)\big) = \frac{1}{2}x^2(t) \leq \frac{1}{2}x^2(0)\ \forall x$, then the origin point is stable. Another case when $x(t){}_0^C D_t^{\beta} x(t) < 0$ and the equation (6) is negative definite, which implies that $\mathrm{V}\big(\mathrm{x(t)}\big) < \mathrm{V}\big(\mathrm{x(0)}\big)$ that gives as the origin point asymptotically stable.

3. The Backstepping Method:

To apply the backstepping method for system FODEs with control functions $u_1, u_2, \dots, u_n$ of the form:

$$\left.\begin{aligned}
{}_0^C\boldsymbol{D}_t^{\boldsymbol{\beta_1}}\boldsymbol{x_1} &= \boldsymbol{f_1(x_1, x_2, \dots, x_n) + u_1} \\
{}_0^C\boldsymbol{D}_t^{\boldsymbol{\beta_2}}\boldsymbol{x_2} &= \boldsymbol{f_2(x_1, x_2, \dots, x_n) + u_2} \\
{}_0^C\boldsymbol{D}_t^{\boldsymbol{\beta_3}}\boldsymbol{x_3} &= \boldsymbol{f_3(x_1, x_2, \dots, x_n) + u_3} \\
&\vdots \\
{}_0^C\boldsymbol{D}_t^{\boldsymbol{\beta_n}}\boldsymbol{x_n} &= \boldsymbol{f_n(x_1, x_2, \dots, x_n) + u_n}
\end{aligned}\right\} \qquad (7)$$

where $x(t) \epsilon R^n$ is the state vector of the system, $\beta_i \in (0{,}1]$, $f_i, i = 1,2, \dots, n$ are continuous functions and $u_i, is = 1,2, \dots, n$ are the controller input functions, which will be introduced for the purpose of stabilizing the original system of FODEs asymptotically as well as to find its solution [27–29].

Now the objective is to apply the backstepping method in order to design a state feedback control function, which asymptotically stabilizes the origin. The design procedure may divide for simplify into steps. It is given in the next theorem.

Theorem 5: Consider the system of FODEs (7) with state variable $x \in R^n$ and controller function $u_i: [0{,}1] \to R, i = 1,2, \dots, n$. If the Lyapunov functions of the subsystems of the FODEs system (7) are supposed to be:

$$\boldsymbol{V_1(z_1) = {z_1}^T p_1 z_1\ ,\ p_1 \in R^+, z_1(t) \in R}$$

$$\boldsymbol{V_i(z_1, z_2, \dots, z_i) = V_{i-1}(z_1, z_2, \dots, z_{i-1}) + {z_i}^T p_1 z_i} \qquad (8)$$

$$\boldsymbol{p_i \in R^+, z_i(t) \in R\ , i = 2, 3, \dots, n}$$

Then there exists nonlinear controller function $u_1, u_2, \dots, u_n$which make system (7) asymptotically stable and solvable, where

$\mathrm{\alpha_i(z_1, z_2, \dots, z_i) = f_i(z_1, z_2, \dots, z_i, x_{i+1}, \dots, x_n) + u_i(t) - z_i}$

Proof:

It is followed in the backstepping method, the proof will be breakdown into steps, for simplicity and comparison purpose. The outline of the proof will be braked into steps as follows:

Step (1): First, consider the stability of the first equation of system (7), namely:

$${}_0^C\boldsymbol{D}_t^{\boldsymbol{\beta_1}}\boldsymbol{x_1} = \boldsymbol{f_1(x_1, x_2, \dots, x_n) + u_1} \qquad (9)$$

where x_2 represents a virtual controller which is introduced for stabilizability, and we define $x_1 = z_1$then we derive this transform both sides with fractional order β_1 by using Caputo derivative with respect to time t, and we get:

$$ {}_0^C D_t^{\beta_1} z_1 = {}_0^C D_t^{\beta_1} x_1 = f_1(z_1, x_2, \dots, x_n) + u_1 \quad (10) $$

Now, construct the first Lyapunov function in quadratic form as:

$$ V_1(z_1) = \frac{1}{2} z_1^2 = z_1^T p_1 z_1 \qquad p_1 \epsilon R^+ \quad (11) $$

