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On P-Essential Submodules

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Abstract

Let R be a commutative ring with identity and let A be an R-module. We call an R-submodule H of A as P-essential if $H \cap L \neq 0$ for each nonzero prime submodule P of A and $0 \neq L \leq P$. Also, we call an R-module A as P-uniform if every non-zero submodule H of A is P-essential. We give some properties of P-essential and introduce many properties to P-uniform R-module. Also, we give conditions under which a submodule H of a multiplication R-module A becomes P-essential. Moreover, various properties of P-essential submodules are considered.

Keywords: Essential submodules, Uniform modules, Fully prime modules, multiplications modules.

حول الفضاءات الجزئية الجوهرية من النمط−P

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الخلاصة

لتكن $\Re$ حلقة إبداليه ذات عنصر محايد وليكن A مقاسا احاديا. نسمي المقاس الجزئي H من A مقاسا . 0 $P \geq$ L ≠بحيث ان Pلكل مقاس جزئي اولي غير صفري $H \cap L \neq 0$ اذا −pجوهريا من النمط وعرفنا مفهوم المقاس المنتظم من النمط −P بانه المقاس الذي يكون فيه كل مقاس جزئي غير صفري هو مقاس جوهري من النمط−P. ثم اعطينا بعض الخواص لذلك المقاس وكذلك تقديم عدد من خصائص المقاس المنتظم من النمط−P كما درسنا بعض الشروط التي بموجبها يكون اي مقاس جزئي من مقاس جدائي جوهري من النمط−P.

1- Introduction

Let R be a commutative ring with unity and let A be a unitary R-module. A non-zero submodule $H$of A is called essential if $H \cap L \neq 0$ for each non-zero submodule L of A [1]. A is called uniform if every non-zero submodule H of A is essential [1]. In (2019), Ahmad and Ibrahiem studied a new concept, which is named H-essential submodules [2]. Ali and Nada [3] introduced the concept of semi-essential submodules as a generalization of the class of essential submodules. They stated that a nonzero submodule H of A is called semi-essential , if $H \cap P \neq 0$ for each nonzero prime submodule P of A. In section two, we introduce a P-essential submodule concept as a generalization of the essential submodule concept. We call an R-submodule H of A as P-essential if $H \cap L \neq 0$ for each nonzero prime submodule P of A and $0 \neq L \leq P$. Our main concerns in this section are to give characterizations for P-

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essential submodules and generalize some known properties of essential submodules to P-essential submodules. In section three, we give conditions under which a submodule H of a faithful multiplication R- module A becomes P-essential. In section four, we present the P-uniform module concept as a generalization of the uniform concept. We also generalize a characterization and some properties of uniform modules to P-uniform modules.

2- P-Essential Submodules

Recall that a non-zero submodule H of an R-module A is called essential if $H \cap L \neq 0$ for each submodule L of A [1].

Definition(2-1)

Let A be an R-module and P be a non-zero prime submodule of A. A submodule H of A is said to be P-essential, written as $\leq_{pe} A$, if for every proper submodule L of P, then $H \cap L = 0$, which implies that $L = 0$.

Or, a non-zero submodule H of A is called P-essential , if $H \cap L \neq 0 \ \forall\ 0 \neq L \subseteq P$.

Remarks and Examples(2-2)

1- Every essential submodule is P- essential submodule, but the converse is not true in general .

For example, consider $A = Z_{24}$ as Z-module , $P =< \bar{3} >= \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \dots, \overline{21}\}$. $H =< \bar{6} > = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\}$, $\leq_{Pe} A$, since $< \bar{0} >, < \bar{6} >, < \overline{12} >$ are proper submodules of $P =< \bar{3} >$, $H \cap L \neq < 0 > \ \forall\ 0 \neq L \leq P$, but $< \bar{6} > \nleq_e A$, since $< \bar{8} > \cap < \bar{6} >= 0$, while $< \bar{8} > \neq < \bar{0} >$.

2- Let $A = Z_{15}$ be a Z-module and the prime submodules of A are : $P_1 =< \bar{3} >, P_2 =< \bar{5} >$.
It follows that $< \bar{3} >, < \bar{5} >$ are P_1-essential and P_2-essential resp. in Z_{15} , but are not essential in Z_{15} , since $< \bar{3} > \cap < \bar{5} >=< \bar{0} >$, but $< \bar{5} > \neq < \bar{0} >$.

