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Solving the Created Equations from Power Function Distribution

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Abstract

In this paper, a new class of ordinary differential equations is designed for some functions such as probability density function, cumulative distribution function, survival function and hazard function of power function distribution, these functions are used of the class under the study. The benefit of our work is that the equations ,which are generated from some probability distributions, are used to model and find the solutions of problems in our lives, and that the solutions of these equations are a solution to these problems, as the solutions of the equations under the study are the closest and the most reliable to reality. The existence and uniqueness of solutions the obtained equations in the current study are discussed. The exact solutions of these obtained differential equations are calculated using some methods. In addition, the approximate solutions are determined by the Variation Iteration Method (VIM) and Runge-Kutta of 4^{th} Order (RK4) method. The chosen approximate methods VIM and RK4 are used in our study because they are reliable, famous, and more suitable for solving such generated equations. Finally, some examples are given to illustrate the behavior of the exact and the approximate solutions of the differential equations with the scale parameters of power function distribution.

keywords: Power function distribution, Ordinary differential equations, Variation iteration method, Runge-Kutta of 4^{th} Order, Survival function, Hazard function.

حل المعادلات التي تم إنشاؤها من توزيع دالة القوة

حسين جبار محمد , مهاعبدالجبارمحمد
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الخلاصة

في هذا البحث، تم تصميم فئة جديدة من المعادلات التفاضلية العادية لبعض الدوال مثل دالة كثافة الاحتمال، ودالة التوزيع التراكمي، ودالة البقاء، ودالة الخطر لتوزيع دالة القوة المستخدمة في الفئة قيد الدراسة.تكمن فائدة عملنا في أن المعادلات التي يتم تكوينها من بعض التوزيعات الاحتمالية هي نمذجة لمشاكل في حياتنا تحتاج إلى حلول، وأن حلول هذه المعادلات هي حل لهذه المشاكل، حيث أن حلول المعادلات قيد الدراسة هي الأقرب والأكثر موثوقية للواقع. تمت مناقشة وجود ووحدانية المعادلات التي تم الحصول عليها في الدراسة الحالية. يتم حساب الحلول المضبوطة لهذه المعادلات التفاضلية باستخدام بعض الطرق. بالإضافة إلى ذلك ، يتم تحديد الحلول التقريبية بواسطة طريقة التكرار المتغير (VIM) وطريقة

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Runge–Kutta من الرتبة الرابعة (RK4). تم استخدام الطرق التقريبية المختارة VIM و RK في دراستنا لأنها موثوقة ومشهورة وأكثر ملاءمة لحل مثل هذه المعادلات المتولدة. أخيرًا ، تم إعطاء بعض الأمثلة لتوضيح سلوك المعادلات التفاضلية الحقيقية والتقريبية للحلول مع معلمات مقياس توزيع دالة القوة.

1. Introduction

The power function distribution is a flexible lifetime model which is the special case of the beta distribution. Sinha et.al.[1] proposed a power function distribution. Also, the power function distribution was derived from the Pareto distribution using the inverse transformation[2].

On the other hand, the ordinary differential equations that are obtained for the probability distributions play an important role in many applied fields in engineering, physics, chemistry, computer science, economy and other sciences [4-6]. Some authors such as Okagbue, et al. [7-16] created ordinary differential equations that are obtained from some distributions such the Burr XII and the Pareto distributions, exponential and truncated distributions exponential, exponentiated generalized exponential distribution, Fréchet distribution, half-normal distribution, Harris exponential distribution extended the Kumaraswamy distribution, distributions of linear failure rate and generalized linear failure rate, the Logistic and the log-Logistic distributions and three-parameters Weibull distribution. RK4 and VIM are reliable approximate methods to solve ODE. A system of non-linear ordinary differential equations was solved by Variation Iteration method (VIM) in [17], [18] and the system of ordinary differential equations was solved using RK4 [19].

In this work, a new class of ordinary differential equations of first and second order is designed by using some functions such as a probability density function, cumulative distribution function, survival function and hazard function of the power function distribution. Also, the exact solutions of these differential equations are calculated by homogeneous, separable, Bernoulli methods, using analytic and numerical methods. In addition, approximate solutions are found analytically by the method of Variation Iteration (VIM) and numerically by Runge-Kutta of 4^{th} order (RK4) method. Finally, the values of exact and approximate solutions are computed according to different values of the parameter θ.

The remainder of present paper is organized as below: In section 2, ordinary differential equations obtained for power function distribution are created. In section 3, the existence and uniqueness of ordinary differential equations that obtained for power function distribution are proved and discussed. In section 4, the formed ordinary differential equations are exactly solved by the homogeneous, separable, and Bernoulli methods, and analytically by (VIM) and numerically by (RK4). The results are clearly discussed by using some tables and figures in section 5. The important results under study are explained and the conclusions are given in section 6.

2. Ordinary Differential Equations Obtained from Power Function Distribution

A random variable $X \in [0,1]$ has a power function distribution denoted by $(x \sim \mathrm{PFD}(\theta))$, and its probability density function is given as follows:

$$f(x) = \theta\, x^{\theta-1}, \qquad 0 \le x \le 1, \theta > 0, \theta \ne 1 \tag{1}$$

The mean, variance, cumulative distribution function (CDF), survival function and hazard function of X are derived by Butt, N.S et. al. [3] as follows, respectively,:

$$E(x) = \frac{\theta}{\theta+1}, \tag{2}$$

$$Var(x) = \frac{\theta}{(\theta+2)(\theta+1)^2}, \tag{3}$$

$$F(x) = x^{\theta}, \tag{4}$$

$$S(x) = 1 - x^{\theta} \tag{5}$$

$$h(x) = \frac{\theta\, x^{\theta-1}}{1-x^{\theta}}. \tag{6}$$

In this section, equations (1) and (4-6) are derived to design a new class of ordinary differential equations.