The derivative with respect to time t is:

$$ \dot{V}_1(z_1) = z_1 \dot{z_1} = -z_1^T Q_1 z_1 < 0 \quad (12) $$

where Q_1 is a positive definite matrix. But we need $z_1\dot{z_1} < 0$ to get $\dot{V}_1(z_1)$ is a negative definite function in R^n. To prove that let

$$ z_1 z_1^{(\beta_1)} = z_1 {}_0^C D_t^{\beta_1} z_1 = z_1(f_1(z_1, x_2, \dots, x_n) + u_1) \quad (13) $$

So that we can choose suitable $u_1$like $u_1 = -f_1(z_1, x_2, \dots, x_n) - k_1 z_1$, if we substitute the *u1*in **z1z1≤z1z1β1** (14) , we get

$z_1 z_1^{(\beta_1)} = -k_1 \ z_1^2$ where $k_1 \in R$, $k_1 > 0$

that means r $z_1 z_1^{(\beta_1)} < 0$ everywhere.

and by using theorem (2), we know that:

$$ z_1 \dot{z_1} \le z_1 z_1^{(\beta_1)} \quad (14) $$

by using theorems (3) and (4), and by Lyapunov stability theory, then the system (9) will be asymptotically stable. It is clear that if we take the virtual control $x_2 = \alpha_1(z_1)$ and the state feedback input function u_1 will render the system (9) asymptotically stable. The function $\alpha_1(z_1)$ ought to be assessed while z_2 is regarded as a controller.

Step (2): To stabilize the second equation of system (7) we define the error variable between x_2 and $\alpha_1(z_1)$ as follows:

$$ z_2 = x_2 - \alpha_1(z_1) \quad (15) $$

hence the time derivative of the error dynamics of subsystem (15) is given by:

$$ {}_0^C D_t^{\beta_1} z_1 = f_1(z_1, x_2, x_3 \dots, x_n) + u_1 $$

$$ {}_0^C D_t^{\beta_2} z_2 = f_1(z_1, z_2 + \alpha_1(z_1), x_3 \dots, x_n) - {}_0^C D_t^{\beta_2} \alpha \ (z_1) + u_2 \quad (16) $$

where x_3 is a virtual controller of subsystem (16), which is chosen to stabilize this subsystem and consider that it is equal to $\alpha_1(z_1, z_2)$, which makes the subsystem (16) asymptotically stable.

In order to find the second Lyapunov function V_2 , that stabilizes asymptotically equation (16), we suppose that V_2 as follows :

$$ V_2(z_1, z_2) = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 = V_1 + z_2^T p_2 z_2 \, , p_2 \epsilon R^+ \quad (17) $$

then the time derivative of V_2 is:

$$ \dot{V}_2(z_1, z_2) = z_1 \dot{z}_1 + z_2 \dot{z}_2 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 < 0 \quad (18) $$

where $Q_1$and Q_2 are positive definite matrices. In order to prove that $z_1\dot{z_1} + z_2\dot{z_2} < 0$ let

$$ z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)} = z_1 {}_0^C D_t^{\beta_1} z_1 + z_2 {}_0^C D_t^{\beta_2} z_2 = -k_1 z_1^2 + z_2(f_2(x_1, x_2, \dots, x_n) + u_2) \quad (19) $$

If we choose $u_2 = -f_2(x_1, x_2, \dots, x_n) - k_2 z_2$ and substituted in equation (23) we get

$z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)} = -k_1 z_1^2 - k_2 z_2^2$where $k_2 \in R$, $k_1 > 0$

that means $z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)} < 0$ everywhere.

And by using theorem (2), we know that:

$$ z_1 \dot{z_1} + z_2 \dot{z_2} < z_1 z_1^{(\beta_1)} + z_2 z_2^{(\beta_2)} < 0 \quad (20) $$

by theorems (3) and (4), we get $\dot{V}_2(z_1, z_2)$ is a negative definite function in R^n and by using Lyapunov stability theory we have subsystem (16) is asymptotically stable. Similarly, if the virtual control is taken to be $x_3 = \alpha_2(z_1, z_2)$ and the state feedback input u_2 is evaluated to make subsystem (16) asymptotically stable.