3- A submodule of a P-essential submodule needs not to be P-essential.
For example, let $A = Z_{24}$ be a Z-module , $H =< \bar{4} >, P =< \bar{2} >$ is a prime submodule of A , $< \bar{4} > \leq_{pe} A$, but $< \bar{8} > \nleq_{pe} A$, since $< \bar{8} > \cap < \bar{6} > = 0$ where $L =< \bar{6} > \subseteq < \bar{2} > = P$ and $< \bar{8} > \leq < \bar{4} >$.

4- If $H_1$and H_2 are P-essential submodules of , then $H_1 \cap H_2$ needs not be to P-essential of A.
For example, let $A = Z_{24}$ and let $P =< \bar{2} >$, $H_1 =< \bar{4}, >$ and $H_2 =< \bar{6} >$ be P-essential of A , but $< \bar{4} > \cap < \bar{6} >=< \bar{0} >$ is not P-essential of Z_{24} .

5- The sum of two P-essential submodules of an R-module A is also P-essential submodule.
Proof: Let A be R-module and let L and K be two P-essential submodules of A. Note that $\leq L + K$, since $L \leq_{pe} A$, implies that $L + K \leq_{pe} A$.

6- A semi-essential submodule needs not to be P-essential submodule, as we see in the following example:
Consider Z_{12} as Z-module . $N =< \bar{3} >$ is semi-essential [3], but it is not P- essential where $P =< \bar{2} >$ and $< \bar{3} > \cap < \bar{4} >=< \bar{0} >$, but $0 \neq < \bar{4} >$.

Proposition (2-3)

Let A be an R-module, P be a prime submodule of A, and K be any submodule of A. If $\leq_{Pe} A$, then $K \leq_{Pe} A$ if and only if $K \leq_e A$.

Proof : Suppose that $K \leq_{Pe} A$. Let P be a prime submodule of A and let $L \leq P$ such that $K \cap L =< 0 >$, implies that $K \cap (P \cap L) =< 0 >$. Since $P \cap L \leq P$ and $K \leq_{Pe} A$, then $P \cap L =< 0 >$. By hypothesis, $P \leq_e A$, thus $L =< 0 >$ which implies that $K \leq_e A$. The converse is obvious.

Poposition (2-4)

A non-zero submodule K of A is P-essential if and only if for each non-zero submodule L of a submodule P, $\exists\ x \in L$ and $r \in R$ such that $0 \neq rx \in K$, where P is a prime submodule of A. The proof is easy and hence is omitted.

Proposition(2-5)

Let A be an R-module and let H_1, H_2 be submodules of A such that $H_1 \leq H_2$. If H_1 is P-essential submodule of A, then H_2 is a P-essential submodule of A.

Proof

Let P be a prime submodule of A, $0 \neq L \leq P$. By using proposition (2-4), $x \in L$, $r \in R$. Since $H_1 \leq_{pe} A$, then $0 \neq rx \in H_1 \leq H_2$, then $0 \neq rx \in H_2$, implies that $H_2 \leq_{Pe} A$.

The converse of prop.(2-5) is not true in general; for example :

Consider Z_{24} as a Z-module and $< \bar{8} >$ is a submodule of $< \bar{4} >$. By remarks and example (2-2)(3), $< \bar{4} > \leq_{pe} Z_{24}$, but $< \bar{8} > \nleq_{pe} Z_{24}$, since $< \bar{8} > \cap < \bar{6} > = < \bar{0} >$ and $< \bar{6} > \neq < \bar{0} >$.

Corollary(2-6)

Let H_1 and H_2 be submodules of A. If $H_1 \cap H_2$ is P-essential submodule of A, then H_1 and H_2 are P-essential.

Proof

By using proposition (2-5), since $H_1 \cap H_2 \leq H_1$ and $H_1 \cap H_2 \leq_{Pe} A$, so $H_1 \leq_{Pe} A$. In the same way, $H_2 \leq_{Pe} A$.

The converse of the previous corollary is not true in general, as shown in remarks and examples(2-2)(5).

Proposition(2-7)

Let A be an R-module and let H_1 and H_2 be submodules of A. If H_1 is an essential submodule of A and H_2 is a P-essential submodule of A, then $H_1 \cap H_2$ is also P-essential submodule of A.