2.1 Probability Density Function

By differentiating the equation (1) under the condition $\theta \neq 1$, we get

$$f'(x) = \frac{\theta-1}{x} f(x),\ x \in (0,1]. \tag{7}$$

After simplification, we obtain

$$x\, f'(x) - (\theta-1)f(x) = 0 \quad . \tag{8}$$

From the equation (1), the initial condition at $x_0 \in (0,1]$ of ordinary differential equation (7) can be written as

$$f(x_0) = \theta(x_0)^{\theta-1} \quad . \tag{9}$$

Again, by differentiating the equation (7) and simplification under the condition $\theta \neq 2$, we obtain

$$xf''(x) - (\theta-2)f'(x) = 0,\ x \in (0,1] \tag{10}$$

To find initial condition for equation (10), we use the substituting equation (9) and by x_0 in equation (7), we get:

$$f'(x_0) = \frac{\theta-1}{x} f(x_0) \tag{11}$$

2.2 Cumulative Distribution Function (CDF)

By differentiating both sides of equation (4), then we have

$$F'(x) = \theta \frac{F(x)}{x} \tag{12}$$

After simplification to obtain;

$$xF'(x) = \theta F(x)\ ,\ \ x \in (0,1] \tag{13}$$

Furthermore, the initial condition at $x_0 \in (0,1]$ of ordinary differential equation (8) is

$$F(x_0) = (x_0)^{\theta} \tag{14}$$

We differentiate equation (12) to obtain the differential equation of the second-order, this becomes as:

$$xF''(x) - (\theta-1)F'(x) = 0, \tag{15}$$

with the following conditions

$$F(x_0) = (x_0)^{\theta} \ . \tag{16}$$

$$F'(x_0) = \theta(x_0)^{\theta-1} \ . \tag{17}$$

2.3 Survival Function (SF)

In the same technique that is used in 2.1 and 2.2, the obtained ordinary differential equation from survival function can be written as follows:

$$xS'(x) - \theta S(x) + \theta = 0 \quad . \tag{18}$$

With the initial condition at $x_0 \in (0,1]$

$$S_0 = 1 - {x_0}^{\theta} \ . \tag{19}$$

2.4 Hazard Function (HF)

From equation (6), we have

$$h'(x) = \frac{(\theta-1)}{x} h(x) + h^2(x) \tag{20}$$

After simplification, we obtain;

$$xh'(x) = (\theta-1)h(x) + xh^2(x),\ x \in (0,1) \tag{21}$$

The ordinary differential equation of the first order (ODE) for the hazard function can be defined as follows

$$xh'(x) - xh^2(x) - (\theta-1)h(x) = 0 \tag{22}$$

From equation (6) the initial condition at $x_0 \in (0,1)$ may be written as follows

$$h_0 = \frac{\theta\, {x_0}^{\theta-1}}{1-{x_0}^{\theta}} \tag{23}$$

3. The Existence and Uniqueness

In present section, Picard-Lindelöf Theorem is used to prove the existence and uniqueness for equations (8), (13), (18) and (22). This is done by Martha L. Abell, James P. Braselton [20, page26] as follows;

The condition necessary for the existence of the equation (8) is $0 < x \leq 1$ and $\theta > 0$ such that $\theta \neq 1$

Let $(x) = y$, we obtain $y' = \frac{(\theta-1)}{x} y$, $y(x_0) = y_0$

$y' = g(x, y) = \frac{(\theta-1)}{x} y$ and $\frac{\partial g}{\partial y} = \frac{(\theta-1)}{x}$ are both continuous when $x \neq 0$,thus the existence and uniqueness of the solution are exist through any point (x_0, y_0) with $x_0 \neq 0$.

In the same method and by the existence and uniqueness theorem, the necessary condition for the existence and uniqueness of the solution in equations (12) and (18) are $x > 0$ $and\ \theta > 0$. While the necessary condition for the existence and uniqueness of the solution in equation (20) are $0 < x < 1$ and $\theta > 0$.

4. Solutions of Ordinary Differential Equations Obtained for Power Function Distribution

In the current section, some exact methods, namely homogeneous, separable, Bernoulli and approximate methods (VIM and RK4) are presented to determine the solution of the ordinary differential equations which are referred to section 2.

4.1 The Exact Solution

In this subsection, the homogeneous, separable, and Bernoulli methods are used to solve equations (8), (13), (18) and (22)

4.1.1 Solve the equations in homogeneous method

In the current subsection, the exact solutions of equation (8) and equation (12) have been computed using the homogeneous method.

To solve equation (8) in a way homogeneous. We use the notation $\tilde{\theta} = (\theta - 1)$ and the transformation $f(x) = vx$ to get

$$v + x\frac{dv}{dx} = \tilde{\theta} v \tag{24}$$

By integrating both sides and separating the variables, we have

$$v = b x^{\tilde{\theta}-1} \quad , \tag{25}$$

where, b is a constant.