Step (n): Continuing in the same approach that is given in the above steps, we arrive at the n^th step by defining the error variable z_n as:

$$z_n = x_n - \alpha_{n-1}(z_1, z_2, \dots, z_n) \quad \textbf{(21)}$$

and suppose that $z_1, z_2, \dots, z_n$ state variables of the subsystem is given by:

$$\left.\begin{aligned}
&{}_0^C\boldsymbol{D}_t^{\beta_1}\boldsymbol{z_1} = \boldsymbol{f_1}(\boldsymbol{z_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \dots, \boldsymbol{x_n}) + \boldsymbol{u_1}\\
&{}_0^C\boldsymbol{D}_t^{\beta_2}\boldsymbol{z_2} = \boldsymbol{f_1}(\boldsymbol{z_1}, \boldsymbol{z_2}+\boldsymbol{\alpha_1}(\boldsymbol{z_1}), \boldsymbol{x_3} \dots, \boldsymbol{x_n}) - {}_0^C\boldsymbol{D}_t^{\beta_2}\boldsymbol{\alpha_1}(\boldsymbol{z_1}) + \boldsymbol{u_2}\\
&{}_0^C\boldsymbol{D}_t^{\beta_3}\boldsymbol{z_3} = \boldsymbol{f_3}(\boldsymbol{z_1}, \boldsymbol{z_2}+\boldsymbol{\alpha_1}(\boldsymbol{z_1}), \boldsymbol{z_3}+\boldsymbol{\alpha_2}(\boldsymbol{z_1}, \boldsymbol{z_2}), \dots, \boldsymbol{x_n}) - {}_0^C\boldsymbol{D}_t^{\beta_3}\boldsymbol{\alpha_2}(\boldsymbol{z_1}, \boldsymbol{z_2}) + \boldsymbol{u_3}\\
&\vdots\\
&{}_0^C\boldsymbol{D}_t^{\beta_n}\boldsymbol{z_n} = \boldsymbol{f_n}\big(\boldsymbol{z_1}, \boldsymbol{z_2}+\boldsymbol{\alpha_1}(\boldsymbol{z_1}), \dots, \boldsymbol{z_n}+\boldsymbol{\alpha_{n-1}}(\boldsymbol{z_1}, \boldsymbol{z_2}, \dots, \boldsymbol{z_{n-1}})\big) -\\
&\qquad {}_0^C\boldsymbol{D}_t^{\beta_n}\boldsymbol{\alpha_{n-1}}(\boldsymbol{z_1}, \boldsymbol{z_2}, \dots, \boldsymbol{z_n}) + \boldsymbol{u_n}
\end{aligned}\right\} \quad (22)$$

Therefore, the n^th Lyapunov function is defined as:

$$\boldsymbol{V_n}(\boldsymbol{z_1}, \boldsymbol{z_2}, \dots, \boldsymbol{z_n}) = \tfrac{1}{2}\boldsymbol{z_1^2} + \tfrac{1}{2}\boldsymbol{z_2^2} + \dots + \tfrac{1}{2}\boldsymbol{z_{n-1}^2} + \tfrac{1}{2}\boldsymbol{z_n^2} = \boldsymbol{V_{n-1}}(\boldsymbol{z_1}, \boldsymbol{z_2}, \dots, \boldsymbol{z_{n-1}}) + \boldsymbol{z_n}^T\boldsymbol{p_n}\boldsymbol{z_n} \qquad, \boldsymbol{p_n} \epsilon \boldsymbol{R^+} \quad (23)$$

Also, we derive V_n with respect to time t, this gives:

$$\begin{aligned}\dot{\boldsymbol{V}}_{\boldsymbol{n}}(\boldsymbol{z_1}, \boldsymbol{z_2}, \dots, \boldsymbol{z_n}) &= \boldsymbol{z_1}\dot{\boldsymbol{z_1}} + \boldsymbol{z_2}\dot{\boldsymbol{z_2}} + \dots + \boldsymbol{z_n}\dot{\boldsymbol{z_n}}\\ &= -\boldsymbol{z_1}^T\boldsymbol{Q_1}\boldsymbol{z_1} - \boldsymbol{z_2}^T\boldsymbol{Q_2}\boldsymbol{z_2} - \dots - \boldsymbol{z_n}^T\boldsymbol{Q_n}\boldsymbol{z_n} < \boldsymbol{0}\end{aligned} \quad (24)$$

where $Q_1, Q_2, Q_3, \dots, Q_n$ be a positive definite matrix, we will also prove that $z_1\dot{z}_1 + z_2\dot{z}_2 + \dots + z_n\dot{z}_n < 0$.