Proof

Let P be prime submodule of A and let $0 \neq L$ submodule of P. Since H_2 is P-essential submodule of A, thus $H_2 \cap L \neq < 0 >$. And since H_1 is an essential submodule of A, then $H_1 \cap (H_2 \cap L) \neq < 0 >$, so $(H_1 \cap H_2) \cap L \neq < 0 >$. This implies that $H_1 \cap H_2$ is P-essential submodule of A.

Proposition(2-8)

Let A and B be R-modules and let $f: A \to B$ be an epimorphism. If K is a P-essential submodule of , then $f^{-1}(K)$ is a $f^{-1}(P)$-essential of A.

Proof

We know that if P is a prime submodule of B then $f^{-1}(P)$ is a prime submodule of A [4]. Let $0 \neq L \leq f^{-1}(P)$ and $f^{-1}(K) \cap L = < 0 >$. To prove that $L = 0$, then $K \cap f(L) = < 0 >$. Since K is P-essential in B and $f(L) \leq P$, then $f(L) = 0$, implies $L \subseteq f^{-1}(0) = kerf \leq f^{-1}(K)$. But $f^{-1}(K) \cap L = < 0 >$, that is $L = 0$. Thus $f^{-1}(K)$ is a $f^{-1}(P)$-essential submodule of A.

Remark(2-9):- Let $f: A \to \acute{A}$ be an isomorphism. If $H \leq_{Pe} A$, then $f(H) \leq_{Pe} \acute{A}$.

Proof : Let P be a prime submodule of $\acute{A}$ and let L be a non-zero submodule of P. Since f is an epimorphism, then $f^{-1}(L)$ is a submodule of $f^{-1}(P)$ which is prime submodule of A by [4]. But $\leq_{Pe} A$, then $H \cap f^{-1}(L) \neq < 0 >$. On the other hand, f is a monomorphism, thus $f(H) \cap L \neq < 0 >$. This completes the proof.

Proposition(2-10)

If K is a submodule of an R-module A and P_1 , P_2 are prime submodules of A, such that $0 \leq P_1 \leq P_2$. If $K \leq_{P_1 e} A$, then $K \leq_{P_2 e} A$.

Proof:- Let $L_2 \leq P_2$ such that $K \cap L_2 = < 0 >$. To prove that $L_2 = 0$. $\exists\ i: P_1 \to P_2$, since $L_2 \leq P_2$, hence $i^{-1}(L_2) \leq P_1$. $i^{-1}(K \cap L_2) = i^{-1} < 0 >$, implies that $\cap\, i^{-1}(L_2) = < 0 >$. Since $\leq_{P_1 e} A$, hence $i^{-1}(L_2) = L_2 = < 0 >$.

Proposition(2-11)

Let C , K , P be submodules of an R-module A and P is prime submodule of , $K \leq C$. $K \leq_{Pe} A$ if and only if $K \leq_{(P \cap C)e} A$ and $C \leq_{Pe} A$.

Proof:- ($\Longrightarrow$) Since P is prime in A , $C \leq A$, then $(P \cap C)$ is prime in C [4]. Let $L \leq (P \cap C)$ with $\cap L =< 0 >$. To prove that $L =< 0 >$, since $L \leq P$, $K \leq_{Pe} A$, hence $L =< 0 >$. Let $T \leq P$ with $\cap C =< 0 >$, implies that $T \cap K =< 0 >$ (the hypothesis has been modified in the proposition). Since $\leq_{Pe} A$, then $T = 0$.
($\Longleftarrow$) Let $L \leq P$ such that $L \cap K =< 0 >$, then $(L \cap K) \cap C =< 0 >$, implies that $(L \cap C) \cap K =< 0 >$, $L \cap C \leq P \cap C$ and $K \leq_{(P\cap C)e} A$, hence $L \cap C =< 0 >$. Since $\leq_{Pe} A$, then $L =< 0 >$, thus $K \leq_{Pe} A$.
In the following proposition, we give the transitive property for non-zero P-essential submodules.

Proposition(2-12)

Let A, B , C be R –modules such that $A \leq B \leq C$. If $A \leq_{pe} B$ and $B \leq_{pe} C$, then $A \leq_{Pe} C$.