The last step is to restore the original variables by reversing the substitution.

$$f(x) = b\ (x^{\tilde{\theta}-1})\,.(x)\ . \tag{26}$$

This implies to

$$f(x) = b\ x^{\theta-1}\ . \tag{27}$$

The initial state in equation (9) gives

$$f(x) = \theta\ x^{\theta-1}.$$

By the same method to solve equation (12) with the initial condition in (14) as below:

Firstly, we use the transformation $F(x) = vx$, then we have

$$F'(x) = v + x\frac{dv}{dx}\ . \tag{28}$$

From equation (13), we have

$$F'(x) = \theta\frac{F(x)}{x} = \theta v \tag{29}$$

Combining the equation (28) and the equation (29) to obtain

$$v + x\frac{dv}{dx} = \theta v\ . \tag{30}$$

Rewriting equation (30) in the following form

$$\frac{dv}{v} = (\theta - 1)\frac{dx}{x}\ . \tag{31}$$

By integrating both sides and separating the variables, we have

$$v = bx^{\theta-1} , \tag{32}$$

where, b is a constant. The last step is to restore the original variables by reversing the substitution.

$$F(x) = b\, x^{\theta}. \tag{33}$$

The initial state in equation (14) gives

$$F(x) = x^{\theta}.$$

4.1.2 Solving the equations by the separable method

In this subsection, the notations $S(x) = u$ *and* $S'(x) = \frac{du}{dx}$ are used, so that the equation (18) becomes

$$x\frac{du}{dx} - \theta u + \theta = 0 \tag{34}$$

Consequently, we have

$$\frac{du}{(u-1)} = \frac{\theta}{x}dx \tag{35}$$

By taking an indefinite integration for both sides, we get

$$u = ax^{\theta} + 1 \tag{36}$$

where, a is a constant. Replacing u by $S(x)$, we get

$$S(x) = ax^{\theta} + 1 \tag{37}$$

The initial condition in equation (19) gives us

$$S(x) = 1 - x^{\theta}$$

4.1.3 Solve the equations by the Bernoulli method

To solve the equation (22) by Bernoulli method, we use the notation $\tilde{\theta} = (\theta - 1)$ and transformation $u(x) = \left(h(x)\right)^{-1}$, we get

$$u' + \frac{\tilde{\theta}}{x}u = -1 \ . \tag{38}$$

Assume that $P(x) = \frac{\tilde{\theta}}{x}$ *and* $Q(x) = -1$, hence

$$I = e^{\int P(x)dx} = x^{\tilde{\theta}} \ . \tag{39}$$

But,

$$I.u = \int I.Q(x)dx.$$

Therefore,

$$u = -\frac{x}{\tilde{\theta}+1} + cx^{-\tilde{\theta}} , \tag{40}$$

where c is a constant. Transforming $u(x)$ by $\left(h(x)\right)^{-1}$ and replacing $\tilde{\theta}$ by -1 , we obtain

$$h(x) = \frac{\theta}{x(\theta c x^{-\theta} - 1)} \ . \tag{41}$$

The initial condition in the equation (23) gives us

$$h(x) = \frac{\theta\, x^{\theta-1}}{1 - x^{\theta}} \ .$$

4.2 Approximate Solutions

In this section, the variational iteration method (VIM) and Runge-Kutta 4th order method (RK4) are used to solve equations (8), (12), (18) and (22)

4.2.1 The Variation Iteration Method (VIM)

Consider the following differential equation, [21],[22]

$$Ly + Ny = k(x) \tag{42}$$

Where, L and N are linear and nonlinear operators respectively *k(x)* is the source inhomogeneous term. The variation iteration method (VIM) is defined in the following form

$$y_{k+1}(x) = y_k(x) + \int_{x_0}^{x} \lambda(t)\, (Ly + Ny - k(x))dt, \tag{43}$$

and the solution is

$$y(x) = lim_{k\to\infty}\, y_k(x) ,$$

where λ is a Lagrange multiplier, in this paper we put $\lambda = -1$.
In this section, the equation (8) is solved by VIM. Moreover, we rewrite the equation (8) according to equation (42), we have

$$Du = (\theta - 1)\theta\, x^{\theta-2} \quad . \tag{44}$$

Where $u = f(x)$ and D is a derivative operator. $Ly = Du, Ny = 0, k(x) = (\theta - 1)\theta\, x^{\theta-2}$, if we take the initial state at x_0 ,then the variation iterative can be written as

$$\mathrm{u_{k+1}(x) = u_k(x)} + \int_{\mathrm{x_0}}^{\mathrm{x}} \lambda\,(\mathrm{Du_k(t)} - (\theta - 1)\theta\, \mathrm{t}^{\theta-2})\mathrm{dt}, k \geq 0 \tag{45}$$

where λ is a Lagrange multiplier, in this paper we put $\lambda = -1$ and putting in with initial condition (10): $u(x_0) = \theta(x_0)^{\theta-1}$.

$u_1(x) = u_0(x) + \int_{x_0}^{x} \lambda\,(Du_0(t) - (\theta - 1)\theta\, t^{\theta-2})dt$, *(k=0)*

$u_2(x) = u_1(x) + \int_{x_0}^{x} \lambda\,(Du_1(t) - (\theta - 1)\theta\, t^{\theta-2})dt$, *(k=1)*

$u_3(x) = u_2(x) + \int_{x_0}^{x} \lambda\,(Du_2(t) - (\theta - 1)\theta\, t^{\theta-2})dt$, *(k=2)*

.
.
.

Consequently, the solutions are given by

$$u(x) = lim_{k\to\infty}\, u_k(x) \tag{46}$$

Rewriting the ordinary differential equation (12) according to equation (42), we have

$$Dy = \theta x^{\theta-1} \tag{47}$$

Where, *y=F(x), and D* is a derivative operator. If the initial state at x_0 is taken then the variation iterative can be written as

$$y_{k+1}(x) = y_k(x) + \int_{x_0}^{x} \lambda\,(Dy_k(t) - \theta\, t^{\theta-1})dt, k \geq 0 \;, \tag{48}$$

where λ is a Lagrange multiplier, in this paper we put $\lambda = -1$ and putting in with initial condition (14) that means *k=0;*

$$y_1(x) = y_0(x) + \int_{x_0}^{x} \lambda\,(Dy_0(t) - \theta\, t^{\theta-1})dt$$

$$y_2(x) = y_1(x) + \int_{x_0}^{x} \lambda\,(Dy_1(t) - \theta\, t^{\theta-1})dt$$

$$y_3(x) = y_2(x) + \int_{x_0}^{x} \lambda\,(Dy_2(t) - \theta\, t^{\theta-1})dt$$

.
.