let

$$\boldsymbol{z_1}\boldsymbol{z_1^{(\beta_1)}} + \boldsymbol{z_2}\boldsymbol{z_2^{(\beta_2)}} + \dots + \boldsymbol{z_n}z_n^{(\beta_n)} = \boldsymbol{z_1}{}_0^C\boldsymbol{D}_t^{\beta_1}\boldsymbol{z_1} + \boldsymbol{z_2}{}_0^C\boldsymbol{D}_t^{\beta_2}\boldsymbol{z_2} + \dots + \boldsymbol{z_n}{}_0^C\boldsymbol{D}_t^{\beta_n}\boldsymbol{z_n} \quad \dots(25)$$
$$= -k_1z_1^2 - k_2z_2^2 - \dots - z_n(f_n(x_1, x_2, \dots, x_n) + u_n)$$

If we choose $u_n = -f_n(x_1, x_2, \dots, x_n) - k_n$ when we substituted in equation (29) we get

$\boldsymbol{z_1}\boldsymbol{z_1^{(\beta_1)}} + \boldsymbol{z_2}\boldsymbol{z_2^{(\beta_2)}} + \dots + \boldsymbol{z_n}z_n^{(\beta_n)} = -k_1z_1^2 - k_2z_2^2 - \dots - k_nz_n^2 < 0$ where $k_n \in R, k_n > 0$

that is mean $z_1z_1^{(\beta_1)} + z_2z_2^{(\beta_2)} + \dots + z_nz_n^{(\beta_n)} < 0$ everywhere.

And by using theorem (3), we know that:

$$\boldsymbol{z_1}\dot{\boldsymbol{z_1}} + \boldsymbol{z_2}\dot{\boldsymbol{z_2}} + \dots + \boldsymbol{z_n}\dot{\boldsymbol{z_n}} < \boldsymbol{z_1}\boldsymbol{z_1^{(\beta_1)}} + \boldsymbol{z_2}\boldsymbol{z_2^{(\beta_2)}} + \dots + \boldsymbol{z_n}\boldsymbol{z_n^{(\beta_n)}} < \boldsymbol{0} \quad (26)$$

By using theorems (3) and (4), then $\dot{V}_n$ is a negative definite function in R^n and similarly by Lyapunov stability theory subsystem (22) is asymptotically stable. The virtual control $x_n = \alpha_{n-1}(z_1, z_2, \dots, z_{n-1})$ and the state feedback input u_n may be evaluated which makes subsystem (22) is asymptotically stable.

Thus, from the result of the previous steps, we get that the system (7) is globally asymptotically stable for all initial conditions $x_i(0) \epsilon R^n$, $i = 1,2, \dots, n.$

4. Illustrative Examples

In this section, two examples are considered in order to illustrate the proposed an approach of stabilizability for systems of fractional order.

Example 1:

Consider the Caputo fractional order differential system

$$\begin{aligned}&{}_0^C\boldsymbol{D}_t^{\beta_1}\boldsymbol{x_1}(\boldsymbol{t}) = \boldsymbol{2x_1}\\ &{}_0^C\boldsymbol{D}_t^{\beta_2}\,\boldsymbol{x_2}(\boldsymbol{t}) = -\boldsymbol{x_1} - \boldsymbol{3x_2}\end{aligned} \quad (27)$$

where $\beta_1, \beta_2 \in (0,1), x_1, x_2 \in R^n$ by using backstepping method, and in order to stabilize the system asymptotically, either it is stable or not, we introduce the controller functions u_1 and u_2 as follows:

$${}_0^C\boldsymbol{D}_t^{\beta_1}\,\boldsymbol{x_1}(\boldsymbol{t}) = \boldsymbol{2x_1} + \boldsymbol{u_1}$$

$$\boldsymbol{{}_{0}^{C}D_{t}^{\beta_2}\,x_2(t) = -x_1 - 3x_2 + u_2} \quad (28)$$