Proof:- Let P be a prime submodule of C and let L be a submodule of P such that $A \cap L = 0$. Note that $0 = A \cap L = (A \cap L) \cap B = A \cap (L \cap B)$. If $B \leq L$ then $0 = A \cap (L \cap B) = A \cap B$, hence $A \cap B = 0$, but $A \leq B$, so $A \cap B = A$, which implies that A=0. But this is a contradiction. Thus $B \nleq L$ and $L \cap B \leq P$. But $A \leq_{Pe} B$, therefore $L \cap B = 0$, and since $B \leq_{Pe} C$, then $L = 0$, that is $A \leq_{pe} C$.
The converse of proposition (2-12) is not true in general, as the following example shows: Consider Z_{24} as Z-module , the submodule $< \bar{6} >$ is P-essential of Z_{24}, by remarks and examples(2-2). But $< \bar{6} >$ is not P-essential submodule of $< \bar{2} >$ where $< \bar{2} > \leq_e Z_{24}$.
Recall that an R-module A is fully prime, if every proper submodule of A is a prime submodule [2].

Proposition(2-13)

Let $A = A_1 \oplus A_2$ be a fully prime R- module where A_1 and A_2 are submodules of , and let $0 \neq K_1 \leq A_1$ and $0 \neq K_2 \leq A_2$. Then $K_1 \oplus K_2$ is P-essential of $A_1 \oplus A_2$ if and only if K_1 is a P-essential submodule of A_1 and K_2 is a P-essential submodule of A_2.

Proof

($\Rightarrow$) Since A is a fully prime module, then by [5], $K_1 \oplus K_2$ is an essential submodule of $A_1 \oplus A_2$ and by [6, proposition(5-20)], K_1 is an essential submodule A_1 and K_2 is an essential submodule of A_2. But since every essential submodule is a P-essential, so we are done. ($\Leftarrow$) It follows similarly.

Proposition(2-14)

Let A be an R-module and let H_1 and H_2 be P-essential submodules of A such that $H_1 \cap H_2 \neq 0$, then $H_1 \cap H_2$ is P-essential submodule of A.

Proof

Let P be a prime submodule of A and let $L \leq P$ such that $(H_1 \cap H_2) \cap L = 0$. This implies that $H_2 \cap (H_1 \cap L) = 0$. If $H_1 \leq L$, then we have a contradiction with the assumption, thus $H_1 \nleq L$. This implies that $H_1 \cap L$ is a submodule of A [5]. Since H_2 is P-essential submodule of A and, by our assumption, $H_1 \cap L$ is a submodule of A, then $H_1 \cap L = 0$. But H_1 is P-essential submodule of , therefore $L = 0$, hence $H_1 \cap H_2$ is P-essential submodule of A.

3- P-Essential Submodules in Multiplication Modules

An R- module A is called multiplication if every submodule H of A is of the form IA for some ideal I of R [7] and an R-module A is called faithfull if $ann(A) = 0$. In this section , we give a condition under which a submodule H of A is a faithful multiplication R-module that becomes P-essential.

Theorem(3-1)

Let A be a faithful multiplication R-module and H be a submodule of A . Then H is P-essential of A if and only if I is P-essential of R.

Proof

Assume that H is P-essential submodule of A, let P be a prime ideal of R and $L \leq P$ such that $I \cap L = 0$. Since A is a faithful multiplication R-module , then $(I \cap L)A = IA \cap LA = 0$. Now, PA is a prime submodule of , $LA \leq PA$ and ($IA = H$ is P-essential submodule of A), implies that $LA = 0$. Since A is finitely generated faithful multiplication R-module , then $L = 0$. Therefore, I is a P-essential . Conversely, let P be a prime submodule of A and L be a submodule of P such that $H \cap L = 0$. Since A is multiplication , then there exists an ideal B of R such that $L = BA$ [8] . Hence $H \cap L = IA \cap BA = (I \cap B)A = 0$. But A is faithful , so $I \cap B = 0$. Since I is a P-essential ideal of R, then $B = 0$, therefore $L = BA = 0$, thus H is a P-essential submodule of A.

Theorem(3-2)

Let A be a faithful multiplication R-module. Then H is a P-essential submodule of A if and only if $[H:< x >]$ is a P-essential ideal of R for each $x \in A$.