Consequently, the solutions are given by

$$y(x) = lim_{k\to\infty}\, y_k(x) \tag{49}$$

4.2.2 The Runge-Kutta of 4th Order (RK4) Method

The RK4 method is a numerical technique which is used to solve the ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y)\,, y(0) = y_0 \tag{50}$$

So, first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method [23].
The RK4 is one of the most accurate iteration numerical methods that has the general form;

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{51}$$

Where

$k_1 = f(x_i, y_i)$

$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1)$$

$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2)$
$k_4 = f(x_i + h, y_i + hk_3)$
Moreover, we rewrite the ordinary differential equations (18) and (22) in the form of equation (51) ,respectively. We have for the equation (18);

$$y' = \frac{\theta(y-1)}{x} \tag{52}$$

$y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$, *i=0,1,2,...*
When *i=0*

$$k_1 = f(x_0, y_0) = \frac{\theta(y_0 - 1)}{x_0}$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right) = \frac{\theta(y_0 + \frac{h}{2}k_1 - 1)}{x_0 + \frac{h}{2}}$$

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2\right) = \frac{\theta(y_0 + \frac{h}{2}k_2 - 1)}{x_0 + \frac{h}{2}}$$

$$k_4 = f(x_0 + h, y_0 + hk_3) = \frac{\theta(y_0 + hk_3 - 1)}{x_0 + h}$$

When *i*=1

$$k_1 = f(x_1, y_1) = \frac{\theta(y_1 - 1)}{x_1}$$

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1\right) = \frac{\theta(y_1 + \frac{h}{2}k_1 - 1)}{x_1 + \frac{h}{2}}$$

$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_2\right) = \frac{\theta(y_1 + \frac{h}{2}k_2 - 1)}{x_1 + \frac{h}{2}}$$

$$k_4 = f(x_1 + h, y_1 + hk_3) = \frac{\theta(y_1 + hk_3 - 1)}{x_1 + h}$$

.
.

For Eq. (22),

$$u' = \frac{xu^2 + (\theta - 1)u}{x} \tag{53}$$

$u_{i+1} = u_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$, *i=0,1,2, ...*
When *i=0*

$$k_1 = f(x_0, u_0) = \frac{x_0 {u_0}^2 + (\theta - 1)u_0}{x_0}$$

$$k_2 = f\left(x_0 + \frac{h}{2}, u_0 + \frac{h}{2}k_1\right) = \frac{(x_0 + \frac{h}{2})(u_0 + \frac{h}{2}k_1)^2 + (\theta - 1)(u_0 + \frac{h}{2}k_1)}{x_0 + \frac{h}{2}}$$

$$k_3 = f\left(x_0 + \frac{h}{2}, u_0 + \frac{h}{2}k_2\right) = \frac{(x_0 + \frac{h}{2})(u_0 + \frac{h}{2}k_2)^2 + (\theta - 1)(u_0 + \frac{h}{2}k_2)}{x_0 + \frac{h}{2}}$$

$$k_4 = f(x_0 + h, u_0 + hk_3) = \frac{(x_0 + h)(u_0 + hk_3)^2 + (\theta - 1)(u_0 + hk_3)}{x_0 + h}$$

When *i=1*

$$k_1 = f(x_1, u_1) = \frac{x_1 {u_1}^2 + (\theta - 1)u_1}{x_1}$$

$$k_2 = f\left(x_1 + \frac{h}{2}, u_1 + \frac{h}{2}k_1\right) = \frac{(x_1 + \frac{h}{2})(u_1 + \frac{h}{2}k_1)^2 + (\theta - 1)(u_1 + \frac{h}{2}k_1)}{x_1 + \frac{h}{2}}$$

$$k_3 = f\left(x_1 + \frac{h}{2}, u_1 + \frac{h}{2}k_2\right) = \frac{(x_1 + \frac{h}{2})(u_1 + \frac{h}{2}k_2)^2 + (\theta - 1)(u_1 + \frac{h}{2}k_2)}{x_1 + \frac{h}{2}}$$

$$k_4 = f(x_1 + h, u_1 + hk_3) = \frac{(x_1 + h)(u_1 + hk_3)^2 + (\theta - 1)(u_1 + hk_3)}{x_1 + h}$$

.

.

where, $y = S(x)$ *and u = h(x)*

5. Results and Discussions

In our study, the ODEs from power function distribution are obtained and solved using VIM. In this section, some results under study are discussed. The Table 1 summarizes the values of approximate and exact solutions of the equation (8) by using VIM of the cases θ = 2,3 and 4 with values *x*= 0.1, 0.2, 0.3, 0.5, 0.7 and 0.9. The solutions have been calculated with different values of parameters. The calculated values show the following results:

(i) There is a convergence in the solutions when x is smaller than 0.3, whenever the value of x increases, the error grows.

(ii) As the values of θ increase, the error increases when $x \geq 0.3$. In any case, the error between the approximate and exact solutions depends upon the parameter θ.

Figure 1 illustrates the relationship between the approximate and exact with the parameter θ in the equation (8). The parameter θ plays an important role in determining the error between the approximate and exact solutions for the equation (8) which increases when the θ increases with $x \geq 0.3$. It can be seen that the effectiveness of θ on the error is clear whenever the value of θ has large and vice versa when it is small.