Step1: Suppose $z_1 = x_1$, then the Caputo derivative of order $\beta_1$with respect to t for the first equation of(28) will be

$$\boldsymbol{{}_{0}^{C}D_{t}^{\beta_1}\,z_1(t) = 2z_1 + u_1} \quad (29)$$

and

$$\boldsymbol{z_1\,{}_{0}^{C}D_{t}^{\beta_1}z_1 = z_1(2z_1 + u_1)} \quad (30)$$

To prove that $z_1 z_1^{(\beta_1)}$is negative definite, we will choose the controller function $u_1 = -2z_1 - k_1 z_1$. If we substitute in (29) then equation (30) will be

$$\boldsymbol{z_1\,{}_{0}^{C}D_{t}^{\beta_1}z_1 = -k_1 z_1^2} \quad (31)$$

which is negative everywhere when $k_1$is positive constant.
since that and by using theorem (2), we know that:

$$\boldsymbol{z_1\dot{z}_1 < z_1\,{}_{0}^{C}D_{t}^{\beta_1}z_1 = -k_1 z_1^2 < 0} \quad (32)$$

By using theorem (3) with equation (36) that gives $z_1\dot{z}_1$will be also negative everywhere, then if we choose the Lyapunov function $V_1 = \frac{1}{2}z_1^2$ and $\dot{V}_1 = z_1\dot{z}_1$ since $z_1\dot{z}_1 < 0$ that means it is asymptotically stable. The function $\alpha_1(z_1)$should be estimated while z_2 is considered as a controller.

Step (2): Suppose $z_2 = x_2 - \alpha_1(z_1)$ or equivalent $x_2 = z_2 + \alpha_1(z_1)$, then

${}_{0}^{C}D_{t}^{\beta_2}x_2 = z_2 + {}_{0}^{C}D_{t}^{\beta_2}\alpha_1(z_1)$

Thus $z_2\,{}_{0}^{C}D_{t}^{\beta_2}z_2 = z_2\left(z_1 - 3z_2 - 3\alpha_1(z_1) - {}_{0}^{C}D_{t}^{\beta_2}\alpha_1(z_1) + u_2\right)$

to make $z_2\,{}_{0}^{C}D_{t}^{\beta_2}z_2$ negative, we choose

$u_2 = -z_1 + 3\alpha_1(z_1) + {}_{0}^{C}D_{t}^{\beta_2}\alpha_1(z_1) - k_2 z_2 \;,\; k_2 \in R^+$

Then the equation is as: $z_2\,{}_{0}^{C}D_{t}^{\beta_2}z_2 = -(3 + k_2)z_2^2$

because that: $z_2\,{}_{0}^{C}D_{t}^{\beta_2}z_2$ is negative and by using by theorem (2), then

$z_2\dot{z}_2 \le z_2\,{}_{0}^{C}D_{t}^{\beta_2}z_2 < 0$

That gives

$z_1\dot{z}_1 + z_2\dot{z}_2 \le z_1\,{}_{0}^{C}D_{t}^{\beta_1}z_1 + z_2\,{}_{0}^{C}D_{t}^{\beta_2}z_2 < 0$

then we can choose Lyapunov function V as

$V = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$

such that

$\dot{V} = z_1\dot{z}_1 + z_2\dot{z}_2 < z_1\,{}_{0}^{C}D_{t}^{\beta_1}z_1 + z_2\,{}_{0}^{C}D_{t}^{\beta_2}z_2 = -k_1 z_1^2 - k_2 z_2^2 < 0$

is negative everywhere which means that solution of the system is asymptotically stable.
Then the equivalent system will be:

$$\boldsymbol{{}_{0}^{C}D_{t}^{\beta}\;\; X(t) = AX} \quad (33)$$

where $\boldsymbol{X(t) = [z_1(t)\; z_2(t)]^T, \beta = [\beta_1, \beta_2],\; A = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix}, k_1, k_2 \in R^+(0)}$

Figure (1) presents the solution of the system of fractional order (31), in which one may see the asymptotic stability of the solutions.