Proof

Assume that H is P-essential . Since A is faithful multiplication R-module , then $[H:A]$ is a P-essential of R, by Theo.(3-1). But $[H:A] \subseteq [H:< x >]$ for each $\in A$, so $H = [H:A]A \subseteq [H:< x >]A$,[7] . Hence $[H:< x >]A$ is P-essential by Proposition (2-5), hence $[H:< x >]$ is a P-essential ideal of R by Theorem (3-1).

Proposition(3-3)

Let A be a finitely generated, faithful and multiplication R- module . If $I \leq_{Pe} J$, then $IA \leq_{Pe} JA$ for every ideals I and J of R.

Proof

Let P be a prime submodule of JA such that $P = KA$ for some prime ideal K of R and $K \subseteq J$,[8] and let L be a submodule of P such that $IA \cap L = 0$. Since A is a multiplication module , then $L = EA$ for some ideal E of R. So $\cap EA = 0$, implies that $(I \cap E)A = 0$. Since A is a faithfull module, then $\cap E = 0$. Since $EA \leq KA$ and A is finitely generated, faithful and multiplication, so by [8], $E \leq K$. Since I is a P-essential ideal of J, then $E = 0$ and hence $L = 0$. That is, $IA \leq_{Pe} JA$.

Proposition(3-4)

Let A be a non-zero multiplication R-module with only one maximal submodule H. If $H \neq 0$, then H is an essential (hence P-essential) submodule of A.

Proof

Let L be a submodule of A with $L \cap H = 0$. If $= A$, then $H \cap A = 0$, hence $H = 0$, which is a contradiction. Thus L is a proper submodule of A, and since A is a non-zero multiplication module , so by [8] , L is contained in some maximal submodule of A. But A has only one maximal submodule, which is H. Thus $L \subseteq H$, implies that $L = 0$, that is H is an essential (hence P-essentianl) submodule of A.

Recall that a non-zero R-module A is called fully essential if every non-zero semi-essential submodule of A is an essential submodule of A [5].

Definition(3-5): A non-zero R-module A is called fully P-essential if every non-zero P-essential submodule of A is an essential submodule of A. A ring R is called fully P- essential if every non-zero P-essential ideal I of R is essential ideal of R.

Examples(3-6)

1- Z_8 as a Z-module is fully P-essential Z-module.

2- Z_{12} as a Z-module is not fully P-essential , since the submodule $< \bar{6} >$ of Z_{12} is P_2-essential where $P_2 =< \bar{3} >$, but not essential since $< \bar{6} > \cap < \bar{4} > = < \bar{0} >$ but $< \bar{4} > \neq < \bar{0} >$.

3- Every fully essential is fully P-essential.

The following theorem gives the hereditary of fully P-essential property between R-module A and the ring R.

Theorem(3-7)

Let A be a non-zero faithfull and multiplication R-module , then A is a fully P-essential module if and only if R is a fully P-essential ring.

Proof

($\Longrightarrow$) Assume that A is a fully P-essential module and let I be a non-zero P-essential ideal of R, then IA is a submodule of A, say H. This implies that H is a P-essential submodule of A. Since $I \neq 0$ and A is faithful module, then $H \neq 0$. But A is a fully P-essential module , thus H is an essential submodule of A. Since A is a faithful and multiplication module, therefore I is an essential ideal of R [8], that is R is a fully P-essential ring.
($\Longleftarrow$) Suppose that R is a fully P-essential ring and let $0 \neq H \leq_{Pe} A$. Since A is a multiplication module, then $H = IA$ for some P-essential ideal of R. By assumption, I is an essential ideal of R. But A is faithful and multiplication module, then H is an essential submodule of A [8]. Thus A is fully P-essential module.

4- P-Uniform Modules

Recall that a non-zero R-module A is called uniform if every non-zero submodule of A is essential [9]. Recall that a non-zero R-module A is called semi-uniform if every non-zero submodule of A is semi-essential [10]. In this section , we give a P-uniform module concept as a generalization of the uniform module concept. We also generalize some properties of uniform modules to P-uniform modules.

Definition(4-1)

A non-zero R-module A is called P-uniform if every non-zero submodule of A is P-essential . A ring R is called P-uniform if R is a P-uniform R-module.