Table 2 summarizes the values of approximate and exact solutions of the equation (13) by using VIM of the cases θ = 2,3 and 4 with values *x*= 0.1, 0.2, 0.3, 0.5, 0.7 and 0.9. The calculated values show the following results:

(i) There is a convergence in the solutions when *x* has small values close to 0.1, while the error grows as the value of x increases.

(ii) The larger values of θ have less the errors that occur when $x < 0.5$, and the errors increases as θ increases when $x \geq 0.5$. In any case, the error between the approximate and the exact solutions depends upon the parameter θ.

Figure 2 illustrates the relationship between the approximate and the exact solutions with the parameter θ in the equation (13). In Figure 2, the curve with different values of θ is plotted. We can see that the highest curve of equation (13) with the greatest error and the value of parameter θ appears when $x \geq 0.5$.

On the other hand, the RK4 results under the study are discussed in this section. Table 3 presents the approximate and the exact solutions of the differential equation (18) are

computed by using RK4 method of the cases θ = 2,3 and 4 in the values of x= 0.1, 0.2, …, 0.9. The computed values indicate that when the value of x increases, the error increases, and when the values of θ increase, the error increases when $x \geq 0.5$.
Figure 3 illustrates the relationship between the RK4 and the exact with the parameter θ in the equation (18), so we can see the equation (18) is a decreasing function, and the greater the error is the greater the value of the parameter θ.
Table 4 presents the approximate and the exact solutions of the equation (22) are computed by using the RK4 method of the cases θ = 2,3 and 4 in the values of x= 0.1, 0.2, …, 0.9. The calculated values show the following results: The error increases with an increasing in values of x and the values of θ. The best results are obtained when values of x and θ are small.
Figure 4 shows the relationship between the approximate and the exact solutions with the parameter
θ in the equation (22) which is an increasing function, and the greater the error is the greater value of θ. All results of programs are done by Matlab R2016b.

Table 1- Exact and VIM solutions of eq. (8) when θ = 2,3 and 4 of x= 0.1, 0.2, 0.3, 0.5, 0.7 and 0.9.

$\theta > 0$ $\theta \neq 1$	x ∈ (0,1]	VIM	Exact	Error	Time
2	0.1	0.2	0.2	0	00:00:00
	0.2	0.399	0.4	0.001	00:00:03
	0.3	0.5979	0.6	0.0021	00:00:02
	0.5	0.9959	1	0.0041	00:00:02
	0.7	1.3938	1.4	0.0062	00:00:02
	0.9	1.7918	1.8	0.0082	00:00:03
3	0.1	0.03	0.03	0	00:00:06
	0.2	0.1187	0.12	0.0013	00:00:06
	0.3	0.2653	0.27	0.0047	00:00:03
	0.5	0.7325	0.75	0.0175	00:00:03
	0.7	1.4317	1.47	0.0383	00:00:04
	0.9	2.3627	2.43	0.0673	00:00:03
4	0.1	0.004	0.004	0	00:00:07
	0.2	0.0311	0.032	0.0009	00:00:03
	0.3	0.1027	0.1080	0.0053	00:00:03
	0.5	0.4647	0.5000	0.0353	00:00:03
	0.7	1.2605	1.3720	0.1115	00:00:03
	0.9	2.6607	2.9160	0.2553	00:00:03
10	0.1	1.0000e-08	1.0000e-08	0.0000	00:00:08
	0.2	3.2716e-06	5.1200e-06	1.8484e-06	00:00:02
	0.3	7.9933e-05	1.9683e-04	1.1690e-04	00:00:02
	0.5	0.0046	0.0195	0.0150	00:00:02
	0.7	0.0695	0.4035	0.3341	00:00:02
	0.9	0.5479	3.8742	3.3263	00:00:02
15	0.1	1.5000e-13	1.5000e-13	0.0000	00:00:10
	0.2	7.1515e-10	2.4576e-09	1.7424e-09	00:00:02
	0.3	6.1919e-08	7.1745e-07	6.5553e-07	00:00:02
	0.5	1.7911e-05	9.1553e-04	8.9762e-04	00:00:02
	0.7	8.4487e-04	0.1017	0.1009	00:00:02
	0.9	0.0163	3.4315	3.4153	00:00:02

Table 2-Exact and VIM solutions of Eq. (13) when θ = 2,3 and 4 and x= 0.1, 0.2, 0.3, 0.5, 0.7 and 0.9.

$\theta > 0$	$x \in (0,1]$	VIM	Exact	Error	Time
2	0.1	0.01	0.01	0	00:00:07
	0.2	0.0396	0.04	0.0004	00:00:03
	0.3	0.0884	0.09	0.0016	00:00:03
	0.5	0.2442	0.25	0.0058	00:00:03
	0.7	0.4772	0.49	0.0128	00:00:03
	0.9	0.7876	0.81	0.0224	00:00:00
3	0.1	0.001	0.001	0	00:00:16
	0.2	0.0078	0.008	0.0002	00:00:03
	0.3	0.0257	0.027	0.0013	00:00:03
	0.5	0.1162	0.125	0.0088	00:00:03
	0.7	0.3151	0.343	0.0279	00:00:03
	0.9	0.6652	0.729	0.0638	00:00:03
4	0.1	0.0001	0.0001	0	00:00:09
	0.2	0.0015	0.0016	0.0001	00:00:03
	0.3	0.0073	0.0081	0.0008	00:00:03
	0.5	0.0529	0.0625	0.0096	00:00:04
	0.7	0.1973	0.2401	0.0428	00:00:04
	0.9	0.5294	0.6561	0.1267	00:00:04

Table 3-Exact and RK4 solutions of Eq. (18) θ = 2,3 and 4 of x= 0.1, 0.2, …, 0.9.