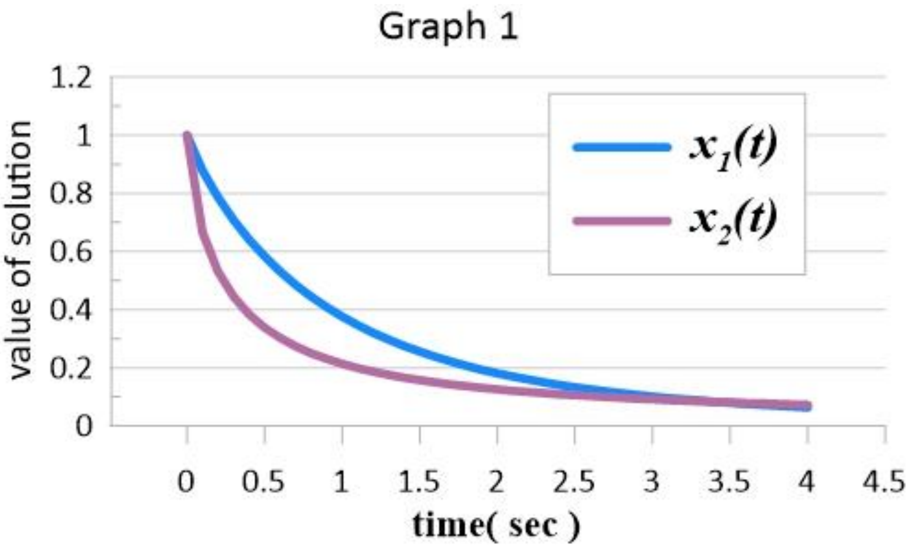

Figure 1-Solution of example 1, where $\beta_1 = 0.9$, $\beta_2 = 0.7$, $k_1 = 1$ and $k_2 = 2$.

Example 2:
Consider the Caputo Fractional differential system

$${}_0^C D_t^{\beta_1}\, x(t) = 3x - 4y + u_1$$
$${}_0^C D_t^{\beta_2}\, y(t) = 4x - 7y + u_2 \qquad (34)$$

Where $x(0) \neq 0, y(0) \neq 0, 0 < \beta_1, \beta_2 < 1, \beta_1, \beta_2 \in R, x, y \in R$

Step1: let $z_1 = x_1$ then the first equation of the system (34) will be as

$${}_0^C D_t^{\beta_1}\, z_1(t) = 3z_1 - 4y + u_1$$

such that

$$z_1\, {}_0^C D_t^{\beta_1} z_1 = z_1(3z_1 - 4y + u_1)$$

to do it negative define we choose $u_1$as:

$$u_1 = -3z_1 + 4y - k_1 z_1$$

then

$$z_1\, {}_0^C D_t^{\beta_1} z_1 = -k_1 z_1^2 < 0$$

which is negative everywhere when k_1 is positive constant..
because that $z_1\, {}_0^C D_t^{\beta_1} z_1$ is negative everywhere and by using theorems (2) then

$$z_1 \dot{z}_1 < z_1\, {}_0^C D_t^{\beta_1} z_1 < 0$$

that gives $z_1 \dot{z}_1 < z_1\, {}_0^C D_t^{\beta_1} z_1 = -k_1 z_1^2 < 0$ will be also negative everywhere then if we chose the Lyapunov function $V_1 = \frac{1}{2} z_1^2$ and $\dot{V}_1 = z_1 \dot{z}_1$ since $z_1 \dot{z}_1 < 0$ by theorem (1) that means the solutions will be asymptotically stable. The function $\alpha_1(z_1)$should be estimated while z_2 is considered as a controller.