Remarks(4-2)

1- Each uniform R-module is P-uniform, but the converse is not true in general . For example, Z_{15} as a Z-module is P-uniform but not uniform since $<\bar{3}>\cap<\bar{5}>=<\bar{0}>$, while $<\bar{5}>\neq<\bar{0}>$; see remarks and examples(2,2),(2).

2- Each simple R-module A is P-uniform. But the converse is not true in general. For example, Z_9 is a P-uniform Z-module where $=<\bar{3}>$, but not simple Z-module.

3- Z_{12} as a Z-module is not P-uniform , where $P=<\bar{2}>$ is prime submodule of Z_{12} , $<\bar{3}>\cap<\bar{4}>=<\bar{0}>$ and $<\bar{4}>\nleq_{Pe}<\bar{2}>$.

4- We can note that a semi-uniform R-module needs not to be P-uniform, as shown in the following example:

The Z-module Z_{36} is semi-uniform [3], but not P_1-uniform and not P_2-uniform, where $P_1=<\bar{2}>, P_2=<\bar{3}>$, since $<\overline{18}>\cap<\overline{12}>=<\bar{0}>$, but $<\overline{12}>\neq<\bar{0}>$, as in the following table:

$\mathcal{H} \subseteq \mathcal{A}$	ess	$\mathcal{P}_2 - ess$	$\mathcal{P}_2 - ess$	Semi$-ess$
$\mathbb{Z}_{36}$	√	√	√	√
$(\bar{2})$	√	√	√	√
$(\bar{3})$	√	√	√	√
$(\bar{4})$	×	√	×	√
$(\bar{6})$	√	√	√	√
$(\bar{9})$	×	×	√	√
$(\overline{12})$	×	×	×	√
$(\overline{18})$	×	×	×	√

Proposition(4-3)

Let A be an R-module , then A is uniform if and only if A is P-uniform and fully P-essential.

Proof:- ($\Longrightarrow$) It is clear.

($\Leftarrow$) Let H be a non-zero submodule of A. since A is P-uniform module, then $H \leq_{Pe} A$. But A is fully essential module , then $H \leq_e A$, implies that A is uniform module.

Theorem(4-4)

Let A be a faithful multiplication R-module , then A is a P-uniform R-module if and only if R is a P-uniform ring.

Proof

Suppose that A is P-uniform and let E be a non-zero ideal of R. Thus EA is P-essential submodule of A. By theorem (3-1), E is a P-essential ideal of R. Conversely, assume that R is P-uniform and H is a submodule of A. Since A is multiplication, then there exists an ideal B of R such that $H = BA$. But R is P-uniform, so B is P-essential . Thus H is P-essential by theorem(3-1).

Proposition(4-5)

Let A_1 and A_2 be two R-modules and let $f: A_1 \to A_2$ be an epimorphism. Then:

1- If A_1 is P-uniform R-module , then A_2 is also P-uniform R-module.
2- If A_2 is P-uniform R-module for each prime submodule P of A_1 , then A_1 is $f^{-1}(P)$-uniform R-module.

Proof

1-Let H_2 be a non-zero submodule of A_2 , then $f^{-1}(H_2)$ is a non-zero submodule of A_1. Since A_1 is P-uniform R-module, thus $f^{-1}(H_2)$ is a P-essential submodule of A_1. By remark(2-9), we get $f\left(f^{-1}(H_2)\right) = H_2$ is a P-essential submodule of A_2. Therefore, A_2 is P-uniform R-module.

2- Let H_1 be a non-zero submodule of A_1, then $f(H_1)$ is a non-zero submodule of A_2. Since A_2 is P-uniform R-module, then $f(H_1)$ is a P-essential submodule of A_2. By proposition(2-8), we get $f^{-1}(f(H_1)) = H_1$ is a $f^{-1}(P)$- essential submodule of A_1. Therefore, A_1 is $f^{-1}(P)$-uniform R-module.

Proposition(4-6)

Let $A = A_1 \oplus A_2$ be R-module, where A_1 and A_2 are R-modules. If A is P-uniform, then A_1 and A_2 are P-uniform modules

Proof

Let H_1 be non-zero submodule of A_1 , so $H_1 \leq A$. But A is a P-uniform, then $H_1$is a P-essential submodule of A. Thus, H_1 is a P-essential submodule of A_1. Therefore, A_1 is P-uniform R-module. In a similar way, we can proof that A_2 is a P-uniform R-module.

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