$\theta > 0$	$x \in (0,1]$	RK4	Exact	Error	Time
2	0.1	0.99	0.99	0	00:00:00
	0.2	0.9606	0.96	0.0006	00:00:00
	0.3	0.9114	0.91	0.0014	00:00:00
	0.4	0.8425	0.84	0.0025	00:00:00
	0.5	0.7539	0.75	0.0039	00:00:00
	0.6	0.6457	0.64	0.0057	00:00:00
	0.7	0.5177	0.51	0.0077	00:00:00
	0.8	0.3701	0.36	0.0101	00:00:00
	0.9	0.2028	0.19	0.0128	00:00:00
3	0.1	0.999	0.999	0	00:00:00
	0.2	0.9925	0.992	0.0005	00:00:00
	0.3	0.9749	0.973	0.0019	00:00:00
	0.4	0.9406	0.936	0.0046	00:00:00
	0.5	0.8841	0.875	0.0091	00:00:00
	0.6	0.7998	0.784	0.0158	00:00:00
	0.7	0.6821	0.657	0.0251	00:00:00
	0.8	0.5255	0.488	0.0375	00:00:00
	0.9	0.3244	0.271	0.0534	00:00:00
4	0.1	0.9999	0.9999	0	00:00:00
	0.2	0.9986	0.9984	0.0002	00:00:00
	0.3	0.9933	0.9919	0.0014	00:00:00
	0.4	0.979	0.9744	0.0046	00:00:00
	0.5	0.9489	0.9375	0.0114	00:00:00
	0.6	0.8941	0.8704	0.0237	00:00:00
	0.7	0.8038	0.7599	0.0439	00:00:00
	0.8	0.6654	0.5904	0.075	00:00:00
	0.9	0.4641	0.3439	0.1202	00:00:00

Table 4-Exact and RK4 solutions of Eq. (22) θ = 2,3 and 4 of x= 0.1, 0.2, …, 0.9.

$\theta > 0$ $\theta \neq 1$	x ∈ (0,1)	RK4	Exact	Error	Time
2	0.1	0.2020	0.2020	0	00:00:00
	0.2	0.4166	0.4167	0.0001	00:00:00
	0.3	0.6592	0.6593	0.0001	00:00:00
	0.4	0.9522	0.9524	0.0002	00:00:00
	0.5	1.3330	1.3333	0.0004	00:00:00
	0.6	1.8743	1.8750	0.0007	00:00:00
	0.7	2.7434	2.7451	0.0017	00:00:00
	0.8	4.4379	4.4444	0.0065	00:00:00
	0.9	9.3889	9.4737	0.0848	00:00:00
3	0.1	0.0300	0.0300	0	00:00:00
	0.2	0.1193	0.1210	0.0017	00:00:00
	0.3	0.2730	0.2775	0.0045	00:00:00
	0.4	0.5040	0.5128	0.0088	00:00:00
	0.5	0.8412	0.8571	0.0159	00:00:00
	0.6	1.3488	1.3776	0.0287	00:00:00
	0.7	2.1815	2.2374	0.0560	00:00:00
	0.8	3.8004	3.9344	0.1341	00:00:00
	0.9	8.3822	8.9668	0.5846	00:00:00
4	0.1	0.0040	0.0040	0	00:00:00
	0.2	0.0300	0.0321	0.0020	00:00:00
	0.3	0.1011	0.1089	0.0077	00:00:00
	0.4	0.2432	0.2627	0.0195	00:00:00
	0.5	0.4918	0.5333	0.0415	00:00:00
	0.6	0.9095	0.9926	0.0832	00:00:00
	0.7	1.6335	1.8055	0.1720	00:00:00
	0.8	3.0523	3.4688	0.4165	00:00:00
	0.9	6.8383	8.4792	1.6409	00:00:00

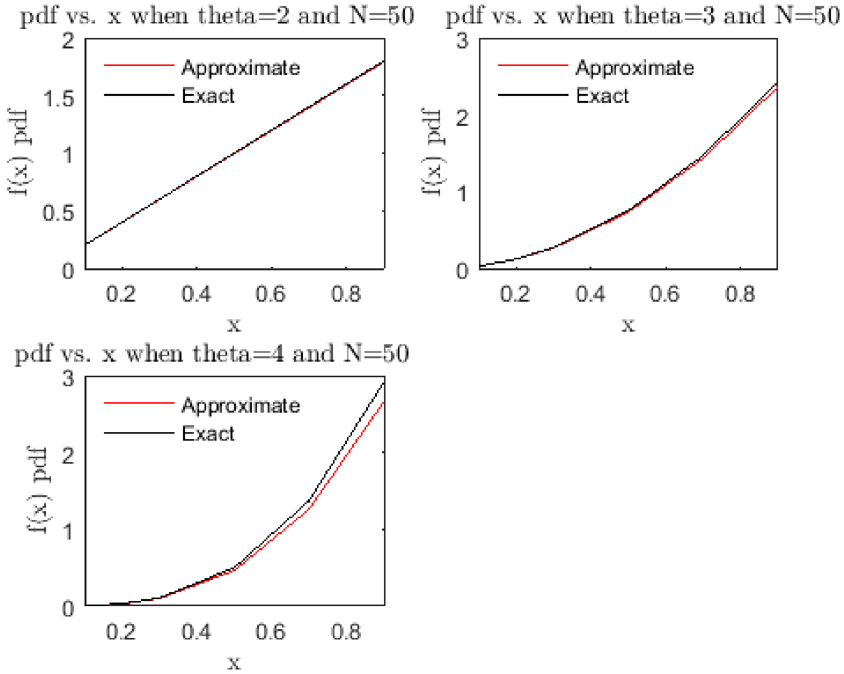

Figure 1-Exact and VIM solutions of Eq.(8) with different values of θ when N=no.of iterations for VIM