Step2: let the error between z_2 and $\alpha_1(z_1)$ be

$$z_2 = y - \alpha_1(z_1)$$

then

$${}_0^C D_t^{\beta_2} z_2 = {}_0^C D_t^{\beta_2}\, y - {}_0^C D_t^{\beta_2} \alpha_1(z_1)$$

that gives as:

$${}_0^C D_t^{\beta_2}\, y = {}_0^C D_t^{\beta_2} z_2 + {}_0^C D_t^{\beta_2} \alpha_1$$
$$= 4z_1 - 7z_2 - 7\alpha_1(z_1) + u_2 - {}_0^C D_t^{\beta_2} \alpha_1(z_1)$$

to evaluate

$$z_2\, {}_0^C D_t^{\beta_2} z_2 = z_2 \left(4z_1 - 7z_2 - 7\alpha_1(z_1) + u_2 - {}_0^C D_t^{\beta_2} \alpha_1(z_1)\right)$$

to do that it is negative definite we choose $u_2 = -4z_1 + 7\alpha_1(z_1) + {}_0^C D_t^{\beta_2}\alpha_1(z_1) - k_2 z_2$, $k_2 \in R^+$
then $z_2\, {}_0^C D_t^{\beta_2} z_2 = -k_2 z_2^2$
because that $z_2\, {}_0^C D_t^{\beta_2} z_2$ is negative everywhere and by using by theorem (2), then
$z_2 \dot{z}_2 < z_2\, {}_0^C D_t^{\beta_2} z_2 = -k_2 z_2^2 < 0$ that gives
$z_1\dot{z}_1 + z_2\dot{z}_2 < z_1\, {}_0^C D_t^{\beta_1} z_1 + z_2\, {}_0^C D_t^{\beta_2} z_2 = -k_1 z_1^2 - k_2 z_2^2 < 0$. Now we can choose Lyapunov function V as $V = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ such that $\dot{V} = z_1\dot{z}_1 + z_2\dot{z}_2$ is negative definte by using theorem (1) that mean the system is asymptotically stable.
Then the equivalent system will be:

$${}_0^C D_t^{\beta} X(t) = AX \qquad (35)$$

Where $X(t) = [z_1(t)\ z_2(t)]^T, \beta = [\beta_1, \beta_2], A = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix}, k_1, k_2 \in R^+$.
Figure (2) presents the solutions of the system of fractional order (37), in which one may see the asymptotic stability of the solutions.

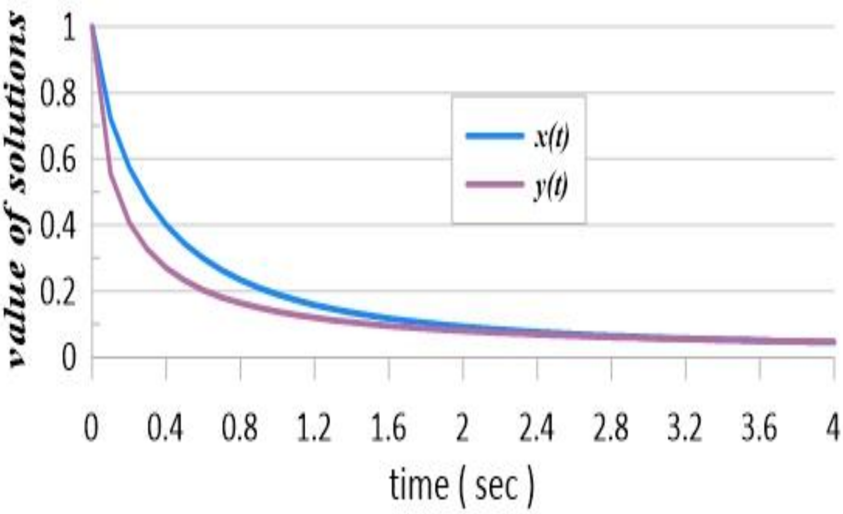

Figure 2-Solution of example 2 where $\beta_1 = 0.8$, $\beta_2 = 0.7$, $k_1 = 2$ and $k_2 = 2$.[21]

Conclusion:

Relation between backstepping of ordinary differential equations and fractional order differential equations are found, as well as some useful theorems are given . We first discuss theoretical side to use the backstepping for fractional order differential equation. We also find the relation between Lyapunov function and fractional order differential equations by helping of previous studied and proved theorem (6). Then after we use the backstepping method for Caputo Fractional order Differential equations. The relation among fractional order differential equations, backstepping method and Lyapunov function are found. We here found the Lyapunov function by iteration few steps (using Backstepping method) to give the controller function that makes all solutions asymptotically stable.

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