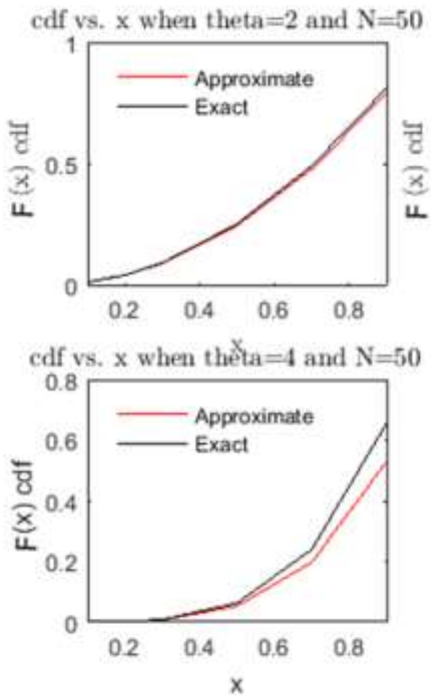

Figure 2-Exact and VIM solutions of Eq. (13) with different values of θ when N=no.of iterations for VIM

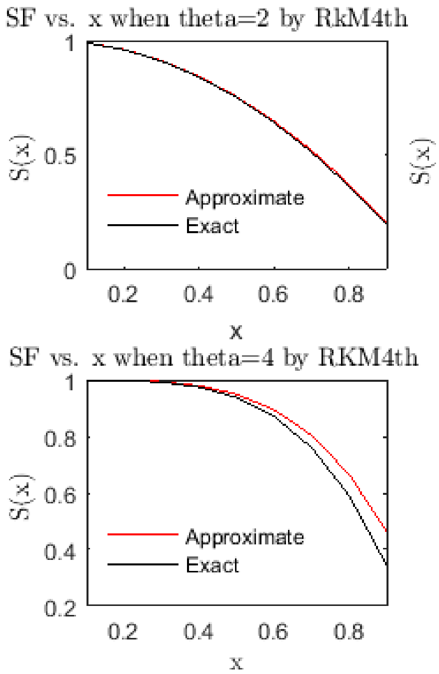

Figure 3- Exact and RK4 solutions with different values of parameter θ of Eq. (18) when N=no.of iterations for RK4

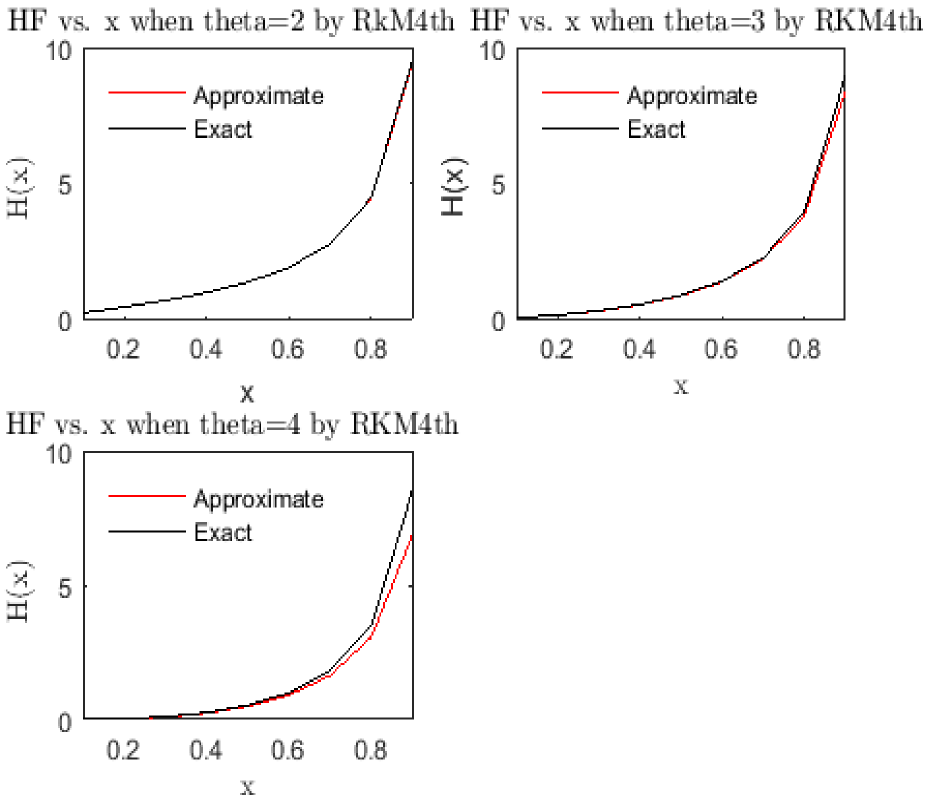

Figure 4- Exact and RK4 solutions with different values of parameter θ of Eq. (22) when N=no.of iterations for RK4

6. Conclusions

Our statistical tools are extended to design the ordinary differential equations from the probability distributions. Some probability functions such as a probability density function, cumulative distribution function, survival function and hazard function of power function distribution are used for this propose. In fact, there are some methods to find the exact solutions to these equations. However, in this work, the methods VIM analytic and RK4 numerical methods are chosen to obtain the approximate results analytically and numerically. In general, the present approximate results have an error that is almost small, the best results are obtained for all equations (8), (13), (18) and (22) when the values of x and θ are small that means the convergence of solutions of the mentioned equations under the study is clear when $\theta < 2$ and $x < 0.3$.

References

[1] S. K. Sinha, P. Singh, and R. Singh, "Preliminary test estimators for the scale parameter of power function distribution," *Journal of Reliability and Statistical Studies*, pp. 18-24, 2008.

[2] A. Dallas, "Characterizing the Pareto and power distributions," *Annals of the Institute of Statistical Mathematics*, vol. 28, pp. 491-497, 1976.

[3] N. S. Butt, M. A. ul Haq, R. M. Usman, and A. A. Fattah, "Transmuted power function distribution," *Gazi University Journal of Science*, vol. 29, pp. 177-185, 2016.

[4] S. Roy, S. Nag, I. K. Maitra, and S. K. Bandyopadhyay, "International journal of advanced research in computer science and software engineering," *International Journal*, vol. 3, 2013.

[5] W. Schappacher, *Differential Equations with Applications in Biology, Physics, and Enqineering*: Routledge, 2017.

[6] G. Scholz and F. Scholz, "First-order differential equations in chemistry," *ChemTexts*, vol. 1, pp. 1-12, 2015.

[7] H. I. Okagbue, M. O. Adamu, and T. A. Anake, "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Burr XII and Pareto Distributions," *Proceedings of the World Congress on Engineering and Computer Science. Lecture Notes in Engineering and Computer Science (San Francisco, USA),* vol. 1, pp. 399–404, 2017.

[8] H. I. Okagbue, M. IAENG, P. E. Oguntunde, A. A. Opanuga, and E. A. Owoloko, "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Frĕchet Distribution," *Proceedings of the World Congress on Engineering and Computer Science. Lecture Notes in Engineering and Computer Science (San Francisco, USA),* vol. 1, pp. 186–191, 2017.

[9] H. I. Okagbue, P. E. Oguntunde, A. A. Opanuga, and P. I. Adamu, "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Linear Failure Rate and Generalized Linear Failure Rate Distributions," *Int. J. Circuits, Systems and Signal Proces,* vol. 12, pp. 596-603, 2018.

[10] H. I. Okagbue, P. E. Oguntunde, A. A. Opanuga, and E. A. Owoloko, "Classes of ordinary differential equations obtained for the probability functions of exponential and truncated exponential distributions," in *Proceedings of the World Congress on Engineering and Computer Science*, 2017.

[11] H. I. Okagbue, P. E. Oguntunde, P. O. Ugwoke, and A. A. Opanuga, "Classes of Ordinary Differential Equations Obtained for The Probability Functions of Exponentiated Generalized Exponential Distribution," *Proceedings of the World Congress on Engineering and Computer Science. Lecture Notes in Engineering and Computer Science (San Francisco, USA),* vol. 1, pp. 192–197, 2017.

[12] H. I. Okagbue, O. A. Odetunmibi, A. A. Opanuga, and P. E. Oguntunde, "Classes of ordinary differential equations obtained for the probability functions of half-normal distribution," in *Proceedings of the World Congress on Engineering and Computer Science*, 2017.

[13] H. I. Okagbue, I. Member, M. O. Adamu, E. A. Owoloko, and E. A. Suleiman, "Classes of Ordinary Differential Equations Obtained for The Probability Functions of Harris Extended Exponential Distribution. ," *Proceedings of the World Congress on Engineering and Computer Science. Lecture Notes in Engineering and Computer Science (San Francisco, USA),* vol. 2, pp. 883–888, 2017.

[14] H. I. Okagbue, M. O. Adamu, T. A. Anake, and A. A. Opanuga, "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Kumaraswamy Distribution," in *Proceedings of the International MultiConference of Engineers and Computer Scientists*, 2018.

[15] H. I. Okagbue, P. E. O. IAENG, A. A. Opanuga, and P. I. Adamu, "Classes of Ordinary Differential Equations Obtained for the Probability Functions of Logistic and Log-Logistic Distributions," in *Proceedings of the World Congress on Engineering and Computer Science*, 2018.

[16] H. I. Okagbue, M. O. Adamu, A. A. Opanuga, and J. G. Oghonyon, "Classes of Ordinary Differential Equations Obtained for the Probability Functions of 3-Parameter Weibull Distribution.," *Proceedings of the World Congress on Engineering and Computer Science. Lecture Notes in Engineering and Computer Science (San Francisco, USA),* vol. 2, pp. 539–545, 2017.

[17] M. A. Sabaa and M. A. Mohammed, "Approximate Solutions of Nonlinear Smoking Habit Model," *Iraqi Journal of Science,* pp. 435-443, 2020.

[18] M. A. Sabaa, M. A. Mohammed, and S. H. Abd Almjeed, "Approximate solutions for alcohol consumption model in spain," *Ibn AL-Haitham Journal For Pure and Applied Science,* vol. 32, pp. 153-164, 2019.

[19] M. A. Mohammed, N. F. M. Noor, Z. Siri, and A. I. N. Ibrahim, "Numerical solution for weight reduction model due to health campaigns in Spain," in *AIP Conference Proceedings*, 2015, p. 020005.

[20] M. L. Abell and J. P. Braselton, *Introductory Differential Equations*: Academic Press, 2018.

[21] A.-M. Wazwaz, *Partial differential equations and solitary waves theory*: Springer Science & Business Media, 2010.

[22] Abdulsahib Ali A., F. S. Fadhel, and S. H. Abid., "Modified Approach for Solving Random Ordinary Differential Equations," *Journal of Theoretical and Applied Information Technology,* vol. 97, 2019.

[23] D. G. Zill, *A first course in differential equations with modeling applications*: Cengage Learning, 2